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Asymp inv of base loci.

X sm proj $/\mathbb{C}$, $L \in \mathrm{Pic}(X)$, $\mathrm{Bs}(L)$

$N^1(X) = \mathrm{Pic}(X)/\equiv$, f.g free ab gp by Neron-Severi

$N^1(X)_{\mathbb{R}} \simeq \mathbb{R}^\rho$

Need to work asymp $B(L) := \bigcap_{m \geq 1} \mathrm{Bs}(L^m)$

$= \mathrm{Bs}(L^p)$

if p is divisible enough by Noetherian induction

Q1: How much of $B(L)$ is detected numerically?

Rmk: $B(L) = B(L^m) \Rightarrow$ can define $B(D)$ for $D \in \mathrm{Div}_{\mathbb{Q}}$

Q2: Define loci for $\mathbb{R}$-divisors, functions on $N^1(X)_{\mathbb{R}}$ describing these loci.

Rmk: Always work with <u>big</u> divisors, via approximations.

D big $\mathbb{R}$-div $\iff \exists\, D = A + E$, A ample, E effective

if $D = \mathbb{Q}$-div, big $\iff |mD|$ gives a gen finite (onto image) map

$\mathrm{Big}(X)_{\mathbb{R}} = \{\text{big } \mathbb{R}\text{-div classes}\} \subseteq N^1(X)_{\mathbb{R}}$

open cone

(interior of $\overline{\{[D] \mid D = \text{eff } \mathbb{R}\text{-div}\}}$)

Rmk: $\exists$ example, $D \equiv E$ but $B(D) \neq B(E)$

trivial one: X curve $g \geq 1$, $\mathcal{O}(D) \in \mathrm{Pic}^0(X)$ ...

$\exists$ also big examples !

Idea : Look at small approx of D.

Def : $B_+(D) := \bigcap_{A \text{ ample}} B(D-A) = B(D-A)$ if A as before, $\|A\|$ small

augmented base locus

$D-A = \mathbb{Q}$-div

for any $D = \mathbb{R}$-div

$\Rightarrow$ closed subset of X.

Rmk 1) $D \equiv E \Rightarrow B(D) = B(E)$

trivial : $D - A = E - (\underbrace{A + \underbrace{E - D}_{\text{num} = 0}}_{\text{ample}})$

2) $B_+(D) = \bigcap_{\substack{D = A + E \\ \text{ample} \quad \text{eff}}} \operatorname{Supp}(E)$

3) $B_+(D) \neq X \iff D = \text{big}$

4) $D = \mathbb{Q}\text{-div} \Rightarrow B(D) \subseteq B_+(D)$.

This is defined by Nakamaya and it is interesting bec of the following thm of him :

when D is nef, notice that D big $\iff D^{\dim X} > 0$.

If $V \subset X$ subvariety st. $D|_V \neq$ big

$\Rightarrow V \subseteq B_+(D)$:

if $V \nsubseteq B_+(D) \Rightarrow D = \underset{\text{ample}}{A} + \underset{\text{ef}}{E}$

$V \nsubseteq \operatorname{Supp} E$

$\Rightarrow D|_V = A|_V + E|_V$ big, $\times$.

Thm (Nakamaya):

$$D = \text{big} + \text{nef} \iff B_+(D) = \bigcup_{D|_V \neq \text{big}} V .$$

Goal: get analogue with D just big.

Rmk: if $D = \text{nef}$, $B_+(D)$ is described by $(V \to (D|_V^{\dim V}))$

Rmk: Suppose $D = \mathbb{Q}$-div with Zariski decomp:

i.e. $D = \underset{\text{nef}}{P} + \underset{\text{effective}}{N}$ both P, N $\mathbb{Q}$-div.

st. $H^0(X, \mathcal{O}(mP)) \xrightarrow{\sim} H^0(X, \mathcal{O}(mD))$ $\forall$ m divisible enough.

(True for dim 2 by Zariski. For higher dim need $\mathbb{R}$-div, need blow-ups (higher model), but still has counter examples.)

Then $B_+(D) = B_+(P) \cup \operatorname{Supp}(N)$.

- Want invariants associated to D, $V \subseteq X$ which if $\exists$ Zariski decomp, are $(P|_V^{\dim V})$.

Question of Vojta:

E = v.b. on X, $\begin{array}{c} \mathbb{P}(E) \\ \pi \downarrow \\ X \end{array}$, assume E = big (i.e. $\mathcal{O}(1)$ big)

If E semi-stable in the sense of Bogomolov, does it follow that $\pi(B_+(\mathcal{O}(1))) \neq X$?

OK if X = curve (easy, by Vojta, since in this case $\mathcal{O}(1)$ is indeed ample).

Eg. $E = \Omega_X$, X = surface of general type.

(This is used in Diophantine approx and hyperbolicity prob.)

Digression on "Restricted base locus":

more elementary if assume vanishing thm's.

Def: D = $\mathbb{R}$-div

$$B_-(D) := \bigcup_{\substack{A \text{ ample} \\ D+A = \mathbb{Q}\text{-div}}} B(D+A)$$

Rmk: 1) $D \equiv E \Rightarrow B_-(D) = B_-(E)$

2) $D = \mathbb{Q}$-div $\Rightarrow B_-(D) \subseteq B(D)$

Q: Is $B_-(D)$ closed? (Not known).

3) D = nef iff $B_-(D) = \phi$.

4) Suppose $D = P + N$ is a Zariski decomp

then $B_-(D) = \underbrace{B_-(P)}_{= \phi} \cup \operatorname{Supp}(N) = \operatorname{Supp}(N)$!

(This is first studied by Nakayama in connection with Zariski decomp).

Can prove: 1) $B_-(D) = \bigcup_{m \geq 1} V(\mathcal{J}(\| mD \|))$ (if D is integral)

asymptotic multiplier ideals.

2) can be described by functions as follows:

Let $x \in X$, $\operatorname{ord}_x(\|D\|) := \lim_{m \to \infty} \frac{\operatorname{mult}_x |mD|}{m}$

$\operatorname{mult}_x |mD| \overset{\text{def}}{=} \operatorname{mult}_x E$, $E \in |mD|$ general.

- $\mathrm{ord}_x(\|\cdot\|)$ conti on $\mathrm{Big}(X)_{\mathbb{R}}$.
- $\mathrm{ord}_x(\|D\|) > 0$ iff $x \in B_-(D)$.

Rmk: $B_-(D) = \{x \mid \{\mathrm{mult}_x |mD|\}_m$ is unbounded $\}$.

Def: D = stable if $B_-(D) = B_+(D)$,

this is equiv to $B(-)$ is constant in a nbd of D.

Proposition: The set of stable classes is open and dense in $\mathrm{Big}(X)_{\mathbb{R}}$, the complement has (Lebesgue) measure 0.

2 cases we know very precise structure
- toric & surfaces cases.

have polyhedron big cone str. — has Zariski decomp.

base loci does not change in each chamber.

Now Going back.

Want: D big $\mathbb{R}$-div, $V \subseteq X$ sub var.

get invariant "should be" $(P|_V^{\dim V})$.

If $V = X$, have $\mathrm{vol}(X, D)$.

Def: $L \in \mathrm{Pic}(X)$, $n = \dim X$,

$$\mathrm{vol}(X, L) := \limsup_{m \to \infty} \frac{n! \cdot h^0(X, L^m)}{m^n}$$

Properties: 1) L big iff $\mathrm{vol}(X, L) > 0$.

2) L = ample $\Rightarrow$ $\mathrm{vol}(X, L) = (L^n)$.

3) $\mathrm{vol}(X, L^m) = m^n \mathrm{vol}(X, L)$.

this $\Rightarrow$ can ext. def to $\mathbb{Q}$-divisors.

Something not so trivial is:

4) $L \equiv M \Rightarrow vol(X,L) = vol(X,M)$

5) $vol(X,-)$ can be extended as conti function to $N^1(X)_{\mathbb{R}}$.

Theorem (Fujita): (Asymp. Zariski decomp.)

D = big $\mathbb{Q}$-div. then

$vol(X,D) = \sup(A^n)$ among all $\rightarrow$ $\left\{ \begin{array}{l} X' \text{ smooth} \\ \downarrow \pi \text{ proper birati} \\ X \\ \pi^* D = \underset{\text{ample}}{A} + \underset{\text{ef}}{E} \end{array} \right.$

Def: $V \hookrightarrow X$ sub var. dim r. $L \in Pic(X)$.

$$Vol_X(V,L) := \limsup \frac{r!\, h^0_X(V,L^m)}{m^r} \leq vol(V, L|_V).$$

$$h^0_X(V,L^m) = \dim \operatorname{Im}\big(H^0(X,L^m) \to H^0(V,L^m)\big).$$

Again some trivial properties:

1) $vol_X(V,L^m) = m^r \cdot vol_X(V,L) \rightsquigarrow$ def for $\mathbb{Q}$-div.

2) L ample $\Rightarrow vol_X(V,L^m) = vol(L|_V) = (L|_V^n)$.

3) If $V \subsetneq B_+(L)$, $\Rightarrow L = \underset{\text{ample}}{A} + \underset{\text{ef}}{E}$, $V \subsetneq \operatorname{Supp} E$.

$$vol_X(V,L) \geq vol_X(V,A) = (A|_V^n) > 0.$$

Idea: properties of $vol(X,-)$ for big div holds for $vol_X(V,-)$ if $V \nsubseteq B_+(-)$.

Now the analogue of Fujita:

Theorem 1: $D = \mathbb{Q}$-div, $V \not\subseteq B_+(D)$,

$\Rightarrow \mathrm{vol}_X(V, D) = \sup\,(A|_{\tilde{V}}^n)$.

$$\begin{tikzcd} \tilde{V} \arrow[r, hook] \arrow[d] & X' = \mathrm{sm} \arrow[d] \\ V \arrow[r, hook] & X \end{tikzcd}$$

proper transf. $\tilde{V}$

prop birat'l isom at generic pt of V

$\pi^* D = A + E$
ample / ef

$\tilde{V} \not\subseteq \mathrm{supp}\, E$.

This proof is based on Lazarsfeld's version of Demailly's pf of Fujita's thm.

Consequences:

① $D \equiv D'$, $V \not\subseteq B_+(D)$ $\Rightarrow$ $\mathrm{vol}_X(V, D) = \mathrm{vol}_X(V, D')$

② $\mathrm{vol}_X(V, -)$ can be extended as a conti fcn to $\{\alpha \in N^1(X)_{\mathbb{R}} \mid V \not\subseteq B_+(\alpha)\}$.

③ $D = \mathrm{nef}$, $V \not\subseteq B_+(D) \Rightarrow \mathrm{vol}_X(V, D) = (D|_V^n)$.

④ If $D = P + N$ a Zariski decomp, $V \not\subseteq B_+(D) \Rightarrow \mathrm{vol}_X(V, D) = \mathrm{vol}_X(V, P) = (P|_V^n)$.

⑤ Tsuji: $L \in \mathrm{Pic}(X)$,
$L \otimes \mathcal{J}(\|L\|) \hookrightarrow L$ "analytic Zariski decomp called by Tsuji" though not very useful.

If $L = P + N$ Zar. decomp with integral div, then $\mathcal{J}(\|L\|) = \mathcal{O}(-N)$.

Prop: If $V \not\subseteq B_+(D) \Rightarrow \mathrm{vol}_X(V, D)$

$$= \lim_{m \to \infty} \frac{r!\, h^0(V, L^m \otimes \mathcal{J}(\|L^m\|)|_V)}{m^r}$$

Main Theorem :

D = big $\mathbb{R}$-div. V = irred. comp in $B_+(D)$

$\Rightarrow \lim\limits_{\substack{D' \to D \\ \mathbb{Q}\text{-div}}} \mathrm{vol}_X(V, D') = 0$

(This is analogue of Nakamaya's thm without nef assumption. It is long but is elementary since no vanishing thm is using, have a hope in char p.)

Idea of pf : Prop : $L \in \mathrm{Pic}(X)$, $V \subsetneq B_+(L)$

$\forall \varepsilon > 0$, $\exists\ x_1, \cdots, x_N \in V, \varepsilon_1, \cdots, \varepsilon_N > 0$ st.

① $H^0(X, L^m) \twoheadrightarrow \bigoplus\limits_{i=1}^{N} H^0(X, L^m \otimes \mathcal{O}_{X,x_i} / \underline{m}_{X,x_i}^{m\varepsilon_i + 1})$

$\forall$ m divisible enough.

② $\left(\mathrm{vol}_X(V, L) \geq \sum\limits_{i=1}^{N} \varepsilon_i^{n}\right) > \mathrm{vol}(V, L) - \varepsilon$.

For, thm, may assume $V \subsetneq B_-(D)$.

By contrad. assume $\exists\ A_m \to 0$ ample

$\mathrm{vol}_X(V, D + A_m) \geq \eta$.

Prop $\Rightarrow$ $\exists$ Large space of sections of $D + A_m$ with small order of vanishing at some pts

Main step : get sections in $K(D + A_m - B)$

(Tricky) st. $B - A_m$ ample.

(with small order of vanishing on V)

- Then an easy arguement to show $\exists\ M > 0$ st. $\mathrm{ord}_V(D - (B - A_m)) \geq M \cdot \|B - A_m\|$. then

$\times$

End