Mustata 12/1 2003 at CTS

Intr. to sing. via Arcs & Jets.

(X, Y), $Y \hookrightarrow X$ closed

$E \hookrightarrow X'$ divisor, $X' \to X$; $\mathcal{O}_{X',E} \hookrightarrow K(X)$ valuation ord_E

2 type of invariants.

Set Up: X sm var $/\mathbb{C}$
$Y \hookrightarrow X$ closed subscheme

Let $\pi : X' \to X$ log resol of (X, Y):
i.e. π proper birat'l, X' sm.
$\pi^{-1}(Y) \cup \mathrm{Exc}(\pi)$ div with s.n.c.

$$\pi^{-1}(Y) = \sum_{i=1}^{n} a_i E_i$$

$$K_{X'/X} = \text{relative Jacobian} = \sum_{i=1}^{n} K_i E_i$$

def by det Jac(π) locally

Def: log-canonical threshold:

$$lc(X, Y) := \min_i \frac{K_i + 1}{a_i}$$

Rmk: Indep on π.

In particular, $lc(X, Y) := \min_{E = \mathrm{div}/X} \dfrac{1 + \mathrm{ord}_E(K_{-/X})}{\mathrm{ord}_E I_{Y/X}}$

$c \le lc(X, Y) \underset{\text{def}}{\iff} (X, c\cdot Y)$ log canonical

Idea: lc is large (in fixed codim of Y)
$\longleftrightarrow$ good singularities.

Eg. 1) Y smooth, codim r

$$\begin{array}{c} X' = Bl_Y X \\ \downarrow \\ X \end{array} \Rightarrow lc(X,Y) = r$$

2) $f = x_1^{a_1} \cdots x_n^{a_n}$; $lc(f) = lc(\mathbb{A}^n, V(f))$
$= \min_i \frac{1}{a_i}$

3) $f = x_1^{a_1} + \cdots + x_n^{a_n}$; $lc(f) = \min\{1, \sum_{i=1} \frac{1}{a_i}\}$

Motivation:

① $f \in \mathbb{C}[x_1, \cdots, x_n]$,
$lc(f) = \sup\{\alpha > 0 \mid \frac{1}{|f|^{2\alpha}}$ loc. integrable on $\mathbb{C}^n\}$

Pf: on $\mathbb{C}$: $\frac{1}{|z|^{2\alpha}}$ is loc. int. iff $\alpha < 1$,

$$\int \frac{1}{|f|^{2\alpha}} = \int \frac{1}{|f(\pi(x))|^{2\alpha}} \cdot |\det(\mathrm{Jac}\,\pi)|^2$$

π : resol. of sing. $\quad f(\pi(x)) \downarrow$ locally $x_1^{a_1} \cdots x_n^{a_n}$; $\quad \det(\mathrm{Jac}\,\pi) \downarrow x_1^{k_1} \cdots x_n^{k_n}$

Locally, we end up with

$$\int \frac{1}{\prod_{i=1}^n |x_i|^{2(a_i\alpha - k_i)}} \quad \text{integrable} \Leftrightarrow a_i\alpha - k_i < 1 \ \forall i$$
$$\Leftrightarrow \alpha < lc(f).$$

② slightly less known: p-adic case

$f \in \mathbb{Z}[x_1, \cdots, x_n]$, $Y = (f=0) \hookrightarrow \mathbb{A}^n$

Igusa's zeta function:

$$Z_f(s) = \int_{(\mathbb{Z}_p)^n} |f(x)|_p^s \, dx \quad ; \quad p: \text{prime}$$

$$|f(x)|_p^s = \left(\frac{1}{p}\right)^{\mathrm{ord}_p(f(x)) \cdot s}$$

$|f(x)|_p \le \left(\frac{1}{p}\right)^m \Leftrightarrow \mathrm{ord}_p(f(x)) \geq m \Leftrightarrow f(\bar{x}) = 0$ in $\mathbb{Z}/p^m\mathbb{Z}$

The measure of $\{ u \in (\mathbb{Z}_p)^n \mid |f(u)| \le \left(\frac{1}{p}\right)^m \}$

$$= C_m \cdot \text{measure}\,(p^m\mathbb{Z})^n = C_m \cdot \left(\frac{1}{p}\right)^{mn}$$

$C_m = \# \{ u \in (\mathbb{Z}/p^m\mathbb{Z})^n \mid f(u) = 0 \}$

Theorem (Igusa): let $c = lc(f)$,

(i) $\exists A > 0$ st. $C_m < A \cdot p^{m(n-c)}$ for all m

(ii) In good situation (can be achieved for general p, finite ext of $\mathbb{Q}_p$)

$\Rightarrow \exists B > 0$ st. $B \cdot p^{m(n-c)} < C_m$, $\forall m$.

i.e. $\log C_m \sim m(n-c) \log p$.

Idea of pf: use resolution of sing + change of variable formula for p-adic int. to reduce to $f = x_1^{a_1} \cdots x_n^{a_n}$, the remaining is trivial □

Notice: $C_m = \# Y(\mathbb{Z}/p^m\mathbb{Z})$

$$\int_{X(\mathbb{Z}_p)} |f(x)|^s$$

Idea: replace $\mathbb{Z}_p$ by $\mathbb{C}[[t]]$

$\mathbb{Z}/p^m\mathbb{Z}$ by $\mathbb{C}[t]/(t^{m+1})$

Trouble: $X(\mathbb{C}[[t]])$ not locally compact!

This is known for sometime and this difficulity is the reason to wait Kontsevich.

Def: Let Y = scheme, f. type / $\mathbb{C}$

Y_∞ defined by

A : $\mathbb{C}$-alg $\Rightarrow$ $\mathrm{Hom}(\mathrm{Spec}\, A, Y_\infty) \simeq \mathrm{Hom}(\mathrm{Spec}\, A[[t]], Y)$

$A = \mathbb{C} \Rightarrow Y_\infty(\mathbb{C}) = \mathrm{Hom}(\mathrm{Spec}\, \mathbb{C}[[t]], Y)$

As a set, $Y_m = \mathrm{Hom}\left(\mathrm{Spec}\, \frac{\mathbb{C}[t]}{t^{m+1}}, Y\right)$

Eg. $m=0$, $Y_0 = Y$

$m = 1$, $Y_1 = TY$

jet

$\mathbb{C}[t]/t^{m+1} \to \mathbb{C}[t]/t^m \rightsquigarrow Y_m \to Y_{m-1} \to \cdots \to Y_1 \to Y$

& $Y_\infty = \varprojlim Y_m$.

Existence:

Suppose $Y \hookrightarrow \mathbb{A}^n$ def by $(f_\alpha)_\alpha$

$$\mathrm{Spec}\, \frac{\mathbb{C}[t]}{t^{m+1}} \to Y \iff \frac{\mathbb{C}[x_1, \cdots, x_n]}{(f_\alpha)_\alpha} \to \mathbb{C}[t]/t^{m+1}$$

$$\iff x_j \to \sum_{i=0}^{m} \frac{a_j^{(i)}}{i!} t^i \text{ st. } f_\alpha \to 0$$

$f \in \mathbb{C}[x_1, \cdots, x_n]$

$f \to \sum_{i=0}^{m} \frac{f^{(i)}(a)}{i!} t^i$ where $f^{(i)}(x_j, x_j', \cdots, x_j^{(i)})$ i-th derivative of f st $x_j^{(p)} \to x_j^{(p+1)}$

Example:

$f: x^2 - y^3$; $f' = 2xx' - 3y^2 y'$

$f'' = 2(x')^2 + 2xx'' - 6y(y')^2 - 3y^2 y''$

$Y = (f = 0) \hookrightarrow \mathbb{A}^2$,

$Y_2 = (f = f' = f'' = 0) \hookrightarrow \mathbb{A}^6$.

$\Rightarrow \quad Y_m \hookrightarrow \mathbb{A}^{(m+1)n}$
def. by $f_\alpha = f_\alpha' = \cdots = f_\alpha^{(m)} = 0 \quad \forall \alpha$.

This gives very concrete eq'n for Y_m.

In particular, Y def by p eq'ns

$\Rightarrow Y_m \hookrightarrow \mathbb{A}^{(m+1)n}$ def by $p(m+1)$ eq'ns

Rmk: X smooth, $\Rightarrow$ $X_m \to X_{m-1}$ loc. trivial in Zariski top, fiber $\mathbb{A}^{\dim X}$

$\Rightarrow \dim X_m = (m+1) \dim X$

$Y \hookrightarrow X$ (sing, sm) the corr. picture of Y implies information on singularities.

X sm. dim n, $\quad X_\infty \xrightarrow[\psi_m]{} X_m \to \cdots \to X$

cylinders: $C = \psi_m^{-1}(S)$, $S \subseteq X_m$ constructible

Def: C irred. (loc closed) if S is.
$\operatorname{codim}(C) := \operatorname{codim}(S, X_m) = (m+1)\dim X - \dim S$

Another type of subset:

Let $Y \hookrightarrow X$ closed subscheme, $\quad$ $Y_\infty \hookrightarrow X_\infty$, $Y_m \hookrightarrow X_m$

Fact: $Y \neq X \Rightarrow \infty$-codim
$\Rightarrow Y_\infty \not\supseteq$ non-empty cylinders.

$F_Y : X_\infty \to \mathbb{N} \cup \{\infty\}$, $\quad \gamma : \operatorname{Spec} \mathbb{C}[[t]] \to X$

$F_Y(\gamma) := \operatorname{ord}(\gamma^{-1} I_{Y/X})$

$\operatorname{Cont}^p(Y) := F_Y^{-1}(p)$, $\quad \operatorname{Cont}^{\geq p}(Y) := F_Y^{-1}(\geq p)$
contact loci $\qquad = \psi_{p-1}^{-1}(Y_{p-1})$ closed cylinder

($\operatorname{Cont}^p(Y)$:) loc. closed set.

p.6 Hodge realization of motivic measure on cylinders would imply:

C cylinder: $\mu(C) \in \mathbb{Z}[u^{\pm 1}, v^{\pm 1}]$ has degree $-2\operatorname{codim}(C)$

$$I = \int_{X_\infty} (uv)^{sF_Y} := \sum_m \mu(F_Y^{-1}(m)) \cdot (uv)^{sm} \in \mathbb{Z}[[u^{\pm\frac{1}{N}}, v^{\pm\frac{1}{N}}]]$$

if s is some rat'l number.

very similar to Igusa's zeta fcn.
|
Kontsevich
Batyrev
Denef-Loeser

convergence: powers in u, v go to $-\infty$

(A) I conv. "should mean" $2(sm - \operatorname{codim} \operatorname{Cont}^m Y) \to -\infty$ for $m \to \infty$.

It also has a CVF: if $\pi: X' \to X$, log res.

$$\Rightarrow \quad I = \int_{X'_\infty} (uv)^{sF_Y \circ \pi_\infty - F_{K_{X'/X}}}$$

$$sF_Y \circ \pi_\infty - F_{K_{X'/X}} = sF_{\pi^{-1}(Y)} - F_{K_{X'/X}} = s \cdot \sum a_i F_{E_i} - \sum_i k_i F_{E_i}.$$

$$\begin{array}{ccccc} X' & \leftarrow & X'_\infty & & \\ \pi \downarrow & & \downarrow \pi_\infty & & \\ X & \leftarrow & X_\infty & \xrightarrow{F_Y} & \mathbb{N} \cup \{\infty\} \end{array}$$

$$I = \sum_{\nu \in \mathbb{N}^r} \underbrace{\mu\Big(\bigcap_i F_{E_i}^{-1}(\nu_i)\Big)}_{\deg = -2 \cdot \sum \nu_i} \cdot (uv)^{s \sum a_i \nu_i - \sum k_i \nu_i}$$

I convergent "should mean"

(B) $\sum \nu_i (k_i + 1) - s \sum a_i \nu_i \to +\infty$ if $\nu_i \to \infty$

$\Leftrightarrow \quad k_i + 1 - s a_i > 0 \quad \forall i$

$\Leftrightarrow \quad s < lc(X, Y)$.

If we put together these 2 lim. conditions (A) & (B), then get; This suggests:

$$lc(X,Y) = \inf_p \frac{\operatorname{codim}(\operatorname{Cont}^p(Y))}{p}.$$

But indeed stronger results holds:
joint work with L. Ein, R. Lazarsfeld:

Let $C \subseteq X_\infty$ loc. closed. irred. cylinder
$X = \operatorname{Spec} A$, $f \in A \Rightarrow \operatorname{ord}_C(f) = \min_{\gamma \in C} F_f(\gamma) \in \mathbb{N}$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \operatorname{ord}_{\gamma^{-1}}(f)$

ord_C can be extended to a valuation of $K(X)$.
(Always assume) if $C \neq$ dominate $X \Rightarrow$ discrete valuation
examples: 1) $E \hookrightarrow X$ sm divisor

$C = \operatorname{Cont}^p(E) \Rightarrow \operatorname{ord}_C = p \cdot \operatorname{ord}_E \quad (p \geq 1)$

2) Let E be a divisor over X,

$$\begin{array}{ccccc} E & \hookrightarrow & X' & \leftarrow & X'_\infty \\ & & \downarrow \pi & & \downarrow \pi_\infty \\ & & X & \leftarrow & X_\infty \end{array}$$

Proposition: $\overline{\pi_\infty(\operatorname{Cont}^p(E))} =: \mathcal{C}^p(E)$
is an irreducible cylinder.

Rmk: $\operatorname{ord}_{\mathcal{C}^p(E)} = p \cdot \operatorname{ord}_E$.

Theorem 1. For any C (irr. loc. closed, does not dominant X)
$\Rightarrow \exists\, p, E$ st. 1) $C \hookrightarrow \mathcal{C}_p(E)$,
2) $\operatorname{ord}_C = p \cdot \operatorname{ord}_E$.

Theorem 2: $\operatorname{codim}(\mathcal{C}_p(E)) = p\cdot(\operatorname{ord}_E(K_{-/X})+1)$.

This is the "Key" to allow applications.

(The advantage is "codim" is more geometric)

Application:

$$lc(X,Y) := \min_{E/X} \frac{1+\operatorname{ord}_E(K_{-/X})}{\operatorname{ord}_E(I_{Y/X})}$$

if $C = \mathcal{C}_p(E)$, get $= \dfrac{\operatorname{codim}(C)}{\operatorname{ord}_C(I_{Y/X})}$ by thm 2.

If C' is arbitrary cylinder (irr, ...)

Thm 1 $\Rightarrow$ $C' \hookrightarrow \mathcal{C}_p(E)$ for some p, E

$\operatorname{codim}(C') \geq \operatorname{codim}(\mathcal{C}_p(E))$

$$\Rightarrow lc(X,Y) = \min_{C} \frac{\operatorname{codim}(C)}{\operatorname{ord}_C(I_{Y/X})}.$$

Take C, let $p = \operatorname{ord}_C(I_{Y/X}) - 1$

$\Rightarrow C \subseteq \psi_p^{-1}(Y_p) \Rightarrow \exists$ irred comp of the same order but with smaller codim

Cor: $$lc(X,Y) = \min_p \frac{\operatorname{codim} \psi_p^{-1}(Y_p)}{p+1} = \dim X - \max_p \frac{\dim Y_p}{p+1}.$$

semi-continuity follows easily.

& also inversion of adjunction.