The First

NCTS Summer School on Algebraic Geometry

July 19 — 30, 1999

Professor Yujiro Kawamata
Tokyo University

(Notes by Chin-Lung Wang)

NCTS Summer School in Algebraic Geometry (1999)

Prof. Yujiro Kawamata

Lecture I. 7/19 at Academia Sinica

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* Algebraic Fiber Spaces, overview

complex algebraic varieties:

reduced, irreducible, finite type scheme / $\mathbb{C}$

Relative situation: alg. fiber space

$$\begin{array}{ccc} f: X & \longrightarrow & Y \\ \cup & & \Psi \\ X_\eta & \longrightarrow & \eta \end{array}$$

morphism

generic point

generic fiber / $\mathbb{C}(Y)$ = rat'l function field of Y

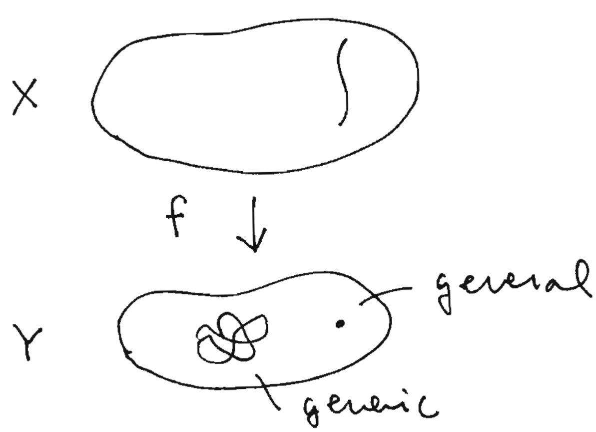
$\mathbb{C}(Y) \subset K$ (alg. closed) eg. $K = \overline{\mathbb{C}(Y)}$

$X_{\bar\eta}$ geometric generic fiber

Def: f alg-fiber space

$\iff X_{\bar\eta}$ algebraic variety

$\iff$ general fiber are alg. v. / $\mathbb{C}$



$Y \supset U \ni \forall p$
open

$X_p = f^{-1}(p) =$ variety

morphism (for complete varieties) — alg. fiber space; finite morphism $\left\{\begin{array}{l}\text{normalization (simplest singularities)} \\ \text{covering with ramification} \\ \text{closed immersion}\end{array}\right.$

We consider only alg. fiber space.

fiber bundle $\Rightarrow$ no degeneration

good model of alg. f. space $\ni$ controlled degeneration

Very important example:

Elliptic surface (Kodaira theory)

$f: X \to Y \qquad \dim X = 2, \ \dim Y = 1$

X_p generic fiber = elliptic curve

good model $\leftrightarrow$ X, Y smooth, no (-1) curve.

good model for a variety "=" smooth

$Z \subset X$, X var. Z = closed subset

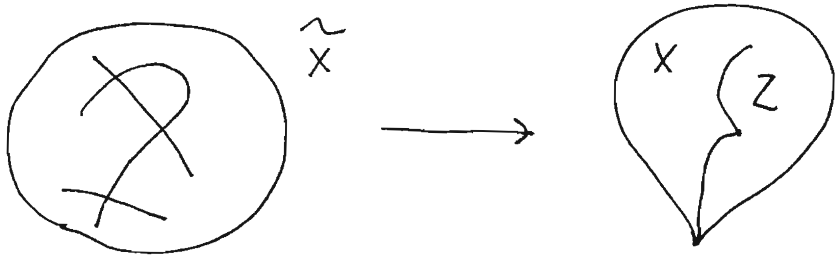
Theorem: $\exists\, \mu : \tilde{X} \to X$ birational morphism

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X \\ \cup & & \cup \ \text{open} \\ \tilde{U} & \xrightarrow{\sim} & U \end{array}$$

$\tilde{X}$ smooth, $\tilde{Z} = \mu^{-1}(Z)$ = normal crossing divisor

$(\tilde{X}, \tilde{Z})$ smooth pair.

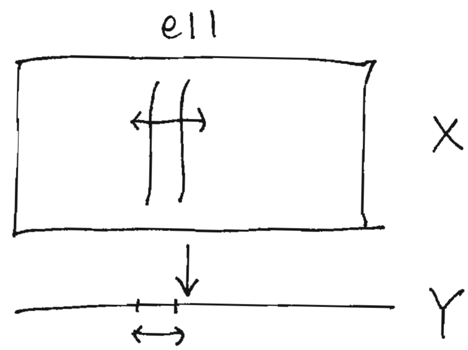
locally $(\tilde{X}, \tilde{Z})$ looks like $\left(\mathbb{C}^n, \text{union of coor. hyperplanes } x_1 x_2 \cdots x_r = 0\right)$.



This is "birational geometry" since

$\mathbb{C}(X) \cong \mathbb{C}(\tilde{X})$.

For elliptic surface



① moduli for general fibers

$X_p \cong \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau$

$J : Y \longrightarrow \mathbb{P}^1 \setminus \{\infty\}$, J function

$\longleftrightarrow$ positivity of curvature (Griffiths)

② Degenerate fibers (singular fibers)

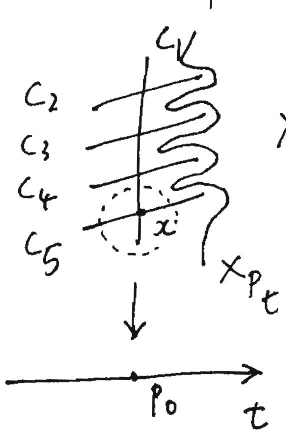
completely classified by Kodaira

(-1)-curves : $C \subset X$, $C \cong \mathbb{P}^1$, $N_{C/X} \cong \mathcal{O}(-1)$

$\rightsquigarrow$ contractible to a smooth point

now we assume no such curves.

One typical case :



$C_i \cong \mathbb{P}^1$

$X_{p_0} = F = 2C_1 + C_2 + C_3 + C_4 + C_5$ — scheme-theoretic fiber

$f^*t = x^2 y$

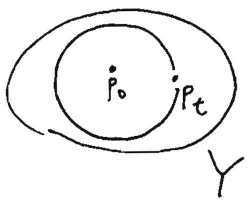
$x = 0 \rightarrow C_1$

$y = 0 \rightarrow C_5$

X_{p_t} = general fiber = smooth elliptic curve.

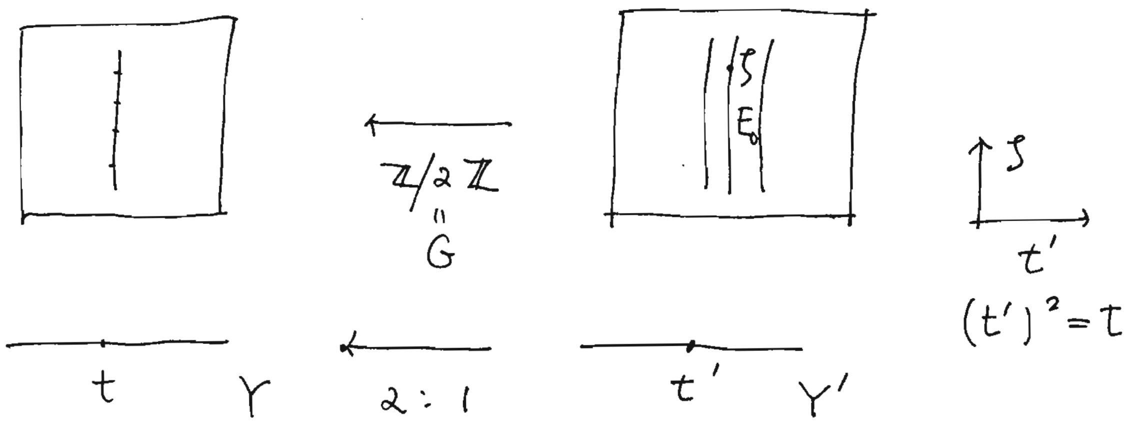
: monodromy

$H^1(X_{p_t}, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$



$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

this can be constructed by



$E_0/G \cong \mathbb{P}^1$, $\mathbb{C}^2 \left\{ \begin{array}{l} \zeta \longrightarrow -\zeta \\ t' \longrightarrow -t \end{array} \right.$

degenerate fibers $\longrightarrow$ $\left\{ \begin{array}{l} \text{monodromy} \\ \text{multiple fiber} \\ \infty\text{-moduli point} \end{array} \right.$

multiple fiber: $F_1 = m F_0$, F_0 elliptic
$m \in \mathbb{N}$, multiplicities

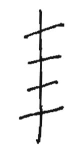
∞-moduli $J(p_0) = \infty$

$F_1 =$  nodal $\mathbb{P}^1$,  chain of $\mathbb{P}^1$'s.

Kodaira's canonical bundle formula :

$$K_X = f^*(K_Y + \text{positive } \mathbb{Q}\text{-divisor } \Theta)$$

$\Theta = \sum a_i P_i$, $f^{-1}(P_i)$ singular fiber , $a_i \in \mathbb{Q}^+$

eg. F_i :   mF_0

$a = 1/2$ $\quad a = \frac{1}{12}$ $\quad a = \frac{m-1}{m}$

K = canonical divisor

= div (differential n-form) $= \sum n_i D_i$

D_i : codim 1 sub. v. $\quad n_i$ order of 0 or ∞ .

Example: $X = \mathbb{C} \xrightarrow[m:1]{\mu} \mathbb{C} = Y$, $t \to t^m = s$

$\mathbb{C} \ni P_0$

$$K_X = \mathrm{div}(\mu^* ds) = \mathrm{div}(m t^{m-1} dt) = (m-1) P_0 + K_Y$$

so $\quad K_X = \mu^* K_Y + (m-1) P_0$.

More Examples :

① Albanese fiber Space

X smooth complete variety

$H^0(X, \Omega^1) = \{\text{hol. 1-forms}\} \neq 0 \quad q$-dim

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "\text{alg. f. space}"'] & & A \\ & Y \arrow[ur, "\text{finite}"'] & \end{tikzcd}$$

$\Rightarrow$ Stein factorization

$\dim A = q$, A abelian v.

② Iitaka fiber space

$H^0(X, mK) = \{ m\text{-ple canonical forms} \} \neq 0$

P_m-dim, basis $\omega_0, \cdots, \omega_{P_m-1}$

$$\begin{array}{ccc} X & \dashrightarrow & \mathbb{P}^{P_m-1} \\ x & \longmapsto & (\omega_0(x) : \omega_1(x) : \cdots : \omega_{P_m-1}(x)) \end{array}$$

$$\begin{tikzcd} X \arrow[r, dashed] & \mathbb{P}^{P_m-1} \\ & X' \arrow[ul, "\text{bi-rat'l}"] \arrow[u, "\text{morphism}"'] \end{tikzcd}$$

rat'l map

morphism : Φ_{mK} m-canonical map.

$\max(\dim \Phi_{mK}(X)) = \kappa(X)$ Kodaira dimension

maximal dim is attained at m

$$\begin{tikzcd} X' \arrow[rr] \arrow[dr, "\text{Iitaka fiber}"'] \arrow[dr, phantom, "\text{Stein fact.}", shift left=4] & & \mathbb{P}^{P_m-1} \\ & Y \arrow[ur] & \end{tikzcd}$$

$$\begin{cases} \kappa(X) = \dim Y \\ \kappa(X_\eta) = 0 \end{cases}$$

Both examples are constructed using differential forms. Now another extreme :

③ Mori fiber space :

assume K_X is not nef :

$\deg K|_C = K.C < 0$ $\exists C$ curve on X

$\Rightarrow$ $\exists$ contraction associated to an extremal ray

$f : X \to Y$. alg. fiber space (including birational morphism)

④ Moduli space & universal family

$$U \longrightarrow G(n, m)$$

universal Grassmannian

$\mathcal{C} \longrightarrow \mathcal{M}$ moduli of stable curves

$\mathcal{U} \longrightarrow \mathrm{Hilb}$ etc.

⑤ Twistor space for HyperKähler manifold

X : cpt Kähler mfd, $2n$-dim

$\mathbb{C} \simeq H^0(X, \Omega^2) \ni \omega$ hol. 2-form

$\wedge^n \omega =$ nowhere vanishing $2n$-form

g : Kähler metric

$(X, g, \omega) \rightsquigarrow \mathcal{X} \xrightarrow{f} \mathbb{P}^1$ family of HK mfds

For previous algebraic examples (alg. fiber space) we always have "positivity". For this analytic case, it is false :

$\mathcal{X} \simeq X \times S^2 \quad (C^\infty)$

f : smooth morphism of cpx manifolds

$f_* \omega_{\mathcal{X}/\mathbb{P}^1} := f_* \mathcal{O}(K_{\mathcal{X}} - f^* K_{\mathbb{P}^1}) = \mathcal{O}(-n), \ n > 0.$

Abramovich - Karn

weak semi-stable model

cf. variety = resolution to smooth model
birational morphism

fiber space alteration
= birat'l morphism + finite covering.

$$\begin{tikzcd}
X \arrow[d, "f"'] & & X' \arrow[ll, "\text{alteration}"'] \arrow[d, "f'"] \arrow[r, phantom, "\underset{\text{birat'l}}{\sim}"] & X \times_Y Y' \\
Y & & Y' \arrow[ll, "\text{alteration}"]
\end{tikzcd}$$

birat'l to fiber product

f' — weak s.s.

case : $\dim Y = 1$:

semi-stable : $\Big\{$ X, Y smooth, all fibers are <u>reduced</u> normal crossing div.

$\operatorname{div}(f^* t) = D_1 + D_2 + \cdots$

$\cup D_i$ NCD. eg. not.

General case :

X, Y smooth, $Y \supset E$ NCD
$X \supset D$

$$\begin{array}{ccc} x & \in & X \\ & & \downarrow f \\ y & \in & Y \end{array}$$

$x \in X$, $y = f(x) \in Y$, local coor.

$x_1 \dots x_n$, $y_1, \cdots, y_m$

$D = \operatorname{div}(x_1 \cdots x_s) = D_1 + \cdots + D_s$ at x

$E = \operatorname{div}(y_1 \cdots y_t) = E_1 + \cdots + E_t$ at y

$f^* E_i = D_{j_{i-1}+1} + \cdots + D_{j_i}$;

locally product of s.s. with $\dim Y = 1$.

This is called semi-stable model which is conjectured to exist, but not proved yet.

Rmk: the necessity to use alteration is already clear even in $\dim = 1$, eg. prev. example.

Weakly semi-stable model :

X : only Gorenstein quotient singularities

$$\begin{array}{l} X \\ \Psi \\ x \end{array} \qquad \begin{array}{ccc} \tilde{X} & \xrightarrow{\pi} & X \\ \text{smooth} & & \| \\ & & \tilde{X}/G \end{array} \qquad \begin{array}{l} (\text{local covering}) \\ G : \text{finite gp} \end{array}$$

$\tilde{X}$: coor. $x_1, \cdots, x_n$

$\cup$

$(\mathbb{C}^\times)^n = \{ x_1 \cdots x_n \neq 0 \}$ mult gp str. $\supset G$

$g \in G$, $g = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix}$; α_i roots of unity

$\tilde{X}, Y$ smooth, $Y \supset E$, $\tilde{X} \supset \tilde{D}$, NCD, $\tilde{f} = f \circ \pi$

$x \in X$, $y \in \tilde{f}(x) \in Y$, local coor. $x_1 \cdots x_n$, $y_1 \cdots y_m$

$$\begin{tikzcd} \tilde{X} \arrow[d] \arrow[dd, bend right, "\tilde{f}"'] & \supset & \tilde{D} \arrow[d] \\ X \arrow[d] & \supset & D \arrow[d] \\ Y & \supset & E \end{tikzcd}$$

$\tilde{D} = \mathrm{div}(x_1 \cdots x_s) = D_1 + \cdots + D_s$ at x

$E = \mathrm{div}(y_1 \cdots y_t) = E_1 + \cdots + E_t$ at y

$\tilde{f}^* E_i = \tilde{D}_{j_{i-1}+1} + \cdots + \tilde{D}_{j_i}$

$E_1 \rightsquigarrow \tilde{D}_1 + \tilde{D}_2$, $E_2 \rightsquigarrow \tilde{D}_3 + \tilde{D}_4$ etc..

Now the condition reads : $f^* E_i = D_{k_{i-1}+1} + \cdots + D_{k_i}$ is a <u>reduced</u> divisor, quotient of NCD.

If $\dim Y = 1$:

s.s. reduction theorem (Mumford et. al)

in Book : Toroidal Embeddings I.

By Hironaka, X, Y smooth, fiber normal crossing but not reduced

$$\begin{array}{c} X \\ \downarrow f \\ Y \ni y \end{array} \qquad f^* y = \sum n_i D_i, \quad n_i \in \mathbb{N}.$$

$l.c.m(n_i) = m$

$(y')^m = y$

$$\begin{array}{ccc} X & \longleftarrow & \widetilde{X \times_Y Y'} \\ \downarrow & & \downarrow \\ Y & \underset{m:1}{\overset{\pi}{\longleftarrow}} & Y' \end{array}$$

$\sim$ – normalization

= weakly s.s. model

$y = \prod x_i^{n_i}$

$y = (y')^m$

Real Blow up:

$$\begin{array}{ccc} X & \longleftarrow & X^{re} \\ f \downarrow \text{w.s.s.} & & \downarrow f^{re} \\ Y & \longleftarrow & Y^{re} \end{array}$$

f^{re} = topologically locally trivial (real analytic map)

eg. $(\mathbb{C}, 0)$ smooth pair $\leftarrow \mathbb{C}^{re}$

$G = e^{is}$

$re^{i\theta} \longleftarrow S^1 \times \mathbb{R}_{\geq 0} \ni (\theta, r)$

different from blow-up of real alg. v.

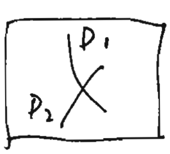
general case is product of this.

$(\mathbb{C}^2, +) = (\mathbb{C}, 0) \times (\mathbb{C}, 0)$

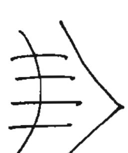
$(\mathbb{C}^2)^{re} = \mathbb{C}^{re} \times \mathbb{C}^{re}$ etc ...

$(X/G)^{re} = X^{re}/G$

$((X, D)/G)^{re} = (X, D)^{re}/G \leftarrow$ free quotient !



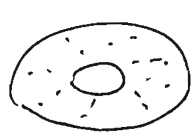
X



mfd with corner $(\mathbb{R}_{\geq 0})^2$

will use this approch to algebraic case to to adjunction theory ... $\square$

$\overset{}{\underset{p}{\times}}$ $Y \leftarrow$



$\overset{r}{\times}$ Y

$r \to 0$

Hodge Theory for algebraic Fiber Spaces 7/23

weakly semi-stable model $f : \mathcal{X} \to S$ = smooth

$p \in \mathcal{X}$, $q = f(p) \in S$

$\exists$ Galois covering $\tilde{U} \xrightarrow[\sigma]{G} U \ni p$

$x_1, \cdots, x_n$ G acts diagonally

$V \ni q$, $y_1, \cdots, y_m$

$\sigma^* f^* y_i = x_{j_{i-1}+1} x_{j_{i-1}+2} \cdots x_{j_i}$

$\rightsquigarrow$ $\begin{cases} f \text{ flat equi-dim'l} \\ X \text{ quotient sing } (\Rightarrow CM) \\ f \text{ has reduced fibers} \end{cases}$

Real Blow-Up:

$$\begin{tikzcd} \mathcal{X}^{re} \arrow[r, "\pi"] \arrow[d, "f^{re}"'] & \mathcal{X} \arrow[d, "f"] \\ S^{re} \arrow[r] & S \end{tikzcd}$$

f^{re} topologically locally trivial family

MONODROMY:

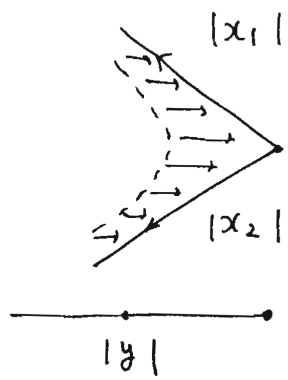
$\pi_1(S^{re}, *) \longrightarrow \mathrm{Aut}(H^p(X_*^{re}, \mathbb{Z}))$ $* \in S^{re}$

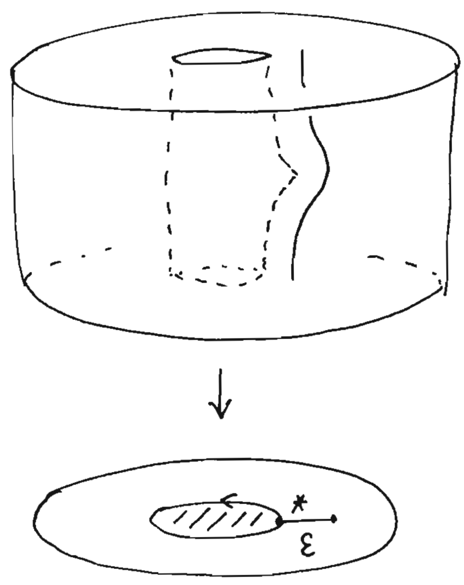
$X_s^{re} = f^{re-1}(s)$, $s \in S^{re}$

Simplest examples:

$y = x_1 x_2$

$|y| = |x_1| \cdot |x_2|$





usually need to take base point with dist $= \varepsilon$

now for real blow up, can take $\varepsilon = 0$. this makes things easier. Will prove:

- locally unipotent monodromy
- globally semi-simple.

$R^p f^{re}_* \mathbb{Z}_{\mathfrak{X}^{re}}$ local system over S^{re}

① Smooth fiber X of $f : \mathfrak{X} \to S$

$$\begin{array}{l} \mathbb{C}_X \xrightarrow[q.i.]{\sim} \Omega^\bullet_X \\ \cup \\ \mathbb{Z}_X \end{array}$$

stupid filtration:

$$\mathcal{O}_X \to \Omega^1_X \to \cdots \to [\, \Omega^p_X \to \Omega^{p+1}_X \to \cdots \,]$$

$$[\, \Omega^p_X \to \Omega^{p+1}_X \to \cdots \,] =: F^p$$

$$0 \to 0 \to 0 \cdots \to \Omega^p_X \to \Omega^{p+1}_X \to \cdots$$

as complexes

$$F^0 = \Omega^\bullet_X \supset F^1 \supset \cdots \supset F^n = \Omega^n_X[-n] \supset 0 \qquad (\text{shifted by } n)$$

filtered complex $\Rightarrow$ spectral sequence

$$E_1^{p,q} = \mathbb{H}^{p+q}(X, Gr^p_F) \Rightarrow H^{p+q}(X, \Omega^\bullet_X)$$

$$Gr^p_F =: F^p/F^{p+1} = \Omega^p_X[-p]$$

$$H^{p+q}(X, \Omega^\bullet_X) = H^{p+q}(X, \mathbb{C})$$

$$E_1^{p,q} = H^{p,q} := H^q(X, \Omega^p_X)$$

Th. degenerate at E_1

$\rightsquigarrow$ Hodge structure on $H^m(X,\mathbb{C}) = \bigoplus_{p+q=m} H^{p,q}$ (arbitrary int.)

Th. $\overline{H^{p,q}} = H^{q,p}$ (this does not follow from deg. at E_1).

② Variation of Hodge Structures

$$\begin{array}{ccc} f: \mathfrak{X} & \longrightarrow & S \\ \cup & & \cup \\ f_0: \mathfrak{X}_0 & \longrightarrow & S_0 \end{array} \quad \text{smooth}$$

relative situation

$\mathfrak{X}_0 = f^{-1}(S_0)$

$$f^{-1}\mathcal{O}_{S_0} \xrightarrow[\text{q.i.}]{\sim} \Omega^{\bullet}_{\mathfrak{X}_0/S_0} = \mathcal{O}_{\mathfrak{X}_0} \to \Omega^1_{\mathfrak{X}_0/S_0} \to \cdots$$

$$E_1^{p,q} = R^q f_{0*}\Omega^p_{\mathfrak{X}_0/S_0} \Rightarrow R^{p+q} f_{0*}\mathbb{C}_{\mathfrak{X}_0} \otimes \mathcal{O}_{S_0}$$

degenerate at E_1

$R^{p+q} f_{0*}\mathbb{C}_{\mathfrak{X}_0} \otimes \mathcal{O}_{S_0} = \mathcal{H}^{p+q}$ locally free

$\mathcal{H}^{p+q} = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \cdots$

holo. subbundles.

$H^m(X^{re}, \mathbb{Z})$: mixed Hodge structure

X smooth projective $\supset D$ NCD

$X_0 = X \setminus D$, $j: X_0 \subset X$

Deligne: $H^m(X_0, \mathbb{Z})$

cohomological mixed Hodge complex:

sheaf of hol. forms with log poles

$$Rj_*\mathbb{C}_{X_0} \xrightarrow[\text{q.i.}]{\sim} j_*\Omega^{\bullet}_{X_0} \xleftarrow[\text{q.i}]{\sim} \Omega^{\bullet}_X(\log D)$$

Though LHS and RHS has no direct connection, but in the derived category, they are equal.

Canonical filtration

$\tau_{\le n}(K^\cdot) \subset K^\cdot$

$$
\begin{array}{ccccccccc}
\cdots \to & K_{n-1} & \to & K_n & \xrightarrow{\alpha} & K_{n+1} & \to & \cdots \\
& \cup & & \cup & & \cup & & \\
\cdots \to & K_{n-1} & \to & \operatorname{Ker}\alpha & \to & 0 & \to & 0 \ \cdots
\end{array}
$$

$\tau_{\le n}(K^\cdot) \subset \tau_{\le n+1}(K^\cdot) \subset \cdots$

then $\qquad Gr^\tau_n(K^\cdot) \cong H^n(K^\cdot)[-n]$

$$R j_* \mathbb{C}_{X_0} \xrightarrow[q.i]{\sim} j_* \Omega^\cdot_{X_0} \xleftarrow[q.i]{\sim} \Omega^\cdot_X(\log D)$$

W = cano.

cano = W
stupid = F

W : weight filtration = cano filt = filtration according to the order of log poles

(can be proved)

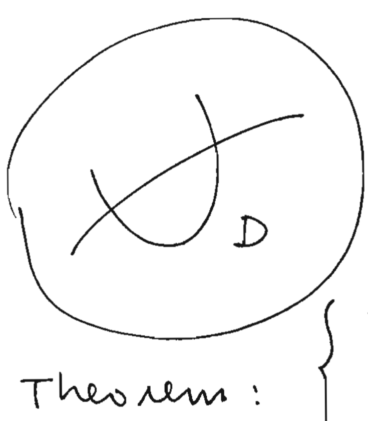
F : Hodge filtration

$$Gr^W_n(\Omega^\cdot_X(\log D)) \xrightarrow[\text{residue}]{\sim} \Omega^\cdot_{X[n]}[-n]$$

disjoint union of intersection of comp of D.

(may apply usual Hodge)

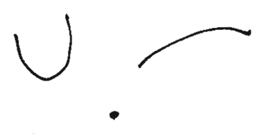
eg. X



X[0] = X

X[1] =

X[2] =



Theorem :
$$\begin{cases} E_1^{p,q} = H^q(X, \Omega^p(\log D)) \underset{\text{deg.}}{\Rightarrow} H^{p+q}(X_0, \mathbb{C}) \\ E_1^{p,q} = H^{p+q}(X, Gr^W_{-p}) \underset{\text{deg. at } E_2}{\Rightarrow} H^{p+q}(X_0, \mathbb{C}) \end{cases}$$

On $H^m(X^{re}, \mathbb{Z})$ mixed Hodge structure

via cohomological mixed Hodge complex

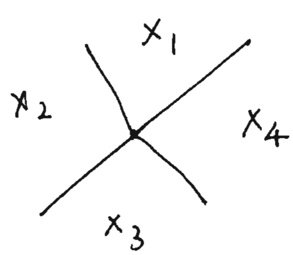
$\mathbb{Z}$ - level: $\quad 0 \to \mathbb{Z}_{X^{re}} \to \mathbb{Z}_{X^{re}[0]} \to \mathbb{Z}_{X^{re}[1]} \to \cdots$

Mayor-Vietoris

$X = f^{-1}(*) = X_1 \cup X_2 \cup \cdots$ irred. comp

generalized normal crossing variety

ex.



$xy = 0$

$zw = 0$

Standard maps

$$X[0] = \coprod X_j \xleftarrow{\text{rel. b-up}} X^{re}[0] \nwarrow$$

$$X[1] = \coprod X_{ij} \text{ of codim } 1 \xleftarrow[\text{+ } S^1 \text{ fiber}]{\text{real}} X^{re}[1] = \coprod (X_j^{re} \cap X_i^{re})$$

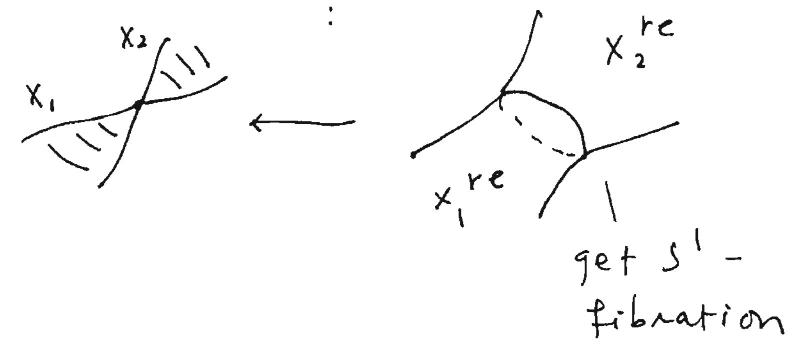
$$\uparrow$$

$$X[2] = \coprod X_{ijk} \text{ codim } 2 \xleftarrow[(S^1)^2\text{- fiber}]{\text{real +}} X^{re}[2]$$

$$\uparrow$$

$$\vdots$$

$$\begin{array}{ccc} X & \xleftarrow{\pi} & X^{re} \\ \uparrow & & \uparrow \\ X[0] & \longleftarrow & X^{re}[0] \end{array}$$



$$W_n = \left(\tau_{\le n} R\pi_* \mathbb{Q}_{X^{re}[0]}\right) \to \tau_{\le (n+1)} R\pi_* \mathbb{Q}_{X^{re}[1]}$$

$$\to \tau_{\le n+2}\left(R\pi_* \mathbb{Q}_{X^{re}[2]}\right) \to \cdots$$

$$Gr_n^W = R^n \pi_* \mathbb{Q}_{X^{re}[0]}[-n] \oplus R^{n+1} \mathbb{Q}_{X^{re}[1]}[-n-2] \oplus$$

$$R^{n+2} \pi_* \mathbb{Q}_{X^{re}[2]}[-n-4] \oplus \cdots$$

$\mathbb{C}$-level:

$$R\pi_* \mathbb{C}_{X^{re}} \xrightarrow{\sim} R\pi_* M \xleftarrow{\sim} \Omega^\cdot_X(\log)$$

real str. $x = re^{i\theta}$

$[\,d\theta,\ \frac{dr}{r},\ \log r\,]$ can also be defined alg'ly.

$\Omega^1_{\mathcal{X}}(\log)$ $\mathcal{X} = U = \tilde{U}/G$, take $\Omega^1_U(\log) = \Omega^1_{\tilde{U}}(\log)^G$

$\Omega^1_S(\log)$ - the usual

is locally free because the action is diag'l. and $\frac{dx}{x}$ is inv.

$\Omega^1_{\mathcal{X}/S}(\log) = \Omega^1_{\mathcal{X}}(\log)/f^*\Omega^1_S(\log)$ loc. free.

$\Omega^1_X(\log) = \Omega^1_{\mathcal{X}/S}(\log) \otimes \mathcal{O}_X$

get

$$\Omega^\cdot_X(\log) \longrightarrow \Omega^\cdot_X(\log) \otimes \mathcal{O}_{X[0]} \longrightarrow \Omega^\cdot_X(\log) \otimes \mathcal{O}_{X[1]} \to \cdots$$

Another Mayer-Vietoris ($\Rightarrow$ weight filt)

Th. ① $E_1^{p,q} = H^q(X, \Omega^p(\log)) \Rightarrow H^{p+q}(X^{re}, \mathbb{C})$

degenerate at E_1.

② $E_1^{p,q} = H^{p+q}(X, Gr^W_{-p}(R\pi_* \mathbb{Q}_{X^{re}})) \Rightarrow H^{p+q}(X^{re}, \mathbb{Q})$

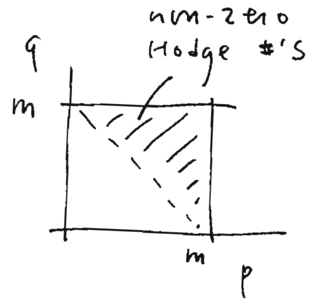
degenerate at E_2.

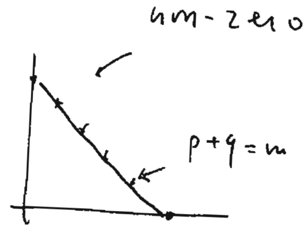
Hodge numbers:

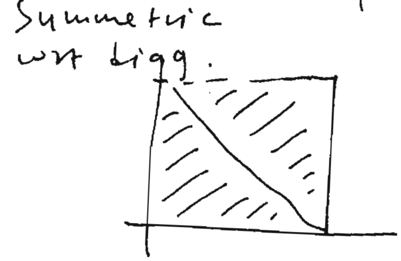
$$Gr^W_n H^m(X_0, \mathbb{C}) = \bigoplus_{p+q=m+n} H^{p,q}, \quad \overline{H^{p,q}} = H^{q,p}$$

ie. $Gr^p_F(Gr^W_n H^m(Y, \mathbb{C})) = H^{p,q}$, $p+q = n+m$

in general.







$X_0 = X \setminus D$ compact X singular cpt X^{re}

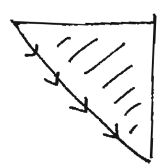
$\tau_{\le n} \to \tau_{\le n+1} \to \cdots$

Cor. $R^q f_* \Omega^p_{\mathcal{X}/S}(\log)$ is locally free over S

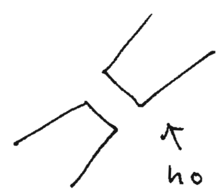
(this follows from de Rham + loc. trivial in part (1) of Th.)

Cor. Monodromy is unipotent (Even easier cor.)

$Gr^W_n (R\pi_* \mathbb{Z}_{X^{re}})$ $\circlearrowleft$



bec.



Non-degenerate Hermitian Sym. Bilinear form h:

$f_* \omega_{\mathcal{X}/S} = f_* \Omega^n_{\mathcal{X}/S}(\log)$ $n = \dim X$

$(R^q f_* \omega_{\mathcal{X}/S})$ $\omega_{\mathcal{X}/S} = \mathcal{O}_{\mathcal{X}}(K_{\mathcal{X}/S}) = \mathcal{O}(K_{\mathcal{X}} - f^* K_S)$

E holo. v.b. on cpx. mfd S $(= S_0)$

$h : E \to \bar{E}^*$ (C^∞-isom.) $h(u, \bar{v}) \in \mathbb{C}$, $u \in E$, $\bar{v} \in \bar{E}$.

$E = R^n f_* \mathbb{C}_X \otimes \mathcal{O}_S \big|_{S_0}$

$h(u, v) = (u \wedge \bar{v})[X_s]$, $u, v \in H^n(X_s, \mathbb{C})$

this is non-definite !

D connection $= h^{-1}\partial h + \bar\partial : E \to \Gamma^1_\infty(E)$

$\Theta = D \circ D$ curvature : $(1,1)$ form with value in $\mathrm{Hom}(E,E)$

$$\varphi = \frac{\sqrt{-1}}{2\pi} \operatorname{Trace} \Theta = \partial\bar\partial(-\log\det h)$$

For $0 \to F \xrightarrow{i} E \xrightarrow{p} G \to 0$ exact

$s\ (C^\infty)$

second fund. form $b = p D i \in \Gamma^1_{C^\infty}(\mathrm{Hom}(F, G))$

Lemma: h non-deg on E, $h|_F$ non-deg $\Rightarrow$

$$h_F(\Theta_F(u), v) = h_E(\Theta_E(u), v) - h_G(b(u), b(v)).$$

pf is by direct computation.

Theorem: (Griffiths)

$$f_* \omega_{X/S}\big|_{S^\circ} \subset R^n f_* \mathbb{C}_X \otimes \mathcal{O}_S\big|_{S^\circ}$$

$\Rightarrow \Theta_F$ is semi-positive (pos. semi-definite)

pf: ① infinitesimal period relation:

$$D(F^p) \subset \Gamma^1_{C^\infty}(F^{p-1})$$

$$F^0 = E \supset F^1 \supset F^2 \supset \cdots \supset F^n = F \supset 0$$

F^p $\quad dz_1 \wedge \cdots \wedge dz_p \wedge d\bar z_{p+1} \wedge \cdots$

(p, q), $p+q = n$

$dz_1 \rightsquigarrow dz_1 + \varepsilon\, d\bar z_\alpha + \cdots$

$d\bar z_{p+1} \rightsquigarrow d\bar z_{p+1} + \varepsilon\, dz_\beta + \cdots$

wedge + the fact "$\varepsilon^2 = 0$".

② Riemann-Hodge bilinear relation

$\dim X = 2n$

$$\mathrm{Ker}\left(H^{n-q}(X,\mathbb{C}) \xrightarrow[\text{Lefschetz}]{\Lambda^q[L]} H^{n+q}(X,\mathbb{C}) \xrightarrow{[L]} H^{p+q+2}(X,\mathbb{C}) \right)$$

$=: H_0^{n-q}(X,\mathbb{C})$ primitive

$$F^n(H^n(X,\mathbb{C})) \subset H_0^n(X,\mathbb{C})$$
$$\shortparallel$$
$$H^0(X,\Omega^n)$$

$$(-1)^{\frac{n(n-1)}{2}+q} (\sqrt{-1})^n h(H_0^{p,q}, H_0^{p,q}) \gg 0 .$$

important part : $(p,q) \to (p+1, q-1)$ change sign

$F^n \longleftrightarrow (F^{n-1} \cap H_0)/F^n$ sign-change.

Now Griffiths' pf follows easily.

$F_1 = F^n$

E flat $\Rightarrow \Theta_E = 0$

and h_G, h_F. diff sign. $\square$

Th: $f_* \omega_{X/S}$ is a numerically semi-positive vector bundle.

<u>To be continued</u>

For Lecture 3: Adjunction theory

Lecture 4: Fano Manifolds.

Prof. Yujiro Kawamata

Lecture III. Adjunction Theory

First we need to finish Lect. II :

Def: X proj. v. E : locally free sheaf
$\mathbb{P}(E)$ proj. bundle over X : $\pi : \mathbb{P}(E) \to X$
$\mathcal{O}(1)$ tautological quotient line bundle of $\pi^* E$.
E is "numerically semi-positive"
$\Leftrightarrow$ $\mathcal{O}(1)$ nef. i.e. $\forall C \subset \mathbb{P}(E)$, $(\mathcal{O}(1), C) \geq 0$.

Th. $f : X \to Y$ alg. fiber sp. X, Y smooth
$Y_0 \subset Y$, $Y - Y_0 = E$ normal crossing divisor
$f_0 = f|_{f^{-1}(Y_0)}$ smooth
$X_0 = f^{-1}(Y_0)$, $n = \dim X - \dim Y$.
Assume : local monodromies of $R^{n+q} f_* \mathbb{Z}_{X_0}$
around E are unipotent
$\Rightarrow$ $R^q f_* \omega_{X/Y}$ is num. semi-positive.

Rmk: This is birational invariant for X
but not for Y.

$$\begin{tikzcd} X' \arrow[dr, "\mu"] \arrow[dd, "f'"'] & \\ & X \arrow[dl, "f"] \\ Y & \end{tikzcd} \qquad R^q f'_* \omega_{X'/Y} \cong R^q f_* \omega_{X/Y}.$$

in fact X has Gorenstein quotient sing.
is OK. just like in smooth case. and
f weakly s.s. $\Rightarrow$ X Gorenstein canonical.

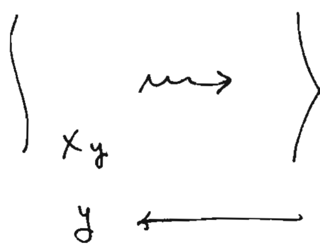
$\mathcal{F}_0 = R^g f_{0*}\omega_{X_0/Y_0}$ has a pos. definite metric with semi-positive curvature, $\mathcal{F}_0 = \mathcal{F}|_{Y_0}$

$\Rightarrow$ $\mathcal{O}(1)|_{\pi^{-1}(Y_0)}$ has a metric with s.p. curvature.
(singular hermitian metric)

Def : Singular hermitian metric h : L line bundle on a variety Y, locally on Y, $h = e^{-\varphi} h_0$
h_0 : C^∞-metric, φ : weight function $\in L^1$.

general fiber of $\mathcal{F}$ = hol. n-forms α

$$h(\alpha, \alpha) = * \int_{X_y} \alpha \wedge \bar{\alpha}$$



$\rightsquigarrow$ $\leftarrow$ log pole at intersections

$$\int \frac{dx}{x} \wedge \frac{d\bar{x}}{\bar{x}} = \infty .$$

h grows at the order of $|\log y|^m$,
$\varphi \sim \log \log |y|$.

$\Rightarrow$ curvature form $\Theta = \partial\bar\partial(-\log h)$

As a distribution $\Theta = \partial\bar\partial(-\log h) + \partial\bar\partial\varphi$

and 1st chern form : $c_1 = \frac{\sqrt{-1}}{2\pi}\Theta$

there is no boundary contribution to Θ :

$$\Theta = \underset{\text{absolute conti.}}{\Theta_{ab.c}} + \underset{0}{\Theta_{sing}}$$

In general, non-unip monodromy $\Rightarrow$ $\Theta_{sing} \neq 0$.

since $\Theta_{ab.c}$ is s.p. Θ is s.p. !

Remark: There is an algebraic proof due to Kollár via vanishing theorem. which is easier but more tricky. (after the existence of the analytic pf).

Adjunction Theory:

$\mathbb{Q}$-divisors: $D = \sum_{\text{finite}} d_j D_j$, $d_j \in \mathbb{Q}$, D_j prime div

nef $\Leftrightarrow$ $(D.C) \geq 0$ $\forall$ curve

A ample $\mathbb{Q}$-div $\Leftrightarrow$ $\exists$ $m \gg 0$ st. mA very ample

nef divisor is a limit of ample divisors:

$$\left.\begin{array}{l} A \text{ ample} \\ D \text{ nef} \end{array}\right\} \Rightarrow A + D \text{ ample}$$

(eg. Nakai criterion)

$\varepsilon A + D$ ample $\varepsilon > 0$, $\lim_{\varepsilon \to 0} (\varepsilon A + D) = D$.

Sometimes it will be important to consider also $\mathbb{R}$-divisor $\sum d_j D_j$, $d_j \in \mathbb{R}$.

Remark: semi-ample
$\Rightarrow$ metric semi-positive $\Rightarrow$ nef
other directions are false

semi-ample $\overset{\text{def}}{\Longleftrightarrow}$ $\exists$ $m \gg 0$, $|mD|$ free.

Correction: In Kodaira's formula

$$K_S \sim_{\mathbb{Q}} f^*(K_C + \Theta)$$

K can. div is defined only up to linear equiv. $\sim$

$\omega = \mathcal{O}(K)$, $A \sim_{\mathbb{Q}} B \Leftrightarrow mA \sim mB$, $\exists$ $m > 0$.

In this way, sheaf theoretic expression is stronger than div. notation.

Theory of singularities :

Surface case :

Zariski decomposition (in his book)
(really the starting point of all MMT)

X : sm. proj. surface, D ef. div. $\sum d_j D_j$, $d_j \in \mathbb{N}$

$\exists$ $D = P + N$ as $\mathbb{Q}$-divisors, uniquely st.

P : nef $\mathbb{Q}$-div, N ef. $\mathbb{Q}$-div $= \sum n_j N_j$ st.

① $P N_j = 0$ $\forall j$

② $[(N_j N_k)]_{j,k}$ negative definite.

the pf is a simple linear algebra. But its truth will implies many conjectures. eg. MMP. (in higher dim)

Connection with minimal models :

$$\begin{array}{ll} X & \text{sm. proj. surf. } \kappa \geq 0 \\ f \downarrow & \text{contraction of } (-1) \text{ curves} \\ X_{min} & \text{minimal model} \end{array}$$

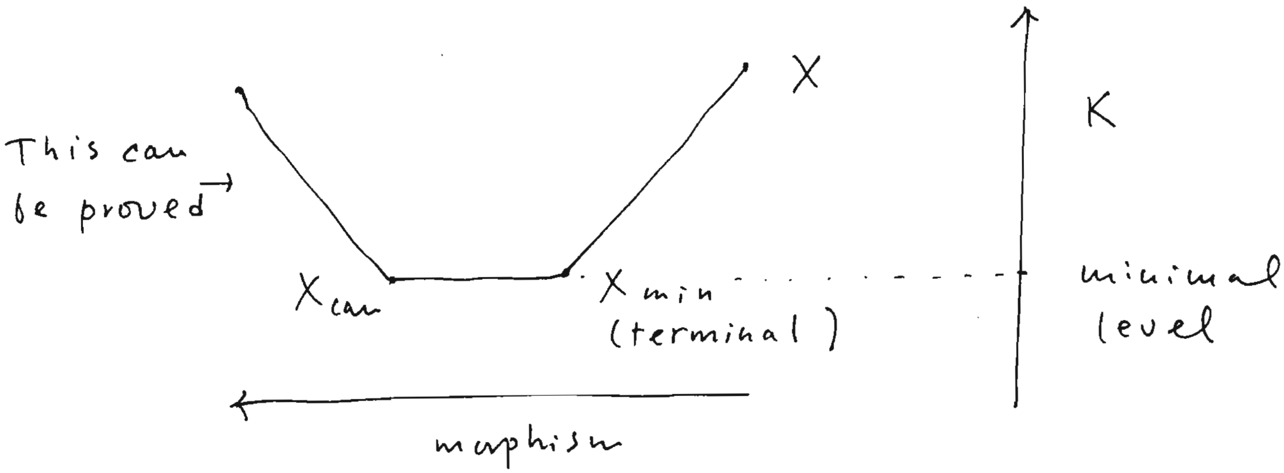
$K_X = f^* K_{X_{min}} + N$ is the Z-decomp.

$\kappa = 2$: $C \subset X_{min}$ $(K_{X_{min}} \cdot C) = 0$ $\Leftrightarrow$ C (-2) curve

$$\begin{array}{ll} X_{min} & \text{(Artin's rational singularities)} \\ g \downarrow & \text{contraction of } (-2) \text{ curves} \\ X_{can} & \text{canonical model (Mumford)} \end{array}$$

$\rightsquigarrow$ can. ring is finitely, generated.

$K_{X_{can}}$ ample, $g^* K_{X_{can}} = K_{X_{min}}$



This picture is also true for higher dimensions!

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X_1 \arrow[rr, dashed] & & X_2 \end{tikzcd}$$

bi-rat'l

may compare:

$$\alpha^* K_{X_1} \gtrless \beta^* K_{X_2}$$

X min $\Longleftrightarrow$ K_X min

(broader sense: categorical sense in terms of morphisms)

X min $\Leftrightarrow$ minimal terminal (narrow sense)

Question: For a fixed birat'l ($K \geq 0$) class, there <u>exists</u> only a <u>finite</u> number (up to isom.) of minimal models?

For dim 3, general type finiteness is proved but for C-Y, it is still unknown.

Adjunction Formula :

X sm. variety , D sm. divisor

$$K_X + D \big|_D = K_D$$

log n-forms $\xrightarrow[\text{residue}]{}$ $(n-1)$ forms

log Pair : (X, B) X normal v. B ef. $\mathbb{Q}$-div

X sm. proj. surface , B n.c.d. (coef 1)

$(X \setminus B)$ open surface $\leftarrow$ Deligne's approach .

Kawamata's approach is really look at the pair :

if $m(K+B)$ effective $\exists\, m > 0$,

$$K_X + B = P + N = f^*(K_{X_{et}} + B_{et}) + N$$

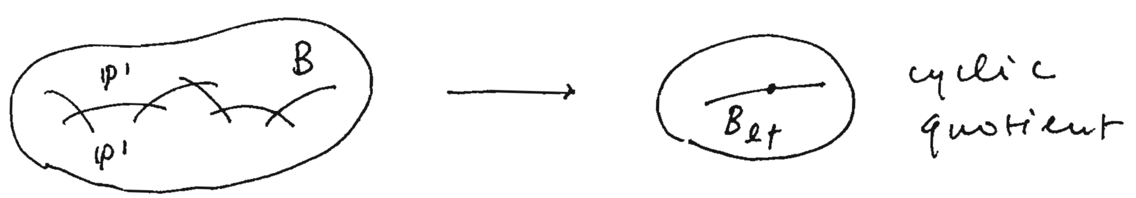
Supp N is contracted , $X \xrightarrow{f} X_{et}$ Grauert contraction Artin (rational singularities) , X_{et} projective .

$B \longrightarrow B_{et} = f_* B$ reduced .

(X_{et}, B_{et}) has singularities classified by

$\rightsquigarrow$ quotient sing. + standard divisor

ex.



Def : (X, B) X normal , B ef. $\mathbb{Q}$-div

① $K_X + B$ $\mathbb{Q}$ Cartier. ie. $\exists\, m > 0$, $m(K_X + B)$ is Cartier div (pull back can be defined) .

② $\mu : Y \longrightarrow X$ resol. of sing.

$$\begin{array}{l} Y \\ \cup \\ E \quad \text{NCD} \quad (\text{log resol}) \\ \| \\ \sum E_j \end{array} \quad \begin{cases} Y - E \cong X - \mu(E) \\ \mu^{-1}(\operatorname{supp} B) \subset E \end{cases}$$

$$\frac{1}{m}\mu^*(m(K_X + B)) = \mu^*(K_X + B) = K_Y + \sum e_j E_j$$

log-canonical : $e_j \leq 1$

klt (kawamata) : $e_j < 1$

dlt (divisorial) : $e_j < 1$ $\forall E_j$ exceptional, $e_j \leq 1$

plt (purely) : dlt and E_j with $e_j = 1$ are disjoint.

Rmk: In 2-dim case Mumford defines the intersection theory in all cases, hence condi ① is not needed. but for higher dim, M. Reid impose it. So that one may pull back etc.. and define int. theory.

Def. X normal

① K_X $\mathbb{Q}$-Cartier

② $\mu : Y \to X$ resol. $\mu^* K_X = K_Y + E$, $E = \sum e_j E_j$

X terminal $\Leftrightarrow$ $e_j < 0$ (sing on ter. min. model)

X cano $\Leftrightarrow$ $e_j \leq 0$ (sing on min model in broder sense)

$$\text{ter} \Rightarrow \text{can} \underset{\substack{\uparrow \\ B = 0}}{\Rightarrow} \text{klt} \Rightarrow \text{plt} \Rightarrow \text{dlt} \Rightarrow \text{lc}$$

Thm: (X, B) dlt $\Rightarrow$ X has only rational sing. hence Cohen-Macaulay.

Rmk: lc may not be CM. eg. cone over abelian surf.

Th. X normal proj. (X, B) klt

L Cartier div. st. $L-(K_X+B)$ nef & big

$\Rightarrow H^p(X, L) = 0 \quad \forall p > 0$.

Also, klt $\Leftrightarrow$ L^2 analytically.

$$(X, B), \quad m(K+B) \longleftrightarrow L \quad \text{line bundle}$$

$$\begin{array}{ccc} \psi & & \psi \\ \varphi & & s \end{array}$$

(rational m-ple n-form)

$$\varphi = \text{—}\,(dx_1 \wedge \dots \wedge dx_n)^{\otimes m} \quad \text{on } X_{reg}$$

$\left|\int_{\text{locally}} \sqrt[m]{\varphi \wedge \bar{\varphi}}\right| < \infty$ easily from definition by calculating in the resolution.

Rmk: the finiteness is false for dlt and also the above vanishing thm is wrong.

(X, B), $\mu : Y \to X$ log resol. $B = \sum b_j B_j$, $0 < b_j < 1$

$$\mu^*(K_X + B) = K_Y + E, \quad E = \sum e_j E_j.$$

consider the sheaf: <u>multiplier ideal sheaf</u>.

$$I = \mu_* \mathcal{O}(\lceil -E_+ \rceil)$$

Ex. (X, B) klt $\Rightarrow \lceil -E_+ \rceil \geq 0 \Rightarrow I = \mathcal{O}$

In general, under assumption on B, get $I \subset \mathcal{O}_X$

With I, would make things L^2:

$I \otimes \mathcal{O}(L)$ sheaf of L^2 sections

Th: X normal proj. (X, B) pair

L Cartier, $L-(K+B)$ big & nef.

$\Rightarrow H^p(X, I \otimes \mathcal{O}(L)) = 0 \quad \forall p > 0$.

Def: (X, B) l.c. (non klt), call
E_j place of lc sing $\Leftrightarrow$ $e_j = 1$
$\mu(E_j)$ center of lc sing.

Prop:
$$\begin{array}{ccc} Y & \xrightarrow{\mu} & X \\ \cup & & \cup \\ E_j & \xrightarrow[\mu_0]{} & \mu(E_j) = W_j \end{array}$$
for E_j a place of l.c. sing. If W_j is minimal center

$\Rightarrow$ μ_0 alg. fiber space, W_j normal.

Prop: W_1, W_2 centers of lc. Then W an irred. comp of $W_1 \cap W_2$ $\Rightarrow$ W center of lc.
so has the notion of minimal center.

Rmk: scheme structure could be defined by $\mathcal{O}/I$.
so lc. $\Rightarrow$ $\mathrm{Spec}(\mathcal{O}/I)$ is reduced.

Th. (X, B^0) klt, X projective, $B > B^0$,
(X, B) lc. $x \in X$, W min center for (X, B) through x. H ample, $\varepsilon > 0$

$$\Rightarrow K + B + \varepsilon H|_W \sim_{\mathbb{Q}} K_W + \underline{\overset{\exists}{B_W}} \quad \text{klt}.$$

This is the Main Result (very non-trivial).
which generalizes " $K + D|_D = K_D$ " via residue.

This appears in many intermediate steps in induction
eg. $\Rightarrow$ rational $\Rightarrow$ boundedness of mult. etc.

<u>To be continued.</u>

Fano Manifolds

Defined by Iskovskih : X proj. smooth, $-K_X$ ample

and by him : classification of Fano 3-folds

ex). 1) quartic 3-fold : $\mathrm{Bir}(X) = \mathrm{Aut}(X)$

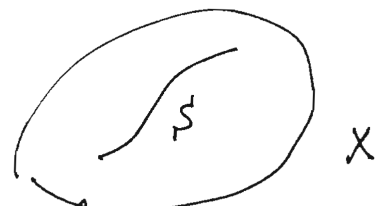
2) $\mathbb{P}^3$: $\mathrm{Bir}(\mathbb{P}^2)$ very large $\gg \mathrm{Aut}(X)$

existence of a good member in $|-K|$,

$\exists S \in |-K|$, $S = K3$ surface

and use knowledge of S to

classify X.



Mukai : use $C \in |(-K)|_S|$: canonical curve

ladder : $C \subset S \subset X$ (Fujita's idea)

Kawamata's study of good member

— study of linear system

Def : (X, B) log Fano variety

(1) X projective

(2) (X, B) klt

(3) $-(K+B)$ ample

This is the "correct" category for study.

Recall : Base Point Free Theorem :

X proj, (X, B) klt, D Cartier div on X

assume ① D nef ② $D-(K+B)$ nef & big (ample)

$\Rightarrow \exists m_0 > 0$ st. $|mD|$ free $\forall m \geq m_0$.

(X, B), R: extremal ray

$\Rightarrow$ D satisfies condi (1), (2)

$(D.C) = 0 \Leftrightarrow [C] \in R$

$\Rightarrow$ $X \xrightarrow[\Phi_{|mD|}]{\varphi} Y$ contraction of R (alg. fiber sp.)

$\varphi(C) = pt \Leftrightarrow (D.C) = 0$

And ($K+B$ not nef $\Rightarrow \exists R$).

$$\text{variety} \xrightarrow[\log \text{MMP}]{\text{MMP}} \begin{cases} \text{minimal model:} \\ \quad \text{ie. } K+B \text{ nef}, \text{ or} \\ \text{Mori fiber space:} \\ \quad \text{ie. } f: X \to Y \text{ contr. of } R \\ \quad \dim X > \dim Y \end{cases}$$

for $\eta \in Y$, (X_η, B_η) is log Fano.

For minimal model:

$X \xrightarrow[m(K+B)]{} Y$: "biregular" Iitaka fiber space

this is the abundance conjecture:

$\exists\, m > 0$ st. $|m(K+B)|$ free.

For original BPF Thm. The problem now is:

effective bound m_0, or a "weaker - stronger":

Problem: (X, B) klt, D nef, $D-(K+B)$ ample

$? \Rightarrow |D| \neq \emptyset$?

Remark: It is equiv. to assume just big & nef

(exercise).

Fano index := r (largest) st.

$$-(K+B) \sim_{\mathbb{Q}} rH, \quad H \text{ Cartier ample}, \ r \in \mathbb{Q}$$

Truth of Prob $\Rightarrow |H| \neq \phi$, which is very strong!
(Kawamata wish a counterexample for the prob.)

Rmk: $r \leq n+1$ ($n = \dim X$):

$P(t) = \chi(X, tH)$ poly of order n, $t \in \mathbb{Z}$

$H^p(X, tH) = 0 \quad \begin{cases} p > 0 \\ t > -r \end{cases}$

$tH - (K+B) = (t+r)H$

$H^0(X, tH) = 0, \quad t < 0$, hence

$\chi(X, tH) = 0$ for $t = -1, -2, \cdots, -(n+1) \Rightarrow$

$-(n+1) > -r \Rightarrow$ contradiction, so $r \leq n+1$

and equality $\Rightarrow X \cong \mathbb{P}^n$, $B = 0$.

$\left\{ \begin{array}{l} \text{Non-Vanishing } |D| \neq \phi \leftarrow \text{Riemann-Roch formula} \\ \text{shape of general element } S \in |D| \leftarrow \text{adjunction theorem} \end{array} \right.$

Th: (Ambro; ~~Horie~~): (X, B) log-Fano

$-(K+B) \sim_{\mathbb{Q}} rH$, $r > n-3 \Rightarrow H^0(X, H) \neq 0$.

Th: (Kawamata): $\dim X = 2$, (X, B) klt, D nef Cartier,

$D - (K+B)$ ample $\Rightarrow h^0(X, D) \neq 0$.

Th: (—): (X, B), D as in Prob.

Prob is true if D ample $\Rightarrow$ Prob. true in general.

explaination : (reduction)

$\phi : X \xrightarrow{|mD|} Y$ alg. fiber space

D $m \gg 0$, E Cartier ample, $D = \phi^* E$ (BPF thm)

$H^0(D) \neq 0 \Leftrightarrow H^0(E) \neq 0$

$\exists B'$ on Y, (Y, B') klt
$E - (K_Y + B')$ ample
$\rightarrow$ semi.posi thm.

Cor : $\nu(X, D) = \kappa(X, D) \leq 2 \Rightarrow$ Problem (too).

So it is very difficult to find counter examples
(can one find in toric varieties ?)

Proof of Thm 1 (Ambro) :

$$p(t) = \chi(tH) \qquad d = H^n > 0, \; b = H^{n-1}B \geq 0$$

$$= \frac{t^n H^n}{n!} - \frac{t^{n-1} H^{n-1} K}{2(n-1)!} + \cdots + 1$$

$$= \frac{d}{n!} t^n + \frac{rd + b}{2(n-1)!} t^{n-1} + \cdots + 1$$

$$= \frac{d}{n!}(t+1)\cdots(t+n-3)\left(t^3 + At^2 + Bt + \frac{n(n-1)(n-2)}{d}\right)$$

$$A = \frac{n(n-r+s)}{2} - 3 + \frac{bn}{2d}$$

$$0 \leq (-1)^n p(-n+2) = -1 + \frac{d}{n(n-1)}\left((n-2)^2 - A(n-2) + B\right)$$

$$\| $$

$$(-1)^n H^n(X, (-n+2)H) \quad \text{(call it } P_{n-2})$$

then $P(1) = n-1 + P_{n-2} + \dfrac{d(n-r+3)+b}{2} > 0$! $\square$

Rmk : this is first done by Alexeev for the case $r > n-4$.
(already very clever)

In Prob. $D-(K+B)$ ample

$\Rightarrow H^p(X, D) = 0 \ \forall p > 0$. so $h^0(X, D) = \chi(X, D)$

which is numerical! That's one reason why

the prob. may be possible.

Th. $f: X \to C$ = curve, alg. fiber space

(X, B) klt, D cartier on X, $D \sim_{\mathbb{Q}} K_{X/C} + B$

$\Rightarrow f_* \mathcal{O}(D)$ s.p. (log version, via covering tech.)

Proof of thm 2 (Kawamata):

may assume X smooth (min. resol.)

R.R.. $\chi(D) = \frac{1}{2} D(B+H) + \chi(\mathcal{O}_X)$

$\chi(\mathcal{O}_X) \geq 0 \Rightarrow \begin{cases} D \equiv 0 \rightsquigarrow \chi(\mathcal{O}) = 1 \\ D(B+H) > 0; \text{ ok.} \end{cases}$

$\chi(\mathcal{O}_X) = 1 - g < 0$, $X \xrightarrow{f} C$, $g(C) = g$

generically $\mathbb{P}^1$-bundle

$f_* \mathcal{O}(D - f^* K_C)$ s.p.

(condi. satisfied since:

$(D - f^* K_C) \sim_{\mathbb{Q}} K_{X/C} + B + H$, $H \sim_{\mathbb{Q}} B$ small.)

D f-nef $\Rightarrow$ D f-generated. ie.

$f^* f_* \mathcal{O}(D) \twoheadrightarrow \mathcal{O}(D)$

$\underbrace{f^* f_* \mathcal{O}(D - f^* K_C)}_{\text{s.p.}} \twoheadrightarrow \underbrace{\mathcal{O}(D - f^* K_C)}_{\text{s.p.}}$

$(D - f^*K_C)(B+H) \geq 0$

$D(B+H) = 2g-2$, $\deg K_C = 2g-2$

$\Rightarrow$ contradiction. $\square$.

General member of linear system:

Thm (Ambro): (X,B) log-Fano, $r > n-3$,

$Y \in |H|$ general member (since know $\neq \emptyset$),

$\Rightarrow (X, B+Y)$ plt.

ie. $\mu : Z \to X$:

$\mu^*(K_X + B + Y) = K_Z + \mu_*^{-1} B + \mu_*^{-1} Y + \sum_j e_j E_j$

$\forall e_j < 1$

$K_X + B + Y|_Y = K_Y + B_Y$, $B_Y = B|_Y$

since Y Cartier div.

$RHS|_{\mu_*^{-1}Y} = K_{\mu_*^{-1}Y} + \underbrace{\boxed{}|_{\mu_*^{-1}Y}}_{\text{coeff} < 1}$

ie. (Y, B_Y) klt.

$-(K_Y + B_Y) \sim_{\mathbb{Q}} (r-1)H|_Y$, hence $\exists$ ladder,

may proceed ...

Theorem (Kawamata): $\dim X = 4$

X canonical Gorenstein, $-K \sim D$ Cartier ample

$\Rightarrow$ ① $h^0(X, D) \neq 0$

② $Y \in |D|$ gen. $\Rightarrow (X, Y)$ plt ($\rightsquigarrow Y$ can. Gorenstein)

③ $h^0(D|_Y) \neq 0$

④ $Z \in |D|_Y| \Rightarrow (Y, Z)$ plt

⑤ X smooth $\Rightarrow Y$ smooth.

Proof by adjunction theorem:

suppose (X, Y) not plt,

$0 < c = \max\{ t \mid (X, B+tY) \text{ l.c.} \} \leq 1$

lc threshold.

$(X, B+cY)$ properly lc (lc & not klt)

W minimal center

$$K_X + B + cY + \varepsilon H \big|_W \sim_{\mathbb{Q}} K_W + B_W \quad \text{klt}$$

$$\wr_{\mathbb{Q}}$$

$$-rH + cH + \varepsilon H$$

$\dim W \geq 3$, $n \geq 4$, $r > 1$

$r - c - \varepsilon > 0 \quad \therefore (W, B_W)$ is log Fano

so $H^0(W, H_W) \neq 0$ by nonvanishing result proved before (Ambro, Kawamata).

get $*$: because $W \subset \mathrm{Bs}\,|H|$ (Bertini's thm)

$$H^0(X, H) \to H^0(W, H) \to H^1(X, I_W \otimes \mathcal{O}(H)) = 0$$

by perturbation, multiplier $= I_W$

(W is only center)

So $*$. $\square$.

Finally let's prove the adjunction theorem:

Th. $f: X \to Y$ alg. f. space, X, Y smooth

$\quad\ \ \cup \quad\ \ \cup$

$f_0 : X_0 \to Y_0 \quad Y - Y_0 \ NC = E, \quad X_0 = f^{-1}(Y_0)$

$f^{-1}(E) \cup \mathrm{Supp}\, B$ NC. (X, B) sub klt

(coeff < 1) B may be non-ef.

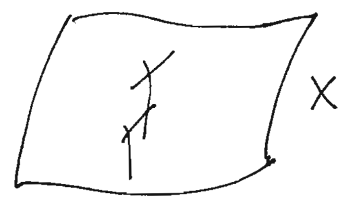
$\mathcal{O}_Y \to f_* \mathcal{O}_X(\lceil -B \rceil)$ surjective over Y_0

$K + B \sim_{\mathbb{Q}} f^*(K_Y + L)$, L: $\mathbb{Q}$-Cartier

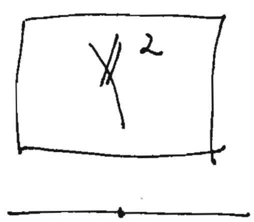
$B' = \sum_i e_j E_j$, e_j min st. $(X, B + f^*(E - B'))$ lc

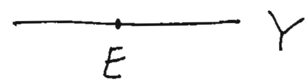
$\Rightarrow$ $L - B'$ nef. (call $M = L - B'$, and $\Delta = B'$ later)

ex. f semi-stable $\Rightarrow$ $B' = 0$.
(& $B = 0$)



But for:





then $e = 1/2$.

So B' is something like how much need to do to the semi stable reduction case.

Proof: Semi-stable case: log version of s.p. thm.
general case: by covering

Th. Y proj. sm. E NCD $= \sum_{j=1}^{N} E_j$

$m_j \in \mathbb{N}$ $(j = 1 \dots N)$, then

$\exists$ $\pi : X \to Y$ finite Galois, X smooth

$\pi^* E_j = m_j \overline{E_j}$, $\overline{E_j}$ reduced and $\sum \overline{E_j}$ NC.

ie. locally via covering is the semi-stable case with additional term B'. (correction term)

$B' \leftrightarrow$ eigenvalues of monodromy ! (rk 1 case)

for V.b. case still don't know the formulation.

Proof of adjunction thm:

$$\begin{array}{ccc} E & \subset & Y \\ {\scriptstyle f}\downarrow & & \\ V & & \downarrow{\scriptstyle \mu} \\ {\scriptstyle \sigma}\downarrow & & \\ W & \subset & X \end{array}$$

E : place of l.c. sing above W.

σ : resol. of W.

f : alg. fiber space.

$$\mu^*(K+B) \sim_{\mathbb{Q}} K_Y + E + F \quad ; \quad \text{eff } F < 1$$

$$\mu^*(K+B)|_E \sim_{\mathbb{Q}} K_E + F|_E \qquad \text{ample}$$

$$\sigma^*(K + B + \varepsilon H|_W) \sim_{\mathbb{Q}} K_V + (M + \varepsilon\sigma^* H - \varepsilon' Q) + \Delta + \varepsilon' Q$$

this formula follows from positivity result.

Q : σ-exceptional eff, $0 < \varepsilon' \ll \varepsilon$

so $K + B + \varepsilon H|_W$ is klt. $\square$.

Remark: If codim $W = 2$, then ε is not necessary.
reason: good moduli theory for S curves with $g = 0$ with marked pts. In this case $f : E \to V$ is good and in fact M is semi-ample.

Finally, Fujita's conjecture (this is the source of all these l.c. center business).

Fujita Conjecture: X sm. proj. dim = n. H ample

$$\Rightarrow \quad K_X + mH \quad \text{free} : m \geq n+1$$
$$\text{very ample} : m \geq n+2.$$

still open!

Idea of proving this:

L ample, $L^n > n^n$ (eg. $L = (n+1)H$)

fix $x \in X$, R.R. $\exists$ $D \in |NL|$ | $\mathrm{mult}_x D > N \cdot n$

$\mathrm{mult} \underset{\overset{\parallel}{B}}{\dfrac{D}{\text{"}N(1+\alpha)\text{"}}} =: n$, (X, B) not klt at x.

l.c. threshold $c \leq 1$:

(X, cB) properly l.c at X.

W minimal center.

Suppose x <u>isolated</u> pt of W

$\Rightarrow H^0(X, K+L) \twoheadrightarrow \underset{\overset{\parallel}{\mathbb{C}}}{H^0(X, K+L|_W)} \longrightarrow \underset{\overset{\parallel}{0}}{H^1(I_W \otimes \mathcal{O}(K+L))}$

$\Rightarrow |K+L|$ free at x.

In this approach, it is necessary to extend Fujita's conj. to singular varieties:

Correct Version:

(X, B) klt, $x \in X$

$Y \xrightarrow{\mu} X$, $\mu^*(K+B) = K_Y + \sum e_j E_j$

minimal log discrepancy

$\sigma := \min \left\{ 1 - e_j \mid \mu(E_j) = \{x\} \text{ for all resolution } \mu \right\} > 0$

ideal condition:

$\forall\, W \ni x$, sub. v. thr x of $\dim = d$

$L^d W > \sigma^d \overset{?}{\Rightarrow}$ free at x. and "maybe"

$L^d W > (\sigma+1)^d \overset{?}{\Rightarrow}$ very ample at x.

Ask for proof in

- surface case ?
- Toric case ?

Partial result on surface :

Kawach - Masek : $L^2 > \mathrm{mult}(X, x)\,\sigma^2 \rightsquigarrow$ R.R.