Lectures on Calabi-Yau Manifolds.

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Lecture 1.

Def: A Calabi-Yau manifold is a compact cpx Kähler mfd (projective) X st. the canonical bundle ω_X is trivial. and usually, $b_1(X) = 0$

Yau $\Rightarrow$ This is equivalent to X admitting a Ricci flat metric.

Basic question: classification?

eg. $\dim_{\mathbb{C}} X = 1$, must be $\mathbb{C}/\mathbb{Z}^2$ elliptic curve

$\dim_{\mathbb{C}} X = 2$: X is a K3 surface, i.e. diffeomorphic to $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$ in $\mathbb{C}P^3$

Fact: There are a countable # of 19 dim'l moduli spaces of algebraic K3 surfaces. considering all K3 surfaces, find only an irreducible 20 dim'l moduli space.

$\dim \geq 3$?

There are known to be 100,000's of families of C-Y 3 folds, often can be distinguished by Betti numbers.

eg. complete intersections in $\mathbb{P}^n$:

$(5) \subseteq \mathbb{P}^4$ $\quad$ $(2,2,3) \subseteq \mathbb{P}^6$

$(2,4) \subseteq \mathbb{P}^5$ $\quad$ $(2,2,2,2) \subseteq \mathbb{P}^7$

$(3,3) \subseteq \mathbb{P}^5$

all have different Euler characteristics.

Question : Are there a finite number of diffeo. type of C-Y 3 folds ?

- A finite number of deformation type ?

Philosophy : new C-Y arise from old C-Y by degenerating and resolving.

$\mathcal{X} \to \Delta$ flat family, Δ : disk st $\mathcal{X}_t$ nonsingular C-Y for $t \neq 0$

$\mathcal{X}_0$ is singular

$\mathcal{X}_0$ is a deg. of $\mathcal{X}_t$.

$Y \to X$ resol. of sing. is called crepant if Y is also C-Y.

eg. ① $\mathcal{X}_t = \{ t x_0^5 + x_1^5 + \cdots + x_4^5 + x_0^2 (x_1^3 + \cdots + x_4^3) = 0 \}$

in $\mathbb{P}^4 \times \Delta \leftarrow$ coor t.

for $t \neq 0$, general $\mathcal{X}_t$ is non-sing.

$t=0$ look at the affine piece of $\mathbb{P}^4$ where $x_0 \neq 0$

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_1^5 + \cdots + x_4^5 = 0$$

singular at $(1:0:\cdots 0) \in \mathbb{P}^4$

$f : Y_1 \to X_0$ is obtained by blowing up $(1:0\cdots 0)$.

$$\begin{array}{ccc} E & \subset & Y_1 \\ \downarrow & & \downarrow \\ p & \in & X_0 \end{array}$$

E is a cubic surface given by

$x_1^3 + \cdots + x_4^3 = 0$ in $\mathbb{P}^3$

Y_1 is C-Y

$b_2(Y_1) = 2$, $b_2(\mathcal{X}_t) = 1$.

topologies changes.

(2) Let X_0 be a quintic containing a $\mathbb{P}^2 \subseteq \mathbb{P}^4$.
X_0 has, in general, 16 singular points which are ODP's ($x^2 + \cdots + w^2 = 0$).
Can't blow up ODP's, since not crepant.
Can blow up $\mathbb{P}^2$

$$\begin{array}{ccccc} E & \subset & Y_2 & \supseteq & \mathbb{P}^1 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}^2 & \subseteq & X_0 & \ni & \cdot \end{array}$$

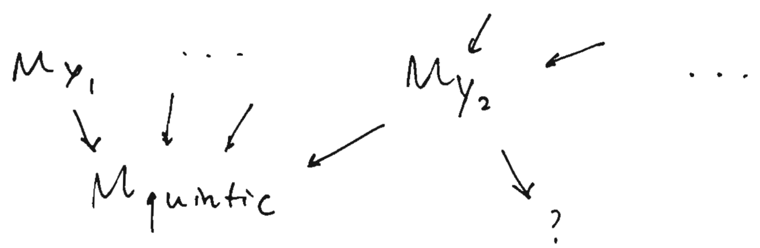
small resol. of ODP
Y_2 is a C-Y
$b_2(Y_2) = 2$.
not diffeom. to Y_1

Def: Let X, Y be 2 C-Y 3folds. We say that X and Y are related by an extremal transition if $\exists$ flat family $\mathcal{X} \to \Delta$ with $\mathcal{X}_t = X$ some t and $Y \to X_0$ a crepant resolution.

e.g. Both Y_1, Y_2 are related by extremal transition to the quintic.

The web of C-Y manifolds:
If X is C-Y, write M_X the moduli space
write $M_Y \to M_X$ if X and Y are related by an extremal transition.

Form graph



Conjecture (Reid's fantasy):

This graph is connected.

Evidence: True for all known examples!
(eg. Greene, Chang, Kanter:
7775 examples of hypersurfaces in weighted $\mathbb{P}^4$.)

Approach: Identify those vertices with no outgoing edges, and connect these together

Def: A Calabi-Yau Y is primitive if there is no birational contraction $f: Y \to X$ st. X is smoothable to a C-Y X' not deformation equivalent to Y.

Question: When is a singular C-Y smoothable?
ie. given Y non-singular, $Y \to X$ birational contraction, when is X smoothable?

Recall: Let X be a cpx mfd with $H^0(T_X) = 0$
(X C-Y, $b_1(X) = 0 \Rightarrow H^0(T_X) = 0$.)

The Kuranishi space of X, or Def X is a germ of a cpx analytic space along with a flat family $\mathcal{X} \to \text{Def } X$ such that given any flat family $\mathcal{X}' \to \mathcal{S} \ni 0$ with $\mathcal{X}'_0 \equiv X$ then $\exists!$ map $\mathcal{S} \to \text{Def } X$ st.

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{S} & \longrightarrow & \text{Def} X \end{array} \qquad \mathcal{X}' = \mathcal{X} \times_{\text{Def} X} \mathcal{S}$$

* Def X exists when:
- X is a cpt cpx mfd
- X singular cpx analytic space
- and certain case of germ of singularities.

General Fact:

- Zariski tangent space to $\text{Def}(X)$ at 0 is $H^1(X, T_X)$
  $\dim H^1(T) = \dim \text{Def } X \iff \text{Def } X$ is nonsingular
- $H^2(X, T) = 0 \Rightarrow$ nonsingular.
  but C-Y 3folds. $\dim H^2(T) = b_2(X) \neq 0$!

Theorem (Bogomolov – Tian – Todorov)

If X is C-Y, Def X is smooth

Tian's pf (sketch):

cpx str. on C-Y mfds $\quad \Lambda^n \Omega_X^1 \cong \mathcal{O}_X$

$\Rightarrow \exists$ nowhere vanishing holomorphic n form Ω

(all choices are related by a const. factor)

locally, $\Omega = dz_1 \wedge \cdots \wedge dz_n$ (cpx coor.)

* Proposition: (opposit direction)

Given a cpx valued n-form Ω on a $2n$-dim'l real mfd st.

① Ω is locally decomposable $\Omega = \theta_1 \wedge \cdots \wedge \theta_n$

② $\Omega \wedge \bar{\Omega}$ is nowhere 0

③ $d\Omega = 0$

determines a cpx structure where Ω is holomorphic.

Pf: (sketch)

$\theta_1, \cdots, \theta_n$ will span the $(1,0)$ forms

$\bar{\theta}_1, \cdots \bar{\theta}_n$ will $\cdots$ $(0,1)$ $\cdots$

This gives the decomposition $\Omega_X^1 \otimes \mathbb{C} = \Omega_X^{1,0} \oplus \Omega_X^{0,1}$

since ② $\Rightarrow$ $\theta_1, \cdots, \theta_n, \bar{\theta}_1, \cdots, \bar{\theta}_n$ are linearly independent. This deter. almost cpx str on X.

③ $\Rightarrow$ this a.c.s. is integrable and Ω is holomorphic. $\square$

This. think of a deformation of cpx str on X by giving a family of n-forms $\Omega(t)$. locally

$$\Omega(0) = dz_1 \wedge \cdots \wedge dz_n$$

$$\Omega(t) = \left(\left(dz_1 + \sum_j \beta_{1j}(t)\, d\bar{z}_j\right) \wedge \cdots \wedge \left(dz_n + \sum \beta_{nj}\, d\bar{z}_j\right)\right)$$

This satisfies ① and ②. $d\Omega(t) = 0$?

Write

$$\beta(t) = \sum_{i,j} \beta_{ij}(t)\, d\bar{z}_j \otimes \frac{\partial}{\partial z_i} \in \Gamma(\Omega^{0,1}_X \otimes T_X)$$

(independent of local coordinates)

$d\Omega(t) = 0 \iff$ Kodaira-Spencer equation:

$$\bar\partial \beta(t) + \frac{1}{2}[\beta(t), \beta(t)] = 0 \quad \text{where}$$

$$\bar\partial : \Gamma(\Omega^{0,1}_X \otimes T_X) \longrightarrow \Gamma(\Omega^{0,2} \otimes T_X)$$

$$\bar\partial\left(\sum \beta_i \otimes \frac{\partial}{\partial z_i}\right) := \sum (\bar\partial \beta_i) \otimes \frac{\partial}{\partial z_i}$$

$$\left[\sum_i d\bar{z}_i \otimes \alpha_i, \sum d\bar{z}_j \otimes \beta_j\right] := \sum_{i,j} d\bar{z}_i \wedge d\bar{z}_j \otimes [\alpha_i, \beta_j].$$

$\beta(0) = 0$

expand β in a Taylor series in t

$\beta = \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \cdots$

looking at t coeff, $\bar\partial \beta_1 = 0$, so β_1 determines a cohomology class in $H^1(X, T_X)$, the tangent space to $\mathrm{Def}\, X$.

looking at t^n term:

$$\bar\partial \beta_n + \frac{1}{2}\sum_{i=1}^{n-1} [\beta_i, \beta_{n-i}] = 0$$

Idea: Pick any β_1, representing a coh. class in $H^1(T_X)$, extends it to an actual deformation. then $\mathrm{Def}\, X$ is smooth.

Solve the Kodaira-Spencer eq^n recursively.

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rest.

(conti of 1st lecture.)

Def: $\iota : \Omega_X^{0,q} \otimes T_X \longrightarrow \Omega_X^{n-1,q}$ by

$\iota(\alpha \otimes \vec{v}) = \alpha \wedge (\vec{v} \lrcorner \Omega)$, is an isomorphism.

$\# : \Omega_X^{n,q} \longrightarrow \Omega_X^{0,q}$ by $\#(\Omega \wedge \alpha) = \alpha$.

Easy to see that $\bar\partial$ commutes with $\iota, \#$.

lemma: for $\alpha_1, \alpha_2 \in \Gamma(\Omega^{0,1} \otimes T_X)$:

$$\iota([\alpha_1, \alpha_2]) = \partial(\alpha_1 \lrcorner \iota(\alpha_2)) - \#(\partial \iota(\alpha_1)) \wedge \iota(\alpha_2)) + \iota(\alpha_1) \wedge \# \partial \iota(\alpha_2).$$

Will inductively, given $\beta_1, \cdots, \beta_{n-1}$, find β_n st.

$$\begin{cases} \bar\partial \beta_n + \frac{1}{2} \sum_{i=1}^{n-1} [\beta_i, \beta_{n-i}] = 0 \\ \partial \beta_n = 0 \end{cases}$$

It's not even clear that we can do this for $n=1$.

Recall $\partial\bar\partial$-lemma:

A $\bar\partial$ closed and ∂ exact form can be written as $\partial\bar\partial\gamma$.

We are given β_1 st $\bar\partial\beta_1 = 0$, $\partial\iota(\beta_1)$ may not be 0.

but is ∂ exact! and

$$\bar\partial(\partial \iota(\beta_1)) = -\partial(\bar\partial(\beta_1)) = 0, \text{ so}$$

$\partial\iota(\beta_1)$ is $\bar\partial$ closed, so $\exists\, \gamma$ st. $\partial\iota(\beta_1) = \partial\bar\partial\gamma$.

Replace β_1 with $\beta_1 - \iota^{-1}(\bar\partial\gamma) = \beta_1 - \bar\partial\iota^{-1}(\gamma)$

(does not change the cohomology class)

then $\partial(\iota\beta_1 - \bar\partial\iota^{-1}\gamma) = \partial\iota\beta_1 - \partial\bar\partial\gamma$

$= \partial\iota\beta_1 - \partial\iota\beta_1 = 0$. ok.

$n=2$: $\iota([\beta_1, \beta_1]) = \partial(\beta_1 \lrcorner \iota(\beta_1))$. so ∂ exact

$\bar\partial[\beta_1, \beta_1] = [\bar\partial\beta_1, \beta_1] + [\beta_1, \bar\partial\beta_1] = 0$, $\bar\partial$ closed by $\partial\bar\partial$ lemma, $\exists\,\psi$ st. $\partial\bar\partial\psi = \frac{1}{2} i[\beta_1, \beta_1]$

Set $\beta_2 = \iota^{-1}(\partial\psi)$. Then $\partial \iota\beta_2 = \partial\partial\psi = 0$

and $\bar\partial\beta_2 = \iota^{-1}(\bar\partial\partial\psi) = -\iota^{-1}(\partial\bar\partial\psi) = -\frac{1}{2}[\beta_1, \beta_1]$

Solves the eq'n for $n=2$.

$n \geq 3$: Can show

$$\bar\partial\left(\sum_{i=1}^{n-1} [\beta_i, \beta_{n-i}]\right) = 0 \quad (\text{Jacobi identity}),$$

Also, $\sum_{i=1}^{n-1} \iota[\beta_i, \beta_{n-i}] = \sum_{i=1}^{n-1} \partial(\beta_i \lrcorner\, \iota\beta_{n-i})$

Hence ∂ exact, $\bar\partial$ closed, so $\exists\,\psi$ st.

$$\partial\bar\partial\psi = \frac{1}{2}\sum_{i=1}^{n-1} [\beta_i, \beta_{n-i}],$$

Take $\beta_n = \iota^{-1}(\partial\psi)$. pf done. $\square$.

History:

Tian (diff. geom. pf 1986)

Ziv Ran: T^1-lifting criterion

Kawamata: extremely simple proof

what about singular varieties with $\omega_X \cong \mathcal{O}_X$?

eg. X has only ODP's?

Tian, Kawamata: ODP

Ran: Kleinian

Namikawa: BTT continues to hold for Calabi-Yau 3-folds with hypersurface canonical singularities and isolated.

(essentially the type of singularities which can arise $f: Y \longrightarrow X$ from smooth C-Y Y are canonical)

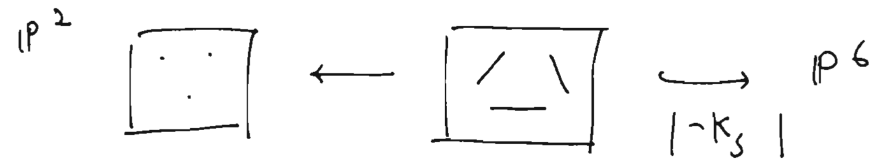
Question: Does BTT extends to C-Y 3folds with arbitrary canonical singularities? End.

Question: Does BTT holds for C-Y 3folds with arbitrary canonical sing.?

Answer: No!

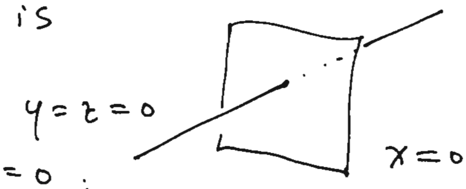
Examples:

① Local version: Take a cone over a Del Pezzo surface of degree 6 in $\mathbb{P}^6$:



Def X (Altman) $\subseteq \mathbb{A}^3$, x, y, z via

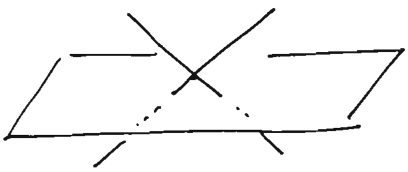
Def $X = \{xy = xz = 0\}$ is singular:



- Deformation corr. to $y=z=0$:

  Take a cone Y over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^7$

  $H \cap \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is a del Pezzo surface of deg 6



  $X = Y \cap H$

  H hyperplane passing through vertex.

  Smooth X by moving H away from the vertex.

- 2nd component:

  Z cone over $\mathbb{P}^2 \times \mathbb{P}^2 \subseteq \mathbb{P}^8$.

  $Z \cap H_1 \cap H_2 \equiv X$ for H_1, H_2 hyperplanes through vertex

  Smooth X by moving H_1, H_2.

  These 2 are topological distinct.

② Global version:

It's not hard to embed the local version into proj case:

Take proj. cone over del Pezzo surface of deg 6

$$\begin{array}{cl} X' & \text{double cover branched over } Q \\ \downarrow & Q = \text{general quartic hypersurface.} \\ X & \text{then } X' \text{ is a } K_{X'} = 0. \text{ Calabi-Yau.} \end{array}$$

Def X' has 2 components.

② Harder example (1st one):
(Essays in Mirror Symmetry II)
X a C-Y 3fold with a singular curve (a $\mathbb{P}^1$), a hypersurface in a smooth 4-fold.
Def X consists of 2 components, X can be smoothed to X_1, X_2 which are not deformation equivalent, But are diffeomorphic!
(can be distinguished using GW inv (–, Ruan)).

* Namikawa: Isolated hypersurface canonical sing.
$\Rightarrow$ BTT holds.

Idea: All obstruction to smoothness are local!

T^1-lifting criterion (Ran, Kawamata):

D a Deformation functor, ie.
D : {Artinian local $\mathbb{C}$-algebras} $\longrightarrow$ Sets
such that $D(\mathbb{C})$ consists of one element

eg. If X is a cpx variety
$$D(A) = \{ X_A \longrightarrow \operatorname{Spec} A : \text{flat deformation with } X_A \otimes \mathbb{C} \equiv X \} / \cong.$$

$A_n = \mathbb{C}[t]/t^{n+1}$

X C-Y mfd, smooth case.. may be thought as

" $D(A_n) = \{ \beta = \beta_1 t + \cdots + \beta_n t^n$ satisfying

$\bar\partial \beta + \frac{1}{2}[\beta, \beta] = 0 \mod t^{n+1} \} / \sim$ "

$B_n = \mathbb{C}[t,y]/(t^{n+1}, y^2) \xrightarrow{\alpha_n} A_n \quad (t \mapsto t, y \to 0)$

Given $X_n \in D(A_n)$, define

$T'(X_n/A_n) = \{ Y \in D(B_n) \text{ s.t. } \alpha_n(Y) = X_n \}$

eg. smooth case :

If X_n is given by $\beta_1 t + \cdots + \beta_n t^n$

$Y \in T'(X_n/A_n)$ is given by $\beta + \beta' y$

$\beta' = \beta_0' + \beta_1' t + \cdots + \beta_n' t^n$

satisfying K-S eq'n. ie.

$\bar\partial(\beta + \beta' y) + \frac{1}{2}[\beta + \beta' y, \beta + \beta' y] = 0$

$\Leftrightarrow \quad \bar\partial \beta + \frac{1}{2}[\beta, \beta] = 0$ (automatically)

$\bar\partial \beta + [\beta, \beta'] = 0$ linear eq'n in β'

so $T'(X_n/A_n)$ is a vector space.

Can show : In smooth case,

$T'(X_n/A_n) \cong H^1(T_{X_n/A_n})$,

in singular case :

$T'(X_n/A_n) \cong \mathrm{Ext}^1(\Omega^1_{X_n/A_n}, \mathcal{O}_{X_n})$

* T'-lifting criterion : Let $X_n \in D(A_n)$, X_{n-1} the image of X_n in $D(A_{n-1})$. Let $\varepsilon_n : A_n \to B_{n-1}$ by $t \mapsto x+y$. Let α = image of X_n in $D(B_{n-1})$ under the map $D(\varepsilon_n)$.

(i.e. $\beta_1 t + \cdots + \beta_n t^n \mapsto \beta_1 (x+y) + \cdots + \beta_n (x+y)^n$)

Then $\alpha \in T^1(X_{n-1}/A_{n-1})$ and there exists a lifting of X_n to $X_{n+1} \in D(A'_{n+1}) \iff \alpha$ is in the image of the map

$$T^1(X_n/A_n) \longrightarrow T^1(X_{n-1}/A_{n-1})$$

$D(A_{n+1}) \to D(A_n)$ is a mapping of sets, not much structures there. but T^1 is a vector space. hence we have the possibility to relate them. using cohomology theory.

Pf: $\alpha = \beta_1 t + \cdots + \beta_{n-1} t^{n-1} + \beta_1 y + 2\beta_2 t y + \cdots + n\beta_n t^{n-1} y$

(via differentiation !)

suppose lifts to

$\alpha' = \beta_1 t + \cdots + \beta_n t^n + \beta_1 y + 2\beta_2 t y + \cdots + n\beta_n t^{n-1} y + \gamma t^n y$

(γ not determined)

look at $t^n y$ term in K-S eq'n :

$$\bar\partial \gamma + \frac{1}{2} \sum_{i=1}^{n} [\beta_i, (n-i+1)\beta_{n-i+1}] = 0$$

$$\iff \bar\partial \gamma + \frac{n}{2} \sum_{i=1}^{n} [\beta_i, \beta_{n+1-i}] = 0$$

Take $\beta_{n+1} = \gamma / n$, so

$$\bar\partial \beta_{n+1} + \frac{1}{2} \sum_{i=1}^{n} [\beta_i, \beta_{n+1-i}] = 0 \quad !!! \quad \square$$

Thus, if $T^1(X_n/A_n) \longrightarrow T^1(X_{n-1}/A_{n-1})$ are surjective $\forall n$, X_n, then BTT holds

Smooth case : this is

$H^1(T_{X_n/A_n}) \longrightarrow H^1(T_{X_{n-1}/A_{n-1}})$

For a C-Y, $T_X \cong \Omega_X^{d-1}$, $d = \dim X$, becomes

$$H^1(\Omega^{d-1}_{X_n/A_n}) \longrightarrow H^1(\Omega^{d-1}_{X_{n-1}/A_{n-1}})$$

this is always true (surjective) by Deligne.

essentially : this is the "base change thm"

i.e. the Hodge bundle (in the infinitesimal sense

$$\begin{array}{ll} X & \\ \downarrow f & R^1 f_* \Omega^{d-1}_{X/S} \text{ a vector bundle !} \\ S & \text{(cf. Deligne's Hodge Theory)} \end{array}$$

(conti.) M. Gross' version of

- Theorem (BTT obstructedness) X a C-Y 3 folds with canonical sing.'s D : deformation functor of X. D_{loc} the def. functor of the germ of sing. of X. Then $\exists$ vector spaces T^2 and T^2_{loc} and a diagram:

$$\begin{tikzcd}
T^1(X_n/A_n) \arrow[r] \arrow[d] & T^1_{loc}(X_n/A_n) \arrow[d] \\
T^1(X_{n-1}/A_{n-1}) \arrow[r] \arrow[d, "\delta"] & T^1_{loc}(X_{n-1}/A_{n-1}) \arrow[d, "\delta'"] \\
T^2 \arrow[r, "\ell"] & T^2_{loc}
\end{tikzcd}$$

The columns are exact. and

$$\ell\big|_{\mathrm{im}\,\delta} : \mathrm{im}\,\delta \longrightarrow T^2_{loc} \text{ is injective.}$$

i.e. local lifting $\Rightarrow$ global lifting.

Rmk: for isolated sing. easier. for general 1-dim'l sing. need strong local duality thm. (same time proved by Lipman + ...).

- Cor: If D_{loc} is unobstructed, so is D.
- Cor: Let Def X_{loc} denote the def. space of germ

If $r = \dim \ker (T^1(X/k) \to T^1_{loc}(X/k))$

$$s = \dim \operatorname{coker} (T^1(X/k) \longrightarrow T^1_{loc}(X/k))$$

Then Def X is set-theoretic defined by s equations in Def $X_{loc} \times \mathbb{C}^r$.

Rmk: Its standard that

$$T^2 = \operatorname{Ext}^2(\Omega^1_X, \mathcal{O}_X).$$

Q: True in higher dimensions ??

Smoothing problem:

Friedman: Let X be a C-Y 3 fold with only ODP's, $f: \hat{X} \to X$ a small resolution ($\hat{X}$ may not be Kähler) let $C_1, \cdots, C_n$ be exceptional curves. Then

$$\sum_{i=1}^n a_i [C_i] = 0 \quad \text{in } H^4(\hat{X}, \mathbb{R}) \qquad a_i \neq 0 \ \forall i$$

$\Longleftrightarrow$ X is smoothable !

It looks like for more general singularities the situation will be worse. Surprisingly, NOT true !

* Thm (Gross)
Let Y be a nonsingular C-Y 3fold, $\pi : Y \to X$ a birat'l contraction st X has isolated complete intersection singularities. Then $\exists$ a deformation of X smoothing all but the ODP's of X. If X has no ODP's, X is smoothable.

* Thm (Namikawa (+ Steenbrink))
If X has canonical complete i. sing's. then X is deformable to something with ODP's.

Idea of pf: look at triple point $\underbrace{x^3+y^3+z^3+w^3}_{f} = 0$

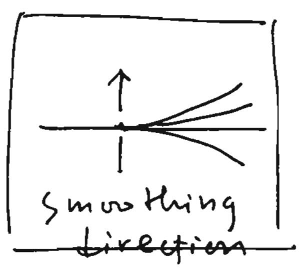
$$T^1_{loc} = \mathbb{C}[x,y,z,w] \Big/ \left(\frac{\partial f}{\partial x}, \dots, \frac{\partial f}{\partial w}, f\right)$$

$$= \mathbb{C}[x,y,z,w] / (x^2, y^2, z^2, w^2)$$

Basis: $x, y, z, w, xy, xz, xw, yz, yw, zw$
" xyz, xyw, xzw, yzw, $xyzw$ " ← This part we do not care. does not change sing.

Universal deformation is

$$f + t_0 1 + t_1 x + t_2 y + \dots + t_{15} xyzw = 0$$



Def X_{loc}
D = Discriminant locus
Tangent space to D is supported on $t_0 = 0$

Schlessinger (or Wahl?) In isolated sing. case for 3-dim's:

$$T^1 \equiv \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) \equiv H^1(U, T_U)$$

$Z \subseteq X$ is sing. locus, $U = X - Z$.

$$H^1(Y,T_Y)\to H^1(U,T_U)\xrightarrow{\ \mathrm{im}\ } H^1_{\pi^{-1}(Z)}(Y,T_Y)\xrightarrow{\ \ker\ } H^2(Y,T_Y)$$

$$\begin{tikzcd} H^1(U,T_U) \arrow[u, "\cong"] \arrow[r] & H^1_Z(X,T_X) \arrow[u] \\ T^1 \arrow[u, equal] \arrow[r] & T^1_{loc} \arrow[u, equal] \end{tikzcd}$$

Want $\ker\left(H^2_{\pi^{-1}(Z)}(T_Y)\to H^2(T_Y)\right)$ to be big.

dual to:

$$\begin{tikzcd} \operatorname{coker}\big(H^1(\Omega^1_Y) \arrow[r] & H^0(R^1\pi_*\Omega^1_Y)\big) \\ \operatorname{Pic} Y\otimes\mathbb{C} \arrow[u, "\wr\wr"] \arrow[r] & \operatorname{Pic}(Y,\pi^{-1}(Z))\otimes\mathbb{C} \arrow[u, "*"] \end{tikzcd}$$

even after ignor insignificant part like $xyz,\dots$ still have $xy, yz, \dots$ which gives the smoothing direction. ※

Rmk: In the ODP case. the argument shows that depends on the geometry globally.
($*$ is an isom. and $H^0(R^1\pi_*\Omega^1_Y)=\mathbb{C}$
$T^1=\mathbb{C}[x,y,z,w]/(x,y,z,w)=\mathbb{C}$).

Primitive contractions: $f: Y\to X$ a birat'l contraction if primitive if can't be factored in the algebraic category.

Type I. Small contraction to hypersurface sing's.

Type II. An irred. del Pezzo surface is contracted to a pt.

Type III. Contraction of a surface to a curve.

* Thm (Gross)

If $f: Y \to X$ is a primitive contraction, it is smoothable unless :

I. f contracts a single $\mathbb{P}^1$ to an odp.

II. f contracts a $\mathbb{P}^1$ or a $\mathbb{P}^2$ blow up in one pt to a point.

III. The exceptional divisor E contracted to a $\mathbb{P}^1$ and $E^3 = 7$ or 8.

* Thm (Nikulin)

If X is a non-singular C-Y 3folds, the one of the following holds :

① rank $\mathrm{Pic}(X) \leq 41$

② X has a type I contraction

③ $\exists$ a nef $\mathbb{R}$-divisor D st. $D^3 = 0$.

* Thm : (Gross)

If X is a primitive C-Y 3fold, then either

① $\mathrm{Pic}\, X = \mathbb{Z}$

② same as in (Nikulin), ③ same.

CONJECTURE : let $f: Y \to X$ be a birat'l contraction with X factorial. Then X is smoothable if all its sing's are smoothable.

Other applications : Four-fold flops (Namikawa)
(in. Proc. Symp. in integrable system)

Morrison – Seiberg :

A – D – E

contracting to n ODP's $\longleftrightarrow$ A_n

— E to $\mathbb{P}^1$ with $E^3 = 8 - n$ $\longleftrightarrow$ D_n

— del Pezzo of degree $9 - n$ $\longleftrightarrow$ E_n

The change in Betti numbers is related to the dual coxeter number of the Lie group.

<u>End</u>.