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1999 1. 8.

Special Lagrangian Geometry II : (I is on Monday)

* X C-Y manifold :
cpt cpx mfd w/ nowhere vanishing hol. n-form Ω and a Ricci-flat metric w/ Kähler form ω, and

$$\frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n \Omega \wedge \bar{\Omega}.$$

Def: $M \subseteq X$ is special Lagrangian if
$\dim_{\mathbb{R}} M = \dim_{\mathbb{C}} X$ and $\omega|_M = 0$, $\operatorname{Im}\Omega|_M = 0$.

SYZ conjecture : If X and $\check{X}$ are a Mirror pair, then there exists fibrations $f : X \to B$, $\check{f} : \check{X} \to B$ whose fibers are s. lag T^n (in general and such that f and $\check{f}$ are dual fibrations

$T^n = V/\Lambda$, $(T^n)^\vee \cong V^\vee/\Lambda^\vee$
$H^1(T^n, \mathbb{R}) \cong V^\vee$; $H^1(T^n, \mathbb{Z}) \cong \Lambda^\vee$.

$$\begin{array}{ccccc} & X_0 & \subseteq & X & \\ & \downarrow f_0 & & \downarrow f & \\ B - \Delta = & B_0 & \hookrightarrow & B & \supseteq \Delta \end{array}$$ discriminant locus

and
$$\begin{array}{ccc} \check{X}_0 & \subseteq & \check{X} \\ \downarrow \check{f}_0 & & \downarrow \check{f} \\ B_0 & \subseteq & B \end{array}$$

f and $\check{f}$ are dual says $\check{f}_0 : \check{X}_0 \to B_0$ coincides with $R^1 f_{0*} \mathbb{R} / R^1 f_{0*} \mathbb{Z}$.

$R^1 f_* \mathbb{Z}$ is a sheaf on B associated to the presheaf: $U \mapsto H^1(f^{-1}(U), \mathbb{Z})$

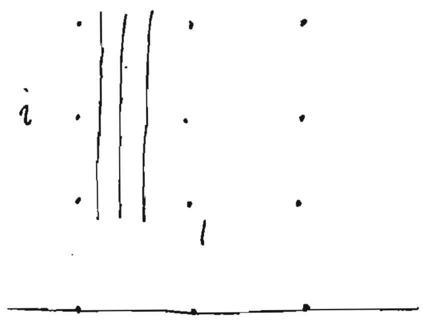
$$(R^1 f_* \mathbb{Z})_b = H^1(X_b, \mathbb{Z})$$

Away from Δ, $R^1 f_* \mathbb{Z}$ is a local system.

Examples: ① Elliptic curve $\mathbb{C}/\mathbb{Z}^2$:

s.lag. fibration

$\mathbb{C}$:



$$\mathbb{C}/\mathbb{Z}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$$
$$\downarrow$$
$$\mathbb{R}/\mathbb{Z}$$

So, The dual of an elliptic curve is an elliptic curve

② K3 surfaces:
Mirror symmetry for K3 surfaces
(Pinkham, Nikulin, Todorov, Dolgachev, Aspinwall + Morrison)

Let S be a K3 surface, $L = H^2(S, \mathbb{Z}) \cong \mathbb{Z}^{22}$
$L \times L \to \mathbb{Z}$ is even unimodular w/ signature $(3, 19)$.
Let $M \subseteq L$ be a primitive sublattice of signature $(1, t)$ for some t.

(primitive $\longleftrightarrow$ L/M is torsion free)

Def: A marked M-polarized K3 surface S is one with a fixed $\cong$ $\phi : H^2(S, \mathbb{Z}) \xrightarrow{\sim} L$ such that $M \subseteq \phi(\mathrm{Pic}\, S)$.

$\exists$ a moduli space of marked M-polarized K3 surfaces.

Now we describe the "Mirror family":

$$T := M^{\perp} \subseteq L$$

suppose T contains $H = \mathbb{Z}^2$ with $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

can then write $T = H \oplus \check{M}$. Then

"The family of M-polarized K3 surfaces is mirror to the family of $\check{M}$-polarized K3 surfaces"

Existence of $H \subseteq T$ is equivalent to existence of a large complex structure limit point in the moduli space of M-polarized K3's.
(actual statement is slightly weaker!)

SYZ picture vs the above:

$$S, \ \Omega, \ \omega$$

$H \subseteq T$ $\qquad$ $E . E' = 1$

$E, E' \in H$ generators $\qquad$ $E^2 = E'^2 = 0$

choose θ so that $(\mathrm{Im}\, e^{i\theta}\Omega) E = 0$.

Replace Ω by $e^{i\theta}\Omega$.

(Recall on Monday): $\exists$ another cpx structure K on S w/ the same metric st.

$$\Omega_K = \mathrm{Im}\, \Omega + i\omega$$
$$\omega_K = \mathrm{Re}\, \Omega$$

$M \subseteq S_K$ holomorphic $\iff$ $M \subseteq S$ S. Lag.

$E . \mathrm{Im}\, \Omega = 0$, $\quad E . \omega = 0 \quad \therefore E . \Omega_K = 0$

standard result in K3 surface $\Rightarrow$

$E \in \mathrm{Pic}\, S_K$, $E^2 = 0$.

Shafarevich : If S is a K3 surface and $\exists\, E \in \mathrm{Pic}\, S$ with $E^2 = 0$, then $\exists$ a map

$$f : S \longrightarrow \mathbb{P}^1 \quad \text{with elliptic fibers.}$$

Thus $\exists\, f : S_K \to \mathbb{P}^1$ elliptic fibration (with fiber $\pm E$ in general)

This gives a s. lag. T^2 fibration $f : S \to S^2$.

In general, f has 24 singular fibers of Kodaira type I_1 :

To see the Mirror, at this moment we only talk about topology :

"Dualizing an elliptic fibration doesn't change it !

for a single elliptic curve $E = V/\Lambda$

$$H^1(E, \mathbb{Z}) \times H^1(E, \mathbb{Z}) \longrightarrow H^2(E, \mathbb{Z}) \cong \mathbb{Z}$$

so $H^1(E, \mathbb{R})^\vee \equiv H^1(E, \mathbb{R})$, so $V^\vee \cong V$.

This is canonical, so extends to give an isomorphism between $f : S \to S^2$ and $\check{f} : \check{S} \to S^2$.

Later we will see for cpx structures, this will lead to new results.

③ 3-dim'l example ; — + Wilson :

Borcea + Voisin :

S a K3 surface with an involution $\iota : S \to S$ such that

$\mathrm{Fix}(\iota) = C_1 \cup \cdots \cup C_N$, $C_1, \cdots, C_N$ non-sing. curves with $C_2, \cdots, C_N \cong \mathbb{P}^1$, C_1 a curve of genus N'. Let E be an elliptic curve:

$$\begin{array}{ccl} Y = (S \times E)/(\iota, -1) & & \text{is singular} \\ \uparrow & & \text{resolution such that} \\ X & & X \text{ is C-Y} \end{array}$$

$h^{1,1}(X) = 11 + 5N - N'$
$h^{1,2}(X) = 11 + 5N' - N$.

want to interchange N and N':

If $\check{S}$ is in the mirror family to S, then $\exists$ a involution $\check{\iota} : \check{S} \to \check{S}$ interchange N and N'.

(Rmk: topologically $\check{S}$ is S, but NOT holomorphically !)

This part B-V use Nikulin's result. Thus

$\check{X} \longrightarrow \check{S} \times E/(\check{\iota}, -1)$ has
$h^{1,1}(X) = h^{1,2}(\check{X})$; $h^{1,2}(X) = h^{1,1}(\check{X})$.

If the cpx structure and metric on S are choosen carefully, $\iota : S_K \longrightarrow S_K$ is anti-holo

$$\begin{tikzcd} S_K \arrow[r, "\iota"] \arrow[d, "f"'] & S_K \arrow[d, "f"] \\ S^2 \arrow[r, "\iota'"] & S^2 \end{tikzcd}$$

ι' interchange the north pole and south pole on S^2

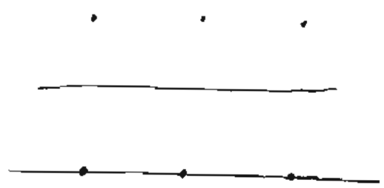
$$\begin{array}{ccccl} \mathrm{Fix}(\iota) \subseteq & S_K & \supseteq & E & \circlearrowleft \iota \text{ anti-holo involution} \\ \downarrow & \downarrow & & \downarrow & \\ \mathrm{Fix}(\iota') \subseteq & S^2 & \ni & p \in \mathrm{Fix}(\iota') & \\ & \text{"equator } S^1\text{"} & & & \end{array}$$

Two sorts of anti-holo involutions

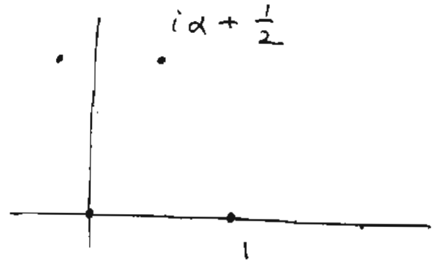
$z \mapsto \bar{z}$ on $\mathbb{C}/\langle 1, \tau\rangle$, $\tau = i\alpha$ (1st case)

$\mathrm{Fix}(\iota) = S^1 \cup S^1 = \{\mathrm{Im}\, z = 0\} \cup \{\mathrm{Im}\, z = \frac{\alpha}{2}\}$

2nd case: $\tau = i\alpha + \frac{1}{2}$, $\mathrm{Fix}(\iota) = S^1 = \{\mathrm{Im}\, z = 0\}$.

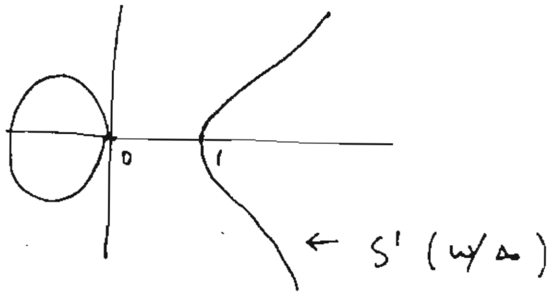


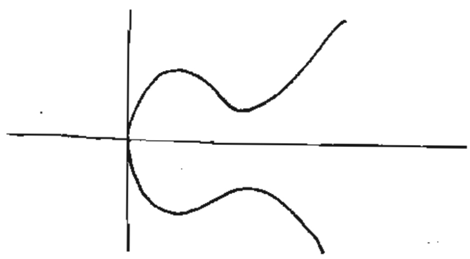
Case 1.



Case 2.

eg. $y^2 = (x-1)x(x+1)$; $y^2 = x(x^2+1)$:





For singular fibers:

$$\begin{array}{ccc} \mathrm{Fix}(\iota) & \subseteq & S_K \\ \downarrow & & \downarrow \\ \mathrm{Fix}(\iota') = S^1 & \subseteq & S^2 \end{array}$$

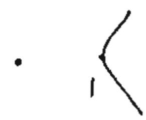
So,

$$\mathrm{Fix}(\iota) = \begin{cases} \text{Figure 8} \\ S^1 \cup \{pt\} \end{cases}.$$

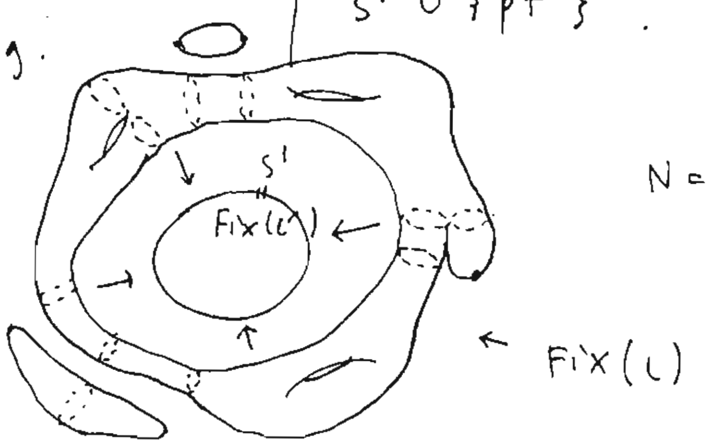
$S_b \cong \alpha$: I_1 fiber

$y^2 = x^2(x+1)$

$S_b \cong y^2 = x^2(x-1)$



eg.



$N = 3$, $N' = 4$.

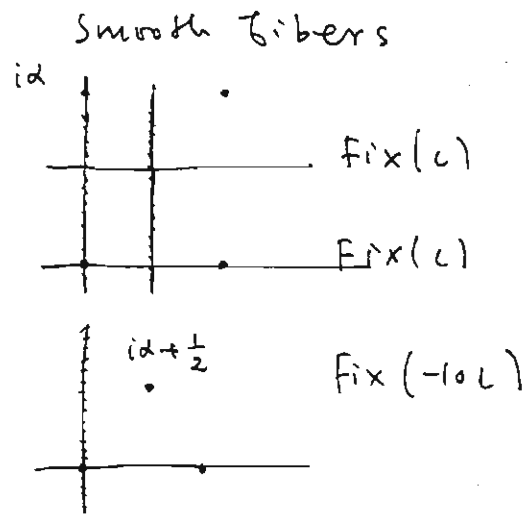
$$\begin{array}{ccc} SK & \xrightarrow{\iota} & SK \\ \downarrow & & \downarrow \\ S^2 & \xrightarrow{\iota'} & S^2 \end{array} \qquad \begin{array}{ccc} SK & \xrightarrow{\check{\iota}} & SK \\ \downarrow & & \downarrow \\ S^2 & \xrightarrow{\iota'} & S^2 \end{array}$$

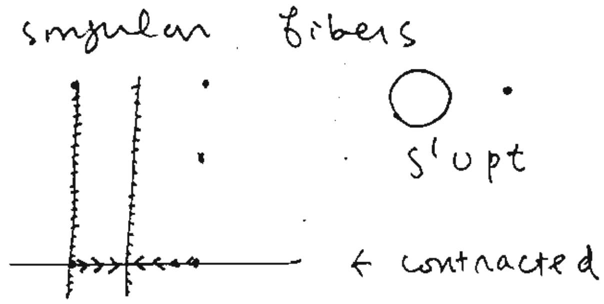
$\check{\iota} = (-1)\circ\iota$ with $-1 : SK \to SK$ negation on fibers.

dual $\to$ $$\begin{array}{ccc} \check{S}K & \xrightarrow{\iota^t} & \check{S}K \\ \downarrow & & \downarrow \\ S^2 & \xrightarrow{\iota'} & S^2 \end{array}$$ ι^t, $\check{\iota}$ coincide with.

Can show: Given the canonical identification between SK and $\check{S}K$, ι^t, $\check{\iota}$ coincide with $\check{\iota}$ is anti-holomorphic. How does Fix($\check{\iota}$) and Fix(ι) diffeomorphic ?

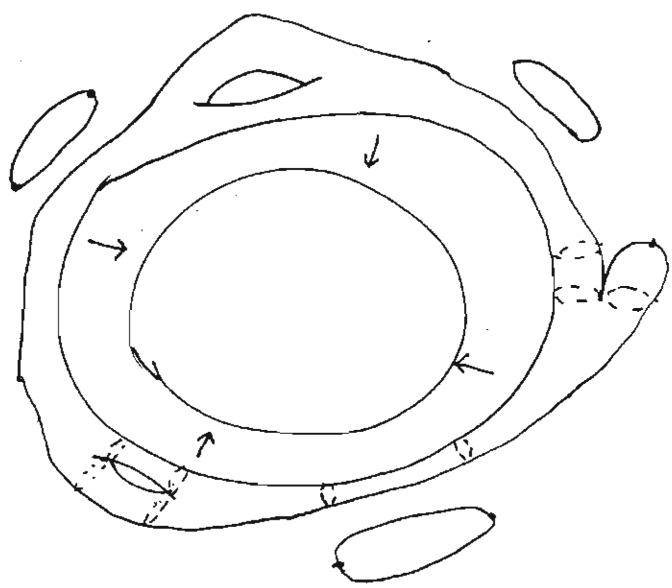
On singular fibers, two sorts of Fixed loci are interchanged!





Use this to get

$N=4$, $N'=3$:



3-dim'l picture :

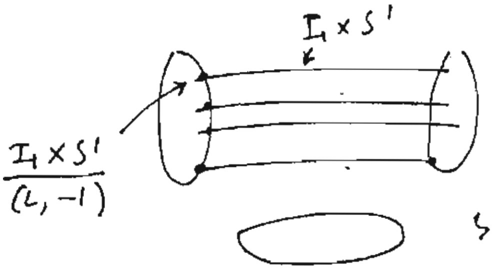
$S \times E$ T^3 fibration slag in product metric
$\downarrow$
$S^2 \times S^1$ has points $p_1, \cdots, p_{24} \in S^2$ st. $f^{-1}(p_i)$ is sing. of type I_1.

The discr. locus of $S \times E \longrightarrow S^2 \times S^1$ is $\bigcup(\{p_i\} \times S^1)$ with singular fibers being $I_1 \times S^1$.

$S \times E/(\iota, -1)$
$\downarrow$
$S^2 \times S^1/(\iota', -1) \cong S^3 \supseteq \Delta$

Δ consists of the image of $\{p_i\} \times S^1$ which is an:
S^1 if $\iota'(p_i) \neq p_i$
interval if $\iota'(p_i) = p_i$.

And the image of $\mathrm{Fix}(\iota, -1)$:



$\leftarrow (T^2 \times S^1)/(\iota, -1)$

singular fiber $I_1 \times S^1$

$\chi(I_1 \times S^1) = 0$

$\chi((T^2 \times S^1)/(\iota, -1)) = \frac{0}{2} + \frac{0}{2} = 0$ since

$\chi(F/\iota) = \frac{1}{2}\chi(F - \mathrm{Fix}(\iota)) + \chi(\mathrm{Fix}(\iota))$

$= \frac{1}{2}\chi(F) + \frac{1}{2}\chi(\mathrm{Fix}(\iota))$.

$\chi(I_1 \times S^1/(\iota, -1)) = \frac{1}{2}\underbrace{\chi(I_1 \times S^1)}_{=\,0} + \frac{1}{2}\chi(\mathrm{Fix}(\iota, -1))$

$\mathrm{Fix}(\iota, -1) = $ "∞" + "∞" $=$ ± 1 depends on the 2 cases.
"$0.$" + "$0.$"

Punch line : dualizing ≡ "divide by transpose"

$$\begin{array}{ccc} S\times E/(\iota,-1) & & \check{S}\times\check{E}/(\check{\iota},-1) \\ \downarrow & & \downarrow \\ S^2\times S^1/(\iota',-1) & & S^2\times S^1/(\iota',-1) \end{array}$$

Thus, ① X, $\check{X}$ are related by dualizing

② Fibers with Euler characteristic ± 1 are interchanged.

This is the 1st 3-dim'l example, and so far the only complete example we know.

One cheating point: these examples are singular! After smoothing, one has no idea whether still has fibration structure.

Question: Does dualizing interchange Hodge numbers? (3-fold case)

Has to understand

$$H^i(X,\mathbb{Q}),\ H^i(\check{X},\mathbb{Q})$$

$f: X\to B$, Use Leray spectral sequence

Problem is that we don't know the details of the singular fiber.

$$\begin{array}{ccc} X_0 & \hookrightarrow & X \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ B_0 & \underset{i}{\hookrightarrow} & B \end{array}$$

Def: We say f is "simple" if $\iota_* R^p f_{0*}\mathbb{Z} \cong R^p f_*\mathbb{Z}$

"Cohomology of the singular fibers is determined by the monodromy about the discriminent locus".

S $\quad T : H^i(X_b, \mathbb{Z}) \circlearrowright$ monodromy

$\downarrow \quad H^i(X_0, \mathbb{Z}) = \ker(T - I)$

in this case, this is the

 vanishing cycles.

Explaination:

$p=0$: $f_{0*}\mathbb{Z} = \mathbb{Z}$,

$f_*\mathbb{Z} = \mathbb{Z}$, if all fibers are connected,

$\iota_*\mathbb{Z} = \mathbb{Z}$.

$p = \dim_{\mathbb{C}} X$: $R^n f_{0*}\mathbb{Z} = \mathbb{Z}$

$(R^n f_* \mathbb{Z})_b = H^n(X_b, \mathbb{Z})$ need not be $\mathbb{Z}$

eg. $n=2$, $H^2($  $, \mathbb{Z}) = \mathbb{Z}^m$

Im

condition holds for n if all fibers are irreducible.

* Theorem: Given certain simple regularity assumptions (eg. f real analytic), and assuming all fibers are irreducible, then in dim 3, f is simple.

$T^n = V/\Lambda$

$H^i(T^n; \mathbb{R}) \equiv \Lambda^i(V^\vee)$

$H^{n-i}((T^n)^\vee; \mathbb{R}) \equiv \Lambda^{n-i}(V)$ and know

$$\Lambda^i(V^\wedge) \cong \Lambda^{n-i}(V)$$

via the perfect pairing $\Lambda^i V \times \Lambda^{n-i} V \to \Lambda^n V = \mathbb{R}$

$\exists$ canonical isom. (Poincaré duality)

$$R^i f_{0*}\mathbb{R} \cong R^{n-i} \check{f}_{0*}\mathbb{R} \quad \text{on } B_0$$

so $\quad i_* R^i f_{0*}\mathbb{R} \cong i_* R^{n-i}\check{f}_{0*}\mathbb{R}$ on B,

Simplicity $\Rightarrow$ $R^i f_*\mathbb{R} \cong R^{n-i} \check{f}_* \mathbb{R}$!

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Special Lagrangian Geometry III

$f : X \to B$; $\check{f} : \check{X} \to B$

$f, \check{f}$ dual fibrations and simple

$\Rightarrow \quad R^i f_* \mathbb{Q} \cong R^{n-i} \check{f}_* \mathbb{Q}$.

Leray Spectral sequence

$$E_2^{p,q} = H^p(B, R^q f_* \mathbb{Q}) \Rightarrow H^n(X, \mathbb{Q})$$

For $n = 3$, $f_* \mathbb{Q} = \mathbb{Q}$, $R^3 f_* \mathbb{Q} = \mathbb{Q}$ (Poincaré dual)

Assume $b_1(X) = 0$ hence $b_1(B) = 0$

(In fact will also implies $b_1(\check{X}) = 0$, let's asume it)

$$\begin{tikzcd}
\mathbb{Q} \arrow[drr] & 0 & 0 & \mathbb{Q} \\
?0 & H^1(R^2 f_* \mathbb{Q}) & H^2(R^2 f_* \mathbb{Q}) & 0* \\
0* & H^1(R^1 f_* \mathbb{Q}) \arrow[drr] & H^2(R^1 f_* \mathbb{Q}) & ?0 \\
\mathbb{Q} & 0 & 0 & \mathbb{Q}
\end{tikzcd}$$

$b_1(B) = 0 \Rightarrow * = 0$

$H^0(R^2 f_* \mathbb{Q}) \cong H^0(R^1 \check{f}_* \mathbb{Q}) = 0$

via the similar picture

in $\check{f} : \check{X} \to B$, $\Rightarrow$ $? = 0$

Now

$H^1(B, R^1 f_* \mathbb{Q}) \xrightarrow{d} H^3(B, \mathbb{Q}) \xrightarrow{f^*} H^3(X, \mathbb{Q})$ is exact

$\mathrm{Vol}(T^3) = \int_{T^3} \Omega \neq 0$ so $[T^3] \in H^3(X, \mathbb{Q})$ is non zero

$\therefore$ f^* is injective. Similarly

$[\mathrm{Re}\,\Omega] \longrightarrow$ something $\neq 0$

$H^3(X, \mathbb{R}) \longrightarrow H^0(B, R^3 f_* \mathbb{R}) = \mathbb{R} \xrightarrow{d} H^2(B, R^1 f_* \mathbb{R})$

$\therefore$ $d = 0$. So Spectral seq. degenerates at E_2 term.

compare with $\check{f} : \check{X} \to B$ gives interchanging of Hodge numbers. In fact, more is true

$$H^{2*}(X, \mathbb{Q}) \cong H^3(\check{X}, \mathbb{Q})$$

but still not canonical yet.

Question: Construct an canonical isomorphism.

Integral coefficients:

$$H^{2*}(X, \mathbb{Z}[\tfrac{1}{2}]) \cong H^3(\check{X}, \mathbb{Z}[\tfrac{1}{2}])$$

Question: what is the meaning of the filtration on $H^3(X, \mathbb{Q})$ induced from the L. s. s.?

(Answer: NOT the Hodge filtration since Hodge filt. is defined $/\mathbb{C}$, not $/\mathbb{Q}$. The possible correct answer is the "monodromy weight filtration". Later will come back to this.)

MacLean: Deformation theory of callibrated submfds. eg $M \subseteq X$ is s. Lag.

what is the moduli space of def. space of M as a s. Lag. submfd?

— Deformation theory is unobstructed,

Tangent space: vector field $\vec{v}$ on M

$$\left\{\begin{array}{ll} M & \\ (\iota(\vec{v})\omega)|_M & \text{is a well-defined 1-form on } M \\ (\iota(\vec{v})\,\mathrm{Im}\,\Omega)|_M & (n-1) \text{ form on } M \end{array}\right.$$

On M, $* \, \iota(\vec{v})\omega = -\iota(\vec{v})\,\mathrm{Im}\,\Omega$.

MacLean showed that if $\vec{v}$ is the normal v.f. to a deformation of M as s. Lag. sub-mfd, Then

$$\begin{cases} d\ \iota(\vec{v})\,\omega = 0 \\ d\ \iota(\vec{v})\,\mathrm{Im}\,\Omega = 0 \end{cases} \iff \iota(\vec{v})\,\omega \text{ is harmonic.}$$

$\Rightarrow$ Tangent Space $= H^1(M, \mathbb{R})$, the space of harmonic 1-forms on M.

$T^n \subseteq X^n$ s. Lag. $H^1(T^n, \mathbb{R}) = n$.

Global action-angle coordinates (Duistermaat)

$f : X \to B$ a s. Lag. fibration

There is an action of T_B^* on X:

Given 1-form α on B, $\exists\ \vec{v}$ on X st.

$\iota(\vec{v})\,\omega = f^*\alpha$, $\vec{v}$ will be tangent to fibers.

Let $T_\alpha : X \longrightarrow X$ be the time 1 flow of $\vec{v}$.

$$\begin{array}{ccc} X & \xrightarrow{T_\alpha} & X \\ & \searrow\ B\ \swarrow & \end{array}$$

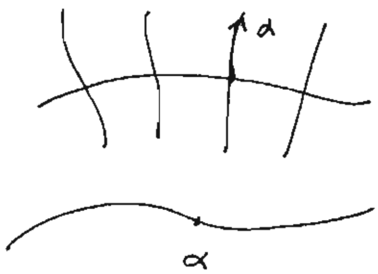
T_α is a symplectomorphism $\iff d\alpha = 0$

Can show that $T_\alpha : X_b \to X_b$ only depends on the value of α at $b \in B$.

Assume that $f : X \to B$ has a Lagrangian section

$\sigma_0 : B \longrightarrow X$,

Define a map $T_B^* \xrightarrow{\pi} X$ by $\alpha \in T_{B,b}^*$
$\mapsto T_\alpha(\sigma_0(b))$:



let $X^\# = X - \mathrm{Crit}(f)$

$\mathrm{Crit}(f) = \{ x \in X \mid f_* : T_{X,x} \longrightarrow T_{B,f(x)} \text{ has } rk < n \}$

π surjects onto $X^\#$ if f is simple.

$\pi^*\omega$ is the canonical symplectic form on T_B^*,

ie. if $y_1, \cdots, y_n$ are loc. coor. on B

$x_1, \cdots, x_n$ " T_B^*

ie. $(y_1, \cdots, y_n, x_1, \cdots, x_n)$ coincides with $\sum x_i dy_i$,

Then $\pi^*\omega = \sum dx_i \wedge dy_i$.

Notice that π depends on the Lag. section σ_0 and the symp. form ω.

(Here we don't need $d\alpha = 0$, still get diffeom. on each fiber)

$$\text{"} \quad T_B^* \xrightarrow{\pi} X^\# \; ; \quad \pi^*\omega = \sum dx_i \wedge dy_i \quad \text{"}$$

Think of $X^\#$ as a sheaf:

$$X^\#(U) = \{ \text{sections } U \longrightarrow X^\# |_U \}, \text{ get}$$

$$0 \to R^{n-1} f_* \mathbb{Z} \longrightarrow T_B^* \longrightarrow X^\# \longrightarrow 0$$

since $0 \to \Lambda \cong H^{n-1}(T^n, \mathbb{Z}) \to V \to V/\Lambda \to 0$.

Given any σ of $X^\#(U)$, get a translation

$T_\sigma : f^{-1}(U) \circlearrowleft$ (Thinking of σ_0 as the zero section

$$\Gamma(B, T_B^*) \longrightarrow \Gamma(B, X^\#)$$

$$\to H^1(B, R^{n-1} f_* \mathbb{Z}) \to H^1(B, T_B^*) = 0$$

Hence, $H^1(B, R^{n-1} f_* \mathbb{Z}) =$ space of sections of f modulo those homotopic to "zero sections".

Given a section σ, $T_\sigma : X \to X$ and

$T_\sigma^* : H^p(X, \mathbb{Q}) \circlearrowleft$

only depends on class of σ in $H^1(B, R^{n-1} f_* \mathbb{Z})$.

Recall, At large cpx str. limit pt : have monodromy transformations $T_1, \dots, T_s$, $s = h^{1,n-1}(X)$

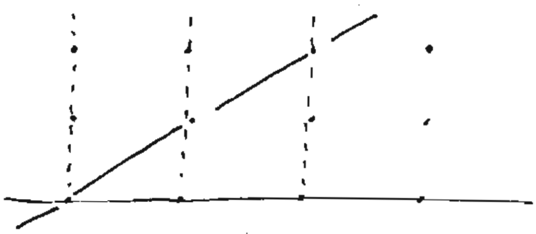
CONJECTURE :

$\exists$ sections $\sigma_1, \dots, \sigma_s \in H^1(B, R^{n-1} f_* \mathbb{Z})$ st.

$T^*_{\sigma_i} = T_i$

(since we haven't proven $\dim H^1(R^{n-1}) = h^{1,n-1}$ yet. So for restrict this to $n \leq 3$.)

eg. Elliptic curves :
this section gives
Dehn twist.



For K3 Surfaces, easy to check that this gives the right action on cohomology :

This T_{σ_i} and the monodromy diffeomorphisms are equivalent up to C^0 isotopy .
(see Dolgachev)

Monodromy weight filtration (let $n=3$) :

$N_i = \log T_i$; $N = \sum a_i N_i$, $a_i > 0$

$\exists!$ filtration $0 \subseteq W_0 \subseteq W_2 \subseteq W_4 \subseteq W_6 = H^3(X, \mathbb{Q})$

$W_0 = \operatorname{Im} N^3$, $W_4 = \ker N^3$

$W_2 / W_0 = \operatorname{Im} N^2$, ...

Cattani - Kaplan $\Rightarrow$ This filtration does not depend on choices of a_i.

Rmk : Part of the data defines MHS .

CONJECTURE (—, Wilson, D. Morrison)

The Leray filt. $\equiv$ The weight filt.

Easy to prove given the monodromy conjecture:

$\sigma \in H^1(B, R^{n-1} f_* \mathbb{Z})$

$T_\sigma^* : H^p(X, \mathbb{Z}) \circlearrowleft$ or $T_\sigma^* - I$

Basic question: understand $T_\sigma^* - I$ in terms of σ, and its action on $R^p f_* \mathbb{Z}$; and Leray filt...

$T_\sigma^* - I : R^p f_* \mathbb{Z} \circlearrowleft$ is zero since action of T_σ on X_b is coh. trivial.

Let $0 \subseteq F_0 \subseteq \cdots \subseteq F_p = H^p(X, \mathbb{Z})$ be the Leray filtration: $\quad (E_\infty^{p-i, i} = Gr_p := F_p / F_{p-1})$

can show: $(T_\sigma^* - I)(F_i) \subseteq F_{i-1}$.

What is the action

$$T_\sigma^* - I : F_i / F_{i-1} \longrightarrow F_{i-1} / F_{i-2} \quad ?$$

$$\begin{array}{ccc} \wr\Vert & & \wr\Vert \\ H^{p-i}(B, R^i f_* \mathbb{Z}) & \longrightarrow & H^{p-i+1}(B, R^{i-1} f_* \mathbb{Z}) \end{array}$$

(if the Leray s.s degenerates at E_2, say $n = 3$)

$\sigma \in H^1(B, R^{n-1} f_* \mathbb{Z}) \cong H^1(B, R^1 \check{f}_* \mathbb{Z})$

construct

$\langle \cdot, \sigma \rangle : H^{p-i}(B, R^i f_* \mathbb{Z}) \to H^{p-i+1}(B, R^{i-1} f_* \mathbb{Z})$ by

$\alpha \in H^{p-i}(B, R^i f_* \mathbb{Z}) \cong H^{p-i}(B, R^{n-i} \check{f}_* \mathbb{Z})$

$\exists$ a pairing

$$H^i(B, R^j \check{f}_* \mathbb{Z}) \times H^{i'}(B, R^{j'} \check{f}_* \mathbb{Z}) \to H^{i+i'}(B, R^{j+j'} \check{f}_* \mathbb{Z})$$

get $\alpha \cup \sigma \in H^{p-i+1}(B, R^{n-i+1} \check{f}_* \mathbb{Z})$

$$\cong H^{p-i+1}(B, R^{i-1} f_* \mathbb{Z})$$

Theorem. The map induced by $T_\sigma^* - I$ is
$\langle \cdot, (-1)^i \sigma \rangle$.

Let $N_\sigma = T_\sigma^* - I$:

$$\begin{array}{ccccccc} \cdot & \overset{N_\sigma}{\searrow} & 0 & & 0 & & \cdot \\ 0 & & \cdot & \overset{N_\sigma}{\searrow} & \cdot & & 0 \\ 0 & & \cdot & & \cdot & \overset{N_\sigma}{\searrow} & 0 \\ \cdot & & 0 & & 0 & & \cdot \end{array}$$

$N_\sigma^3(1) = D^3 \cdot [T^3]$
where $D \in H^1(B, R^1 \check{f}_* \mathbb{Z})$
$= \mathrm{Pic}(\check{X})$
corresponding to σ.

$\Rightarrow$ weight filtration = Leray filtration.

Fact (from Nilpotent orbit thm):
knowledge of the monodromy
$\Rightarrow$ asymptotic behavior of $(1,2)$-Yukawa coupling
$\Rightarrow$ const. term of $(1,2)$-Yukawa coupling
$\Rightarrow$ The const. term agree w/ the const. term of the $(1,1)$-Yukawa coupling of the mirror.

More topological result:

$H^1(B, R^2 f_* \mathbb{Z}) \cong H^1(B, R^1 \check{f}_* \mathbb{Z}) = \mathrm{Pic}(\check{X})$ gives correspondence between line bundles on $\check{X}$ and sections of f: $D \mapsto \sigma_D \in H^1(B, R^2 f_* \mathbb{Z})$.
σ_D is a section, hence defines coh. class in $H^3(X, \mathbb{Z})$.

CONJECTURE (Mirror Riemann-Roch)

$$\sigma_0 \cdot \sigma_D = \chi(\mathcal{O}_{X^\vee}(D)).$$

(predicted by Kontsevich's homological m.s. program.)

休憩

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1999. 1. 11.

$f: X \to B$; construct dual $\rightsquigarrow$ $\check{f}: \check{X} \to B$?
Given Kähler and cpx structure on X, what is the structure on $\check{X}$?

$$\begin{array}{ccc} X & & \check{X} \\ \Omega & & \check{\Omega} \\ B + i\omega \in \dfrac{H^2(X,\mathbb{C})}{H^2(X,\mathbb{Z})} & & \check{B} + i\check{\omega} \end{array}$$

Mirror Symmetry suggests that if we put

$\Omega_n = \Omega / \underbrace{\int_{T^n} \Omega}_{\mathrm{vol}(T^n)}$ (so $\int_{T^n} \Omega_n = 1$)

we want

① $[\Omega_n] - [\sigma_0] \in H^1(B, R^{n-1} f_* \mathbb{C}) \cong H^1(B, R^1 \check{f}_* \mathbb{C})$

in good cases : $\cong H^2(\check{X}, \mathbb{C})$

corresponds to $\check{B} + i\check{\omega}$.

② $B + i\omega \in H^1(B, R^1 f_* \mathbb{C}) \cong H^1(B, R^{n-1} \check{f}_* \mathbb{C})$

should correspond to $[\check{\Omega}_n] - [\check{\sigma}_0]$.

Recall :

$$0 \to R^{n-1} f_* \mathbb{Z} \to T_B^* \xrightarrow{\pi} X^\# \to 0$$

the LHS inclusion has another interpretation :

$$(R^{n-1} f_* \mathbb{Z})_b \cong H^{n-1}_c(X_b^\#, \mathbb{Z}) \cong H_1(X_b^\#, \mathbb{Z})$$

Define a map $H_1(X_b^\#, \mathbb{Z}) \hookrightarrow T^*_{B,b}$ as follows :

$$\begin{array}{c} \in \\ \gamma \end{array} \longmapsto \left(\vec{v} \longmapsto \int_\gamma \iota(\vec{v}) \omega \right)$$

lifting is not unique, but contracts with ω is then unique.

In fact, this embedding coincides w/ embedding of last lecture. Dual side on $\check{X}$ has:

$$0 \to R^1 f_* \mathbb{Z} \to T_B \to \check{X}^\# \to 0$$

Two methods for constructing $\check{\omega}$ on $\check{X}^\#$:

① Embed $R^1 f_* \mathbb{Z} \hookrightarrow T_B^*$ via $(R^1 f_* \mathbb{Z})_b = H_{n-1}(X_b, \mathbb{Z})$

$$H_{n-1}(X_b, \mathbb{Z}) \hookrightarrow T_B^* \text{ via}$$

$$\gamma \longmapsto \left(\vec{v} \longmapsto \int_\gamma \iota(\vec{v}) \operatorname{Im} \Omega_n \right) \quad \text{(due to Hitchin)}$$

② There is a natural metric h on B given by

$$h(\vec{v}, \vec{w}) = -\int_{X_b} \iota(\vec{v})\omega \wedge \iota(\vec{w}) \operatorname{Im} \Omega_n$$

Use this to identify T_B and T_B^*, giving

$$\begin{tikzcd} 0 \arrow[r] & R^1 f_* \mathbb{Z} \arrow[r, hook] \arrow[d, equal] & T_B^* \arrow[r] & \check{X}^\# \arrow[r] & 0 \\ 0 \arrow[r] & R^{n-1} \check{f}_* \mathbb{Z} \arrow[ur] & & & \end{tikzcd}$$

Fact: ①, ② gives the same embedding.

This gives $\check{\omega}$ on $\check{X}_0$ which is of the described cohomology class. In fact, for all $b \in B_0$,

$$\gamma \in H_1(\check{X}_b, \mathbb{Z}) \cong H_{n-1}(X_b, \mathbb{Z}), \quad \vec{v} \in T_{B,b}$$

$$\int_\gamma \iota(\vec{v})\check{\omega} = \int_\gamma \iota(\vec{v}) \operatorname{Im} \Omega_n$$

$\Rightarrow$ $\check{\omega}$ and $\operatorname{Im}\Omega_n$ represent the same class in

$$H^1(B, R^1 \check{f}_* \mathbb{R}) \cong H^1(B, R^{n-1} f_* \mathbb{R}).$$

This should give the symplectic structure on $\check{X}$, at least on the smooth points.

$$0 \to R^{n-1} f_* \mathbb{Z} \to T_B^* \to X^\# \to 0$$
$$\text{0-section} \mapsto \sigma_0$$

This held w/ choice of σ_0. Even w/o a choice,

$$0 \to R^{n-1} f_* \mathbb{Z} \to T_B^* \to J^\# \to 0$$

$X^\# \subseteq J^\#$ iff $X^\#$ had a Lagrangian section

$J^\# \to B$ is the Jacobian of $X^\# \to B$, which clearly has a Lag. section.

The set of fibration $X^\# \to B$ with Jacobian $J^\# \to B$ is in fact equal to $H^1(B, \Lambda(J^\#))$, where $\Lambda(J^\#)$ is the sheaf of Lag. sections of $J^\#$.

Complex structure ?

Recall: To specify cpx str, enough to specify cpx valued n-form Ω st.

① $d\Omega = 0$

② Ω is locally decomposable ($\Omega = \theta_1 \wedge \cdots \wedge \theta_n$)

③ $\Omega \wedge \bar{\Omega}$ should be nowhere zero.

If $y_1, \cdots, y_n$ coor. on base, $x_1, \cdots, x_n$ coor. on T_B^* st. $\omega = \sum dx_i \wedge dy_i$, can write

$$\Omega = V \cdot \Lambda_i \left(dx_i + \textstyle\sum_j \beta_{ij} \, dy_j \right)$$

where $\Omega|_{T^n} = V \, dx_1 \wedge \cdots \wedge dx_n = \mathrm{Vol}(T^n)$.

If (g_{ij}) denotes metric on the fibers, then $V = \sqrt{\det(g_{ij})}$.

* Calculation ① : (simple)

$$(-1)^{n(n-1)/2} \left(\frac{i}{2}\right)^n \Omega \wedge \bar{\Omega} = V^2 \det(\mathrm{Im}\,\beta) \cdot \frac{\omega^n}{n!}$$

So for Ricci-flat metrics, want $\det(\mathrm{Im}\,\beta) = V^{-2}$.

* Calculation ②: ω is a positive $(1,1)$ form $\Leftrightarrow$ $\beta = (\beta_{ij})$ is symmetric and $\mathrm{Im}\,\beta$ is positive definite.

* Calculation ③: $\mathrm{Im}\,\beta_{ij} = g^{ij}$ and the spaces

$$(T X_{b,x})^{\perp} = J(T_{X_{b,x}})$$

are spanned by $\left\{ \frac{\partial}{\partial y_j} - \sum \mathrm{Re}\,\beta_{ij} \frac{\partial}{\partial x_i} \;\middle|\; 1 \le j \le n \right\}$.

$\therefore \Omega$ is determined by β.

$\mathrm{Im}\,\beta \Leftrightarrow$ metric on fibers

$\mathrm{Re}\,\beta \Leftrightarrow$ Horizontal connection, with horizontal spaces being Lagrangian.

Now consider $d\Omega = 0$. Write $\beta = \sum \beta_{ij}\, dy_j \otimes \frac{\partial}{\partial x_i}$

- Theorem: The almost cpx structure determined by Ω (i.e. $\theta_1, \cdots, \theta_n$ span $(1,0)$ forms) is integrable iff $\quad d_y \beta - \frac{1}{2}[\beta, \beta] = 0$.

Here $\beta = \sum \beta_i \frac{\partial}{\partial x_i}$, β_i : 1-forms,

$d_y \beta = \sum (d_y \beta_i) \frac{\partial}{\partial x_i}$; $d_y :=$ exterior diff only involving $y_1, \dots, y_n$.

If $\alpha = \sum \alpha_i dy_i$, $\beta = \sum \beta_j dy_j$. Then

$$[\alpha, \beta] := \sum_{i,j} [\alpha_i, \beta_j]\, dy_i \wedge dy_j.$$

(This is the analog in K–S theory, with $z, \bar{z}$ replaced by y and x)

Now a very "hard calculational" thm:

- Theorem: $d\Omega = 0 \Leftrightarrow d_y\beta - \frac{1}{2}[\beta, \beta] = 0$ and $d\Omega$ contains no terms of the form:

$$dy_1 \wedge dx_1 \wedge \cdots \wedge dx_n.$$

$$\Omega = V dx_1 \wedge \cdots \wedge dx_n + \sum_{i,j} (-1)^{i-1} \beta_{ij}\, dy_j \wedge dx_1 \wedge \cdots \widehat{dx_i} \wedge \cdots \wedge dx_n + \cdots$$

for $\beta = b + ig$, this is equiv. to

① $\underbrace{d_y b - \frac{1}{2}[b,b]}_{\text{curvature of horizontal connection}} + \frac{1}{2}[g,g] = 0$

② $d_y g - [b, g] = 0$

③ The 1-forms $dx_1, \ldots, dx_n$ are harmonic on fibers.

④ The volume form $V dx_1 \wedge \cdots \wedge dx_n$ is inv. under parallel translation by the connection.

This is the recipe for constructing Ricci flat structure from an "symplectic structure".

Cor: If $d\Omega = 0$ and the horizontal connection is flat, then the metric g is constant on fibers (flat on fibers).

pf: Curvature $= 0 \underset{①}{\Rightarrow} [g, g] = 0$.

Let $g^i = \sum g^{ij} \frac{\partial}{\partial x_j}$, then $[g^i, g^j] = 0$

Let $\omega_1, \ldots, \omega_n$ be the dual basis on $T X_b$,

i.e. $\omega_i = \sum g_{ij}\, dx^j$

$d\omega_i(g^j, g^k) = g^j(\omega_i(g^k)) - g^k(\omega_i(g^j)) - \omega_i([g^j, g^k])$
$= 0$

Then $\exists$ a function φ_i on $\mathbb{R}^n$ (univ. cover of X_b)

st. $\frac{\partial \varphi_i}{\partial x_j} = g_{ij}$ $(d\varphi_i = \omega_i)$

since $\partial \varphi_i / \partial x_j = \partial \varphi_j / \partial x_i$, can find φ on $\mathbb{R}^n$ st.

$g_{ij} = \partial^2 \varphi / \partial x_i \partial x_j$.

choose φ to be of the form

$$\varphi_{quad} + \varphi_{per} \quad \text{with } \int_{X_b} \varphi_{per} = 0$$

Can calculate $* dx_i = V^{-1} \omega_1 \wedge \dots \wedge \hat{\omega}_i \wedge \dots \wedge \omega_n$.

By ③, $0 = d(V^{-1}\omega_1 \wedge \dots \hat{\omega}_i \dots \wedge \omega_n)$

$$= \underbrace{\pm g^i (V^{-1})}_{=0} \, \omega_1 \wedge \dots \wedge \omega_n$$

$\forall i$ so V^{-1} is constant on fiber.

But $V = \sqrt{\det(g_{ij})}$, thus $\det\left(\frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right) = C$

Monge-Ampère Eq'n $\Rightarrow$ $\varphi_{per} = 0$ via standard PDE theory.

Recall $\check{\omega}$ satisfied $\int_\gamma c(\vec{v})\check{\omega} = \int_\gamma c(\vec{v}) \operatorname{Im} \Omega_n$, we also want (by symmetry)

$$(*) \quad \int_\gamma c(\vec{v}) \operatorname{Im} \check{\Omega}_n = \int_\gamma c(\vec{v})\, \omega$$

Recall metric h on B, metric $\check{h}$ on B, once $\check{\Omega}$ is constructed, $(*)$ is equiv. to $h = \check{h}$.

This was expected by Hitchin, now was put in a more clear way.

Hitchin Solution: Given $f: X \to B$, can construct $\check{\Omega}$ on $\check{f}_0 : \check{X}_0 \to B$ by taking the connection to be the Gauss-Manin connection. Ie.

∇_{GM} on $(R^1 f_* \mathbb{R}) \otimes C^\infty(B)$, whose horizontal sections are the sections of $R^1 f_* \mathbb{R}$

and take the metric to be $\check{g}_{ij} = h_{ij}$.

(Rmk: compare with Hitchin's original form)

But this can't be true since flat metric can't degenerate to singular fiber by collapsing cycles.

Twisted Hitchin's Solution:

Let $\mathcal{A}$ be the sheaf on B defined by

$$\mathcal{A}(U) = \{ \text{sections } \sigma : U \to \check{f}^{-1}(U) \text{ for which } T_\sigma^* \check{\omega} = \check{\omega} \text{ and } T_\sigma^* \check{\Omega}_n = \check{\Omega}_n \}$$

Can show $\mathcal{A} \cong R^{n-1} \check{f}_*(\mathbb{R}/\mathbb{Z})$, ie. only those sections which are horizontal.

Elements of $H^1(B_0, \mathcal{A})$ give rise to non-isom choices $\check{X}_0' \to B_0$. (Some of these may not have Lagrangian sections.)

Refined Mirror Symmetry Conjecture:

Let X be a compact complex C-Y n-fold, w/ ω, Ω, $f : X \to B$ a simple s. Lag. fibration. Then for each $\underline{B} \in H^1(B, R^1 f_* \mathbb{R}/\mathbb{Z})$ there exists a C-Y n-fold $\check{X}$, $\check{\omega}$, $\check{\Omega}$ and s. Lag fibration $\check{f} : \check{X} \to B$ (not nec. has a section) st.

① for each $U \subseteq B_0 = B - \Delta$ on which f and $\check{f}$ both have sections, $f^{-1}(U) \to U$ and $\check{f}^{-1}(U) \to U$ are topologically dual.

② $\forall\, b \in B_0$, $\gamma \in H_1(\check{X}_b, \mathbb{Z}) \cong H_{n-1}(X_b, \mathbb{Z})$ and $v \in T_{B,b}$,

$$\int_\gamma \iota(v)\check{\omega} = \int_\gamma \iota(v) \operatorname{Im} \Omega_n$$

③ For $\gamma \in H_{n-1}(\check{X}_b, \mathbb{Z}) \cong H_1(X_b, \mathbb{Z})$:

$$\int_\gamma \iota(v)\,\omega = \int_\gamma \iota(v) \operatorname{Im} \check{\Omega}_n$$

④ $\check{f}$ has a C^∞ section if $\underline{B} \in H^1(B, R^1 f_* \mathbb{R}) / H^1(B, R^1 f_* \mathbb{Z})$.

In this case, if $\check{\gamma}$ is a section, then

$$[\mathrm{Re}\,\check{\Omega}_n] - [\check{\gamma}]$$

defines a class in $H^1(B, R^{n-1}\check{f}_* \mathbb{R})$ which is well-defined modulo $H^1(B, R^{n-1}\check{f}_*\mathbb{Z})$ and this element coincides with $\underline{B}$.

⑤ If $\check{J}$ is the Jacobian of $\check{X}$ then $\check{X}$ is obtained from $\check{J}$ as a sympectic manifold via re-gluing using $\underline{B} \in H^1(B, R^{n-1}\check{f}_*\mathbb{R}/\mathbb{Z}) \longrightarrow H^1(B, \Lambda(\check{J}^\#))$.

⑥ Once $\check{X}$, $\check{\omega}$ is fixed, $\check{\Omega}_n$ is unique up to translation by a section.

THEOREM: The conjecture holds for K3 surfaces.

This pf does not use Hodge theory (Torelli thm)! but does Yau's thm on Calabi-Conj!!

what's left?

① Regularity for special Lagrangian currents.

② Regularity of s. Lag. fibrations.

③ Existence of s. Lag. cycles.
(cf. Schoen + Wolfson in 4-dim'l case)

④ Existence of s. Lag. fibrations
(would be the hardest!)

⑤ Classification of singular fibers.

⑥ Topological compactification of the dual.

⑦ Symplectic compactification.

⑧ Complex structure
(solving the eq'ns)

⑨ What does this have to do with counting rational curves.

End