M. Paun 3/29 2013 1, F Campana

Generic semipositivity for orbifold in Viehweg's sense

(X, D) X^n proj mfd, $D = \Sigma Y_i$ SNC.

$T_X^*\langle D\rangle$ = log cotangent bundle $\Omega \subset X$ coor open set, z_i

$D \cap \Omega = (z_1 \cdots z_p = 0)$

$T_X^*\langle D\rangle|_\Omega$ has local generator $\left(\frac{dz_i}{z_i}\right)_{i=1,\dots p}$, $dz_{p+1}, \dots, dz_n$

A : ample line bundle

Thm: Assume $\exists\ A \hookrightarrow S^m T_X^*\langle D\rangle$

gen inj $\Rightarrow K_X + D$ is big

This enters in Viehweg's hyperbolicity conj

In diff geom terms, $A \hookrightarrow S^m T_X^*\langle D\rangle$ means has a Finsler metric with negative curvature, K_X+D means we require Kähler metric with negative Ricci

The difficulty is that only have "Finsler metric"

2 main techniques
- finite generation
- generic semi-positivity

Proof: Analyze particular cases.

- $D = 0$, K_X ample (this looks like a joke!)

  but will show $\Rightarrow$ lower bound of $\mathrm{vol}(K_X)$

  then $\Rightarrow T_X^*$ is K_X s.s $\Rightarrow A\, K_X^{n-1} \leq c(m)\, K_X^n$

Now use convexity ineq $\Rightarrow$ $\vee$ $(A^n)^{1/n}(K_X^n)^{\frac{n-1}{n}}$

$\Rightarrow K_X^n > c(m)\, A^n$

this is really a nice thing

- slightly more serious example:

  $D = 0$, K_X nef

  Thm (Y Miyaoka) K_X is psef $\Rightarrow T_X^*$ is generically semi-positive

ie. $\forall\ \mathcal{F} \hookrightarrow S^m T_X^*$, $\forall H$ = ample

$\Rightarrow \mu(\mathcal{F})\, H^{n-1} \leq c(m)\, K_X\, H^{n-1}$

Miyaoka proves more (about uni-ruledness)
but the above special case generalizes to pairs !

$K_X + \frac{1}{k}A$ ample $\overset{M}{\Rightarrow}$ $A\,(K_X + \frac{1}{k}A)^{n-1} \leq d(m)\, K_X\,(K_X + \frac{1}{k}A)^{n-1}$

$\underset{\text{conv ineq}}{\Rightarrow}$ $A^n \leq c(m)(K_X + \frac{1}{k}A)^n$ $\underset{k\to\infty}{\Rightarrow}$ $K_X^n \geq c(m)\, A^n > 0$

.. Rather seriously

D arbitrary, $K_X + D$ psef (still don't have Zariski decomp)

$\Rightarrow$ $K_X + D + \frac{1}{k}A$ big $\forall k \geq 1$

otherwise can isolate of & nef part

$R_k = \bigoplus_{p \in k\mathbb{Z}_+} H^0(X, p(K_X + D + \frac{1}{k}A))$

= finitely generated, by $\sigma_0^{(p)}, \ldots, \sigma_{N_p}^{(p)}$

" Desingularize the ideal defined by these "

$T_k :=$ curvature of the metric on $K_X + D + \frac{1}{k}A$ induced by $\{\sigma_j^{(p)}\}$

Fact : $\exists\ \mu_k : X_k \to X$ modification of X st

$\mu_k^*(T_k) = P_k + E_k$ where P_k semi-positive, E_k effective

. $\mathrm{Vol}(K_X + D + \frac{1}{k}A) = P_k^n$, $P_k^{n-1}.\,E_k = 0$

this really uses f.g (otherwise only assump)

. $\mathrm{Supp}(K_{X_k/X}) \subset \mathrm{Supp}(E_k)$

zero of inverse image of the ideal

Following the procedure by Wlodarczyk of principalization will find this is really the case

Thm. (X, D) log-canonical st $K_X + D$ psef

$\Rightarrow$ $T^*_{(X,D)}$ is generically s.p. (generalization of Miyaoka)

combining both techniques,

$F_k + \mu_k^*(K_X + D) = K_{X_k} + D_k$, $\mathrm{Supp}(F_k) \subset \mathrm{Supp}(K_{X_k/X}) \subset \mathrm{Supp}(E_k)$

$\mu_k^*(A) \hookrightarrow S^m T^*_{(X_k, D_k)}$

g.s.p $\Rightarrow$ $\mu_k^*(A)\; D_k^{n-1} \leq c(m)\,(K_{X_k} + D_k)\; P_k^{n-1}$

univ. const.

LHS $\mu_k^*(A)\, P_k^{n-1} \geq (A^n)^{1/n} \cdot \mathrm{Vol}(K_X + D + \tfrac{1}{k} A)^{\frac{n-1}{n}}$ 3/4

As for the RHS : ` exactly lower bound we want

$$(K_{X_k} + D_k) . P_k^{n-1} = (F_k + \mu^*(K_X + D))\ P_k^{n-1}$$

` in general separate nef + ef have trouble with exc. div. but now ok with P_k^{n-1}

$$\leq (D_k + E_k)\ P_k^{n-1} = P_k^n$$

$$\Rightarrow\ P_k^n \geq c(n) (P_k^n)^{\frac{n-1}{n}} \Rightarrow P_k^n \geq c(n)$$

$$\Rightarrow\ \mathrm{Vol}(K_X + D + \tfrac{1}{k} A) \geq c(n)$$ $k \to \infty$ get $K_X + D$ big !

Hint for . how to remove the hypothesis $K_X + D$ psef

idea from PDE , ie continuity method

General case . $I = \{ t \in [0,1] : K_X + D + tA \text{ big} \}$

- $1 \in I$ (as we may assume A large)
- open
- closed ? $(t_k)_{k \geq 1} \subset I$, $t_k \to t_\infty$

$$K_X + D + t_k A \text{ big} \ \overset{?}{\Rightarrow}\ K_X + D + t_\infty A \text{ big}$$

but we know at least it is psef

can assume $t_\infty \in \mathbb{Q}$! (using Shokurov's polytope and BCHM)

$$\Delta = \sum_{j=1}^{N} \left(1 - \frac{1}{r_j}\right) Y_j \ ,\ (X, \Delta),\ r_j \in \mathbb{Q}_+$$

using " $\dfrac{dz_j}{z^{1-1/r_j}}$ " but this is multi-valued

consider ramified cover techniques to remove the prob

cotangent / tangent bundle are just inverse images from here

About g.s.p $\exists$ alg geom / diff geom approach

since this is AG conf we use diff geom method !

Thm H ample

$D = 0$, $0 \to L \to T_X \Rightarrow L . H^{n-1} \leq 0$

$K_X \geq 0$, by Yau , $\exists\ \omega \in c(H)$ st $\Theta_\omega(K_X) \geq 0$.

$\exists\ \sigma \in H^0(X, T_X \otimes \underbrace{L^{-1}}_{h_L})$

$$i\partial\bar\partial \log|\sigma|^2 \geq \Theta_h(L) - \frac{\overset{\star}{\langle \Theta(T_X,\omega)\sigma, \sigma\rangle}}{|\sigma|^2}$$

Poincaré-Lelong type formula (well known in diff geom)

want $\Theta_h(L) \wedge \omega^{n-1}$:

- $\int_X i\partial\bar\partial\log|\sigma|^2 \wedge \omega^{n-1} = 0$
- Symmetries of $\Theta(T_X, \omega)$ $\Rightarrow \star = \dfrac{\langle \Theta(K_X)\sigma, \sigma\rangle}{|\sigma|^2} \geq 0$

  wrt any metric

<u>K_X psef</u>

End

M Paun 4/1 2013

(1) Basics on Bergman kernels

(2) Psh variation of twisted KE metrics

(1).1.1. $\Omega \Subset \mathbb{C}^n$ Ψ-convex domain

$\varphi \in Psh(\overline{\Omega})$, eg $c \log(\sum_{i=1}^{n} |f_i|^2)$ $f_i \in \mathcal{O}(\Omega)$

the class is the smallest class closed under taking max and limits

$A^2_\varphi(\Omega) = (f \in \mathcal{O}(\Omega) : \|f\|^2_{L^2} = \int_\Omega |f|^2 e^{-\varphi} d\lambda < \infty)$

= Hilbert space

$z \in \Omega$, $ev_z : A^2_\varphi(\Omega) \longrightarrow \mathbb{C}$ $f \mapsto f(z)$ bounded l.f

$\Rightarrow \exists\ k_z \in A^2_\varphi(\Omega)$ st $f(z) = \int_\Omega f(w) \overline{k_z(w)} e^{-\varphi} d\lambda$

define $K(z,w) := \overline{k_z(w)}$ = Bergman kernel of domain Ω

$\Rightarrow \forall\ h \in A^2_\varphi(\Omega)$

$h(z) = \int_\Omega h(w) K(z,w) e^{-\varphi(w)} d\lambda$

In fact, (a_j) ONB of $A^2_\varphi(\Omega)$, $K(z,w) = \sum_{j \geq 1} a_j(z) \overline{a_j(w)}$

$K(z,z) = \sup_{\|h\|_{L^2} \leq 1} |h(z)|^2$ ← characterization

$\Rightarrow z \mapsto \log K(z,z) \in Psh(\Omega)$

1.2 Family version

can be removed ↓

$\Omega \Subset \mathbb{C}^n \times \mathbb{C}$, Ψ-convex , $\varphi \in Psh(\overline{\Omega}) \cap C^\infty(\Omega)$

$$\begin{tikzcd} \Omega \arrow[d, "p"'] & z, t \\ \mathbb{C} \ni t & \end{tikzcd}$$

$t \mapsto K_t$ = Bergman kernel of $\Omega_t := p^{-1}(t)$, $\varphi|_{\Omega_t}$

Thm (Yamaguchi, Suzuki,)

$(t,z) \longmapsto \log K_t(z)$ is psh in (t,z)

This is remarkable! Since its dependence on t is not holo, 1.3. Relation to alg geom

X sm proj mfd / $\mathbb{C}$

$L \to X$ line bundle, h_L herm metric

local wt. $e^{-\varphi_\Omega}$ on Ω

instead of Lebesgue measure, now use

sections $\underset{u}{\text{of}}$ $K_X + L$ ⟷ elements of $A^2_\varphi(\Omega)$.

$\int_X u \wedge \bar{u}\, e^{-\varphi_L} =: |u|^2_{L^2}$ a natural way of int

Bergman kernel ⟷ metrics on $K_X + L$

$H^0(X, K_X + L)$.

Family version: $\underset{\text{surj}}{p : X \to Y}$, X, Y sm proj

$L \to X$ line bundle

- fiberwise Bergman kernel ⟷ metrics for $K_{X/Y} + L$

$\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{K_X\ p^*K_Y^{-1}}$

$z^1, \dots, z^{n+m}$ local coor at $x_0 \in X$

$w^1, \dots, w^m$ " at $y_0 \in Y$, $= p(x_0)$

$$\varphi_\Omega(x_0) = \log\left(\sup_{\|u\|_{L^2, y_0} \le 1} \left| \frac{\hat{u} \wedge dw}{dz} \right|^2\right)$$

mult. form notation

where $\|u\|^2_{L^2} := \int_{X_{y_0}} |u|^2 e^{-\varphi_L}$ $\hat{u}$ = local expression of u

(φ_Ω) glue as a metric for $K_{X/Y} + L$ $u \in H^0(X_{y_0}, K_{X_{y_0}} + L)$

y_0 = regular value of p

Thm (Griffiths, Fujita, Kawamata, Viehweg, ..)

$K_{X/Y} + L$ is psef the metric given by the fiberwise Bergman kernel is semi-pos curved

(called Hodge metric in diff names)

Tsuji, Siu

X proj mfd, K_X ample, A large enough ample, h_A

$K_X + A$ metric h_1

$K_X + \underbrace{K_X + A}_{L,\ h_1}$ get metric h_2 = associated (Bergman) metric iterate this process :

$K_X + (2K_X + A) \rightsquigarrow h_3$
$\quad L, h_2$

$K_X + (m-1)K_X + A \rightsquigarrow h_m$
$\quad L, h_{m-1}$

Theorem $h_m^{1/m}$ metric for $K_X + \frac{1}{m}A$ element
$m \to \infty$, get h_∞ metric for K_X (i.e. vol. for X)
which is K-E

Yau $\exists !\ \omega \in c(K_X)$ KE

Unfortunately to prove this need to know Yau's thm first
So for the time being still have prob for big & nef case

Consider $p : X \to Y$ sm K_{X_y} ample

$$K_{X/Y} = \lim_{m\to\infty} \frac{1}{m}\big(\underbrace{mK_{X/Y} + A}_{\geq 0 \text{ curved}}\big)$$

when endowed with fiber-wise Bergman kernel

$\Rightarrow$ the fiberwise KE metric has psh variation
(due to G Schumacher)

*

(2) X compact Kähler manifold

- the main point is to twist with (1,1) class, instead of a line bundle L

$\beta = (1,1)$ form, C^∞, ≥ 0 . $d\beta = 0$

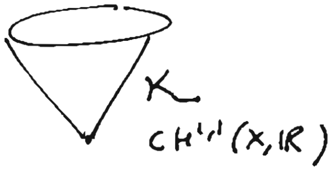
Thm Consider $p : X \to Y$ surj. X, Y cpt Kähler
assume that $c(K_{X_y}) + \{\beta\}|_{X_y}$ is a Kähler class
$y \in Y_0 = Y \setminus W$
proper analytic set ($\supset$ sing value of p)
$\Rightarrow c(K_{X/Y}) + \{\beta\}$ contains Ⓗ closed positive current
= non-singular, "explicit"
on $X_0 = p^{-1}(Y_0)$

Before giving the pt let's see the applications first

- X compact Kähler with $-K_X$ nef

Def $L \to X$ is nef if $c(L) + \varepsilon\{\omega\}$ is a Kähler class $\forall \varepsilon > 0$
or $(1,1)$ class, any Kähler class

Albanese torus



$H^0(X, T_X^*)^* / \operatorname{Im} H_1(X,\mathbb{Z}) =: \operatorname{Alb}(X)$

$-K_X$ nef $\underset{\text{conjecture}}{\Longrightarrow}$ $\alpha_X : X \longrightarrow \operatorname{Alb}(X)$ is an submerssion (and iso-trivial)

Notice that

$-K_X \geq 0 \underset{\text{Yau}}{\Longleftrightarrow} \exists\, \omega_1 \in c(X)$, $\operatorname{Ric}_{\omega_1} \geq 0$, the Bochner formula $\Rightarrow$ conj immediately.

$x \longmapsto \left(\int_{x_0}^{x} \alpha_j\right)_{j=1,\dots,q(X)}$

and prove α_j parallel ($\Rightarrow$ submerssion)

But harder in nef case

Theorem: α_X is surjective. (Q. Zhang if X proj using char p method)

Pf: If not surjective, then

$\alpha_X : X \to Y \hookrightarrow \operatorname{Alb}(X)$ Assume Y sm (can be removed)

Ueno: $\kappa(Y) \geq 1$ (pull back hol. form can't be flat since $Y \neq$ transl of sub torus)

$K_{X/Y} + (\underbrace{-K_X + \varepsilon\omega}_{\beta})$, it is Kähler when restricts to generic fiber

Thm $\Rightarrow$ $c(K_{X/Y}) + c(-K_X) + \varepsilon\{\omega\} \ni T_\varepsilon \geq 0$ (psef)

$\varepsilon \to 0$ $\underset{= -p^* K_Y}{c(K_{X/Y}) - c(K_X)} \ni T \geq 0$ psef

$\Rightarrow -K_Y$ psef $\ast$ to $\kappa(Y) \geq 1$

Notice that it is essential to allow twisting by β $(1,1)$ class, not just a line bundle, Idea of the pf

construction of metric on $K_{X/Y}$

$p : X_0 \to Y_0$ proper submersion

consider ρ = closed $(1,1)$ form on X_0 st $\rho|_{X_y}$ = Kähler metric

Then ρ induces a metric for $K_{X/Y}|_{X_0}$:

local weight : $z^1, \dots, z^{n+m}$ local coor (x, x_0), Ω

$w^1, \dots, w^m$ Y_0, $y_0 = p(x_0)$

$\rho^n \wedge p^*(d\lambda(w))$ is then <u>strictly</u> positive $= e^{\varphi_\Omega} d\lambda(z)$

(φ_Ω) defines a metric on $K_{X/Y}|_{X_0}$ Lebesgue meas

<u>Ex.</u> $p : \hat{X} \to X$ blow-up of X along $Z \subset X$, $K_{X/Y} = *E$

$p^* d\lambda(w) = e^{\varphi_\Omega} d\lambda(z)$ just Jacobian of the map

(φ_Ω) metric induced by the tautological sections of $K_{X/Y}$

ω = Kähler metric on X

h_ω = metric induced on $K_{X/Y}$ by ω "Thm of Yau" $\Rightarrow$

$$\Big(\underbrace{\Theta_{h_\omega}(K_{X/Y}) + \beta}_{\text{Kähler class}}\Big|_{X_y} + \sqrt{-1}\partial\bar\partial\varphi_y\Big)^n = e^{\varphi_y}\omega^n \quad \text{st}$$

$$\Theta_{h_\omega}(K_{X/Y}) + \beta|_{X_y} + \sqrt{-1}\partial\bar\partial\varphi_y > 0$$

↑ Called the twisted KE metric in the Literature

↑ notice that is singular a priori, so need special care

Claims :

K

① $\Theta_{h_\omega}(K_{X/Y}) + \beta + \sqrt{-1}\partial\bar\partial\varphi \geq 0$ on X_0

$\varphi(x) := \varphi_y(x)$, $y = p(x)$ ← even in the bad direction

② Θ extends across $p^{-1}(W)$

$\in c_1(K_{X/Y}) + \{\beta\}$, exactly like the <u>Ex</u> case !

Remark : $\varphi : X_0 \to \mathbb{R}$, $\varphi(x) := \varphi_y(x)$ is smooth

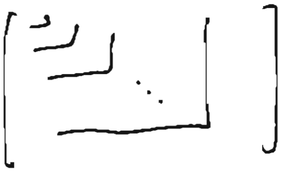
(by Yau's estimate, mobilized with base).

Rmk ! It is enough to prove thm then if $\dim Y_0 = 1$

due to the def'n of Psh (in all 1-dim'l directions)

$(z^1, \cdots, z^n, t)$ coor at $x_0 \in X_0$

$$\Theta = \sqrt{-1}\, g_{t\bar t}\, dt\, d\bar t + 2\,\mathrm{Re}\, g_{t\bar\alpha}\, dt\, d\bar z^\alpha + \sqrt{-1}\, g_{\alpha\bar\beta}\, dz^\alpha\, d\bar z^\beta$$



already have $\Theta|_{X_y} > 0$, So $\Theta \geq 0$ $\Leftrightarrow$ its enough to prove $\Theta^{n+1} \geq 0$ (i.e. det)

$$\Theta^{n+1} = c[\Theta]\, \Theta^n \wedge \sqrt{-1}\, dt\, d\bar t$$

$$c(\Theta) = g_{t\bar t} - g^{\bar\beta\alpha} g_{t\bar\beta}\, g_{\alpha\bar t} \overset{?}{\geq} 0$$

["$c(\Theta)=0$" is exactly the equation for "geodesics" in the space of Kähler metrics]

compute $\Delta_{\Theta_y}(C(\Theta))$: $(\Theta_y := \Theta|_{X_y})$ important remark

$$-g^{\bar i j}\frac{\partial^2}{\partial z^j \partial \bar z^i} = \Delta$$ (sample computation can already see the idea)

locally $\Theta = \sqrt{-1}\,\partial\bar\partial(\text{function})$, hence

$$\frac{\partial^2 g_{t\bar t}}{\partial z \partial \bar z} = \frac{\partial^2 g_{j\bar i}}{\partial t \partial \bar t}$$

$$g^{\bar i j}\frac{\partial^2 g_{j\bar i}}{\partial t\,\partial \bar t} = \frac{\partial}{\partial t}\Big(\underbrace{g^{i\bar j}\frac{\partial g_{i\bar j}}{\partial \bar t}}\Big) - \frac{\partial g^{\bar i j}}{\partial t}\frac{\partial g_{i\bar j}}{\partial \bar t}$$

we know what this is $= \frac{\partial}{\partial \bar t} \log\det \Theta_y$

$$\Rightarrow \frac{\partial^2}{\partial t\,\partial \bar t}\log\det g_{\rho\bar\delta} = \partial\bar\partial(\varphi_y - \varphi_\beta) \cdot g_{t\bar t} - \beta_{t\bar t}$$

$\det g_{\alpha\bar\beta} = e^{\varphi_y - \varphi_\beta}$

$$\Delta\, c(\Theta) = -c(\Theta) + |\bar\partial v|^2 + \overset{\overset{0}{\wedge}}{\beta}(v,\bar v)$$

where v = "horizontal" lift of $\frac{\partial}{\partial t}$

$$= \frac{\partial}{\partial t} - \sum g^{\beta\alpha} g_{t\bar\beta}\frac{\partial}{\partial z^\alpha}$$ C^∞ v.f along fiber

$$(I+\Delta)\,c(\Theta) = |\bar\partial v|^2 + \beta(v,\bar v)$$

$\Theta|_{X_y}$: its Ricci curvature $\geq -C$ (key lower bound)

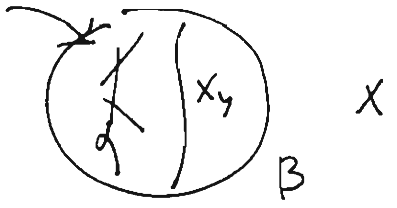
$$\xrightarrow{} \quad c(\Theta) \geq \text{const} \int_{X_y}(|\bar\partial v|^2 + \beta(v,v))\,\Theta_y^n \geq 0$$

Cheeger-Yau gradient estimate

(also we know precisely when $= 0$)

Now the 2nd claim :

the favorite part : singular fiber

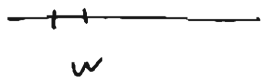


$\Theta = \Theta_{hw}(K_{X/Y}) + \beta + \sqrt{-1}\partial\bar\partial\varphi$

$\varphi|X_y$

- Goal : $\sup_{X_0} \varphi < \infty$.

(just like ext of holo function)



Use Monge-Ampère eq'n, C^0 estimate
but this depends on the "geometry" of mfd X_y
so can't be applied directly

However, hw :

$(**)$ $\omega^n \wedge p^* d\lambda(w) = e^{\varphi_{hw}} d\lambda(z)$

has the same singularity as Jacobian $\varphi_{hw} = \log(|Jac(p)|^2)$

idea : $\underline{\varphi + \varphi_{hw}}$ is "estimable"

Rmk : β sm , $K_{X_y} + \{\beta\}$ = Kähler $\longleftrightarrow$ singular version ?
this is important in alg. geom (eg. big line bd)

$\phi :=$ potential of Θ
$= \varphi_{hw} + \varphi + \varphi_\beta$

$(\sqrt{-1}\partial\bar\partial\phi)^n = e^{\phi - \varphi_\beta} d\lambda$, in the small ball B
really the round ball

$H_m := \{ f \in \mathcal{O}(\Omega \cap X_y) : \int_{\Omega \cap X_y} |f|^2 e^{-m\phi} (\sqrt{-1}\partial\bar\partial\phi)^n < \infty \}$

consider the Bergman kernel associated to these

Then Thm : $\phi(z) = \limsup_{\|f\|_{L^2} \le 1} \frac{1}{m} \log|f(z)|^2$.
due to Demailly

consider $f \in H_m$, $\|f\|_{L^2} \le 1$

In order to get upper bound can only consider such f

$\int_{\Omega_y} |f|^2 e^{-\phi} (\sqrt{-1}\partial\bar\partial\phi)^n \le C_{univ}$
vol of the fiber is fixed (cohomologically).

on fiber

$\int_{\Omega_y} |f|^{2/m} e^{-\phi'} e^{\phi'-\varphi_\beta} \cdot \underline{\underline{d\lambda'}} \le C_{univ}$

plug in MA eq'n   as quotient of 2 Lebesgue measure

then use Jacobian vol bound in (**)

$$\int_{\Omega_y} |f|^{2/m} e^{-\varphi_\beta} \frac{d\lambda(t)}{d\lambda(w)} \le C_{univ}$$

$\Rightarrow$ $\exists\, F \in \mathcal{O}(\Omega)$

by a version of Ohsawa-Takegoshi ext'n thm   holomorphic ext'n

$\int_\Omega |F|^{2/m} e^{-\varphi_\beta} d\lambda \le C_{univ}$ $F|_{\Omega_y} = f$

nothing to do with the geometric property

hence we can go to singular fiber.

Can do Cauchy inequality on $\underline{\Omega}$, but not Ω_y

nice ball

Rmk. The importance of $\Theta^{n+1} \geq 0$
is also related to holomorphic Morse inequality

$L \to X$, $\int_{(X,\le 1)} \Theta(L)^{\dim X} > 0 \Rightarrow$ L is big
under very mild
curvature condition