Y. Kawamata lect 1 . at NTU. P. 1

Hacon McKernan :

- Boundedness of pluricanonical maps
- On the existence of flips

vanishing $\rightsquigarrow$ extension

$\searrow$ { base pt free thm
cone thm

? $\searrow$ termination conjecture

vanishing, also resolution of sing. $\rightarrow$ require char $= 0$

variety $\rightarrow$ $\boxed{\text{MMP}}$ $\rightarrow$ (minimal model
Mori fiber space

can be non-closed. e.g. $\mathbb{Q}$.

contraction of extremal ray

{ divisorial
small
mfs
$<$ conj I : existence
conj II : termination

log variety (X, B), B $\mathbb{R}$-divisor $= \sum b_i B_i$, $\overset{\uparrow}{\mathbb{R}}$ under b_i

- adjunction $K_X + Y|_Y = K_Y$
  key idea: induction on dim.
- deformation of coefficients

$>$ so log theory is better

$f : (X, B) \to S$ proj, B $\mathbb{R}$-div, $K_X + B$ $\mathbb{R}$-Cartier

(X,B) $\underbrace{d}_{*}\underbrace{lt}_{**}$, $\exists\ Y \xrightarrow{\mu} X$ log resolution

$$\mu^*(K_X + B) = K_Y + C, \quad C = \sum c_j C_j$$

(1) $c_j \le 1$, $\underbrace{c_j < 1}_{**}$ if C_j exceptional

(2) $\mathrm{EXC}(\mu)$ divisor

lc $c_j \le 1$ $\forall j$ (largest)

klt $c_j < 1$ $\forall j$ the cat. everything OK. but is open

hence has trouble when take limits !

D $\mathbb{R}$-Cartier on X

f-nef $\quad (D.C) \geq 0 \quad \forall C \quad f(C) = pt$

f-big $\quad D \geq \exists A$ ample

f-ample $\quad D = \sum a_i A_i$, A_i ample, $a_i > 0$.

f-semi-ample

$$\begin{tikzcd} \exists\ X \arrow[d, "f"'] \arrow[dr, "g"] & \\ S & X', A' \arrow[l, "f'"] \end{tikzcd}$$

A' : f'-ample
$D = g^* A'$

Cone Thm : H f-ample, $\varepsilon > 0$
$\exists R_i$ finitely many extremal rays st.
$\overline{NE}(X/S) = \overline{NE}(X/S)_{K+B+\varepsilon H \geq 0} + \sum R_i$ . .
$\exists$ C_i rational curves $R_i = \mathbb{R}_+[C_i]$.
$0 > (K+B).C_i \geq -2n$, $n =$ max dim fiber .

This follows from the $\mathbb{Q}$-div case:

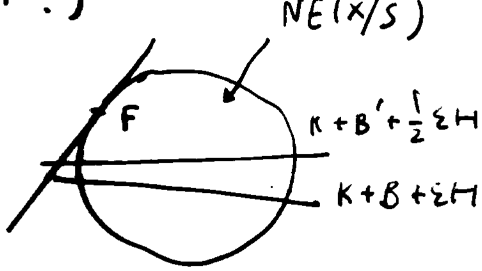
∵) take $\mathbb{Q}$-div B' close to B , (X, B') dlt

$B = \sum b_i B_i$, $\underset{\in \mathbb{Q}}{b'_i} \geq b_i$, $B_i + \underset{\text{eff}}{B'_i} \geq 0$

$b''_i \geq b'_i - b_i$, $B' = \sum (b'_i B_i + b''_i B'_i)$

(if $c_j = 1$ then don't change it .)

$K + B' + \frac{1}{2}\varepsilon H$ cone thm $\Rightarrow$ ok .



Base Point Free Thm :
L $\mathbb{R}$-Cartier div, f-nef ,
$L - (K+B)$ f-ample $\Rightarrow$ L - f s.a.

(∵) $L^\perp \cap \overline{NE}(X/S) = F$ face fig.

$V^\perp = \{ \ell \in N^1(X/S), \ \ell|_F = 0 \}$

$P = V^\perp \cap \mathrm{Nef}(X/S)$. $\quad L \in \mathrm{Int}\, P$.

By BPF thm for $\mathbb{Q}$-div, int. pt in Int P are s.a.
$\Rightarrow$ OK.

Deformation of coefficients of B
influence on MMP process

Thm. Assume MMP (existence and term)

$(X,B) \to S$ klt

K_X+B f-big, $B=\sum_{i=1}^{t} b_i B_i$

$t' \le t \Rightarrow \exists\, \varepsilon>0$, $\exists$ partition $I=\prod_{i=1}^{t'}[b_i-\varepsilon, b_i+\varepsilon] = \coprod_{j=1}^{k} I_j$

$\exists$ sequence $\alpha_j : X=X_{j,0} \dashrightarrow X_{j,1} \dashrightarrow \cdots \dashrightarrow X_{j,t_j}=Y_j/S$

s.t. $\forall C=\sum c_i B_i \in I_j$, α_j is MMP for (X,C)

$\bar{I_j}$ finitely gen. $K_{Y_j}+C_{Y_j}$ nef$/S$

(not known whether I_j is open or closed.)

(Some weak version is in Hacon-McKernan)

$\because$) MMP for (X,B), also MMP for (X,C) near B

$K+B$ not nef $\to$ extr. ray $\to \cdots$

So $K+B$ nef (may assume)

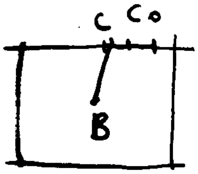
$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "g"] & \\ S & Z \arrow[l, "h"] \end{tikzcd}$$

g birat'l

$K+B=g^*H$, H : h-ample

use induction on dim.

$\partial I = J \ni c_0$ $J_{c_0} = \coprod J_{c_0,k}$



on this polytope has simultaneous MMP$/Z$.

$tc+(1-t)b$, $0<t\le 1$

$b \equiv 0/Z$, so has same MMP over this.

∂I cpt $\Rightarrow$ finite decomposition.

but we need to be over S.

" A g-ample, B h-ample $\Rightarrow$ $A+mB$ f-ample for $m \gg 0$ "

But not for nef div.

p.4 But for MMP is OK: since then nef div is always pull back of ample div.

$$\begin{tikzcd} X \arrow[r, "g_1"] \arrow[dr, "g"] \arrow[dd, "f"'] & X_1 \arrow[d, "h_1"] \\ & Z \arrow[dl, "h"] \\ S & \end{tikzcd}$$

$K+C = g_1^* H_1$, $K+B = g^* H$.

$m h_1^* H + H_1$ $h \circ h_1$-ample.

Reduction Step: Shokurov

(more readable account Fujino http://www.math.nagoya-u.ac.jp/~fujino/index.html)

Thm: (Special Termination)

Assume MMP for dim $< n$, then have sp.term:

S.T. n-dim flips

$$\begin{tikzcd} (X_1,B_1) \arrow[dr, "\phi_1"] & & (X_2,B_2) \arrow[dl, "\phi_2"'] \arrow[dr] & & \cdots \arrow[dl] \\ & Z_1 & & Z_2 & \end{tikzcd}$$

$\Rightarrow \exists\, m_0$ st $m \geq m_0$, $\mathrm{Exc}\, \phi_m \cap \lfloor B_m \rfloor = \phi$. i.e. become klt.

Notice that for termination dlt is a natural cat. so this thm reduces back to klt.

Thm: Existence of pl flip in dim n, sp. termination in dim n $\Big\} \Rightarrow$ existence in dim n.

pl: dlt + $\lfloor B \rfloor = S$ irreducible & $-S$ ϕ ample.

$K_X + B|_S = K_S + \Delta$. very special type of dlt.

Hacon-McKernan: Existence of pl flip in n. assuming MMP$<n$, Remaining: termination for klt.

Termination in dim 3.

http://faculty.ms.u-tokyo.ac.jp/~kawamata/index.html.

KMM, term. dim 3.

Th. (X,B) div contr.

$$\begin{tikzcd} (X,B) \arrow[d] \\ (X',B') \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, dashed, "\text{flip}"] \arrow[dr] & (X',B') \arrow[d] \\ & Z \end{tikzcd}$$

$E = \mathrm{Exc}\,\phi$, common log-resol.

$$\begin{tikzcd} Y \arrow[r, "\mu"] \arrow[dr, "\mu'"'] & X \\ & X' \end{tikzcd}$$

$\mu^*(K_X+B) = K_Y + C$, $C = \sum c_i C_i$

$\mu'^*(K_{X'}+B') = K_Y + C'$, $C_0' = \sum c_i' C_i$

$\Rightarrow$ $c_i \geq c_i'$, and $c_i > c_i' \Leftrightarrow \mu(C_i) \subset \mathrm{Exc}(\phi)$.

This is the key to termination. the problem is that c_i' may $\in \mathbb{R}$, in dim 3 this will not happen.

X 3 dim'l terminal sing. $(X, 0)$ has coeff < 0

$\exists r$, rK_X Cartier, r : index

(We always assume X $\mathbb{Q}$-factorial in the lecture series)

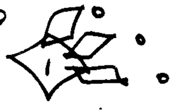
(1) rK_X Cartier $\Rightarrow$ rD Cartier $\forall D$ (this is easier)

(2) r index $\Rightarrow$ $\exists E$, exc. div coeff $= -\frac{1}{r}$ (this is hard. need Mori's classification of ter. sing.)

Thm: In dim 3, there is no infinite sequence of klt flips.

pf: May assume (X, B) klt,

$e = \#\{\text{exc. div. coeff} \geq 0\} < \infty$ ← this is finite only for klt. since



any blow-up.

- terminal case $e = 0$.

in the ∞ sequence may assume $e = $ const.

- induction on e.

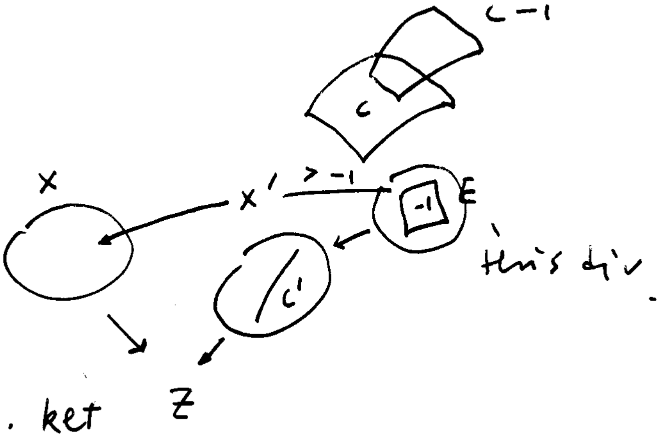
difficulty $= \#\{\text{exc} \mid \text{coeff} > -1\} < \infty$ if $e = 0$ decrease.

(X, B) term $\Rightarrow$ $(X, 0)$ term

coeff E $>$ coeff E

$\geq -\frac{1}{r}$

This gives a pf when $B = 0$.



general case (Shokurov):

(X, B) pair with given e, $\mathbb{Q}$-fact. klt

$\Rightarrow$ $\exists$ $X' \xrightarrow{f} X$ birat'l proj, $e(X', B') = e - 1$.

$f^*(K_X+B) = K_{X'} + B'$, $\mathrm{exc}(f) =$ prime div E, $B' = f_*^{-1}B + cE$.

This called the "divisorial extraction".

(2) $\forall \varepsilon > 0$, $\exists q > 0$,
$e(X,B) = e$, coeff $\cap (-\varepsilon, 0) = \phi$, D prime div on X
$\Rightarrow$ qD Cartier. eg. $e=0$, $\varepsilon = \frac{1}{r}$, $q = r!$

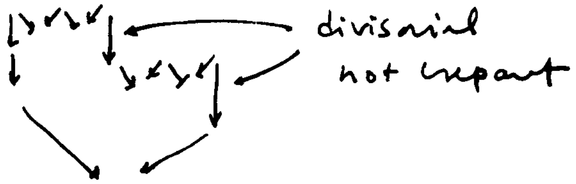
(This 2 statement can be found in the homepage).

Assume a sequence of flips:

$$\begin{tikzcd}
(Y_1,C_1) \arrow[dr] \arrow[dd, dashed, "\text{terminal-ization}"', "e=1"] & & Y_1' \arrow[dl] \arrow[dr] & & Y_1^2 \arrow[dl] \arrow[dr] & & (Y_2,C_2) \arrow[dl] \arrow[rr, bend left] \arrow[ddll, "\text{divisorial contr.}"] & & (y_2,c_2') \arrow[ddllll, bend left, "\text{crepant}"] \\
& W_1 & & W_2 & & W_3 & & & \\
(X_1,B) \arrow[drr] & & & & (X_2,B_2) \arrow[dll] \arrow[drr] & & & & \\
& & Z_1 & & & & Z_2 & &
\end{tikzcd}$$

$K_{Y_2} + C_2$ neg
$C_2 \geq C_2'$.

(for $e=2$ case then will look like



)

Need to show that there is no infinite (sequence) decreasing of coeffi in C.

$l \subset E \subset X'$ extraction of E, $\mu(l) = $ pt.
rat'l curve $\mu \downarrow$ coeff of E becomes stable
(X,B)

$B = \sum b_i B_i$,
$\mu_*^{-1} B_i = B_i'$

E is birat'l ruled surface.

$(K_{X'} + \sum b_i B_i' + bE).l = 0$
〃 Q: finite possibility?
$\mu^*(K_X + B)$

$0 \geq (K_Y + E).l \geq -3$, $(E.l) < 0$, $1-b \geq \varepsilon$ fixed $(= 1 - b_1)$.

by (2), $(K_Y.l), (B_i'.l), (E.l) \in \frac{1}{q}\mathbb{Z}$.

$0 > (1-b)(E.l) = (K_{X'} + E + \sum b_i E_i').l \geq -3$.

$\Rightarrow$ finiteness.

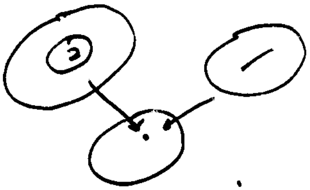
$$b = \frac{(k_X' + \Sigma b_i B_i').\ell}{-(E.\ell)} \quad ; \quad B_i.\ell \geq 0 .$$

notice that b_i are just $\in \mathbb{R}$. so although we get finiteness, but not know exactly the possibility.

Rmk: the statement (1), (2) in dim 4 is NOT TRUE.

① pl-f → difficulty ⇝ , in dim 4 may require :

② dim of cycles :



more invariants.

⇝ termination in dim 4 in terminal case. (only).

canonical done by Fujino & Matsuki.

2nd lect is on extension thm.

↑

vanishing thm.

(1) $(X,B) \xrightarrow{f} S$ dlt, f proj, L Cartier,

$L - (K_X + B)$ f-ample $\Rightarrow R^p f_* \mathcal{O}_X(L) = 0,\ p > 0$.

open & stable under restr.

(2) klt, f-nef & f-big $\Rightarrow$ "

open   not open   in fact (1) ⟺ (2).

some openness is always important.

→ multiplier ideal sheaf :

$$\underset{}{\mathcal{F}} = f_* \mathcal{O}_X(L) = \underbrace{I}_{\text{ideal}} \otimes \underbrace{\mathcal{G}}_{\text{invertible}}$$

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "g:\text{birat'l}"] & \\ S & Y \arrow[l, "h"] \end{tikzcd}$$

$\Rightarrow \begin{cases} R^p g_* \mathcal{O}(L) = 0 & p > 0 \\ R^p h_* \mathcal{F} = 0 & p > 0 \end{cases}$ ← Key point !

log pluri-canonical forms

Tool: vanishing for multiplier ideals
since everything birat'l, may assume X smooth

$$\begin{tikzcd} V \arrow[r, "f"] & S \\ X \arrow[u, phantom, "\cup"] & \end{tikzcd}$$ (affine)

A E
means ample + eff
as $\mathbb{Q}$-div

Theorem. (non-log) (1) $K_V + X$ big wrt (V, X)
$\Rightarrow H^0(m(K_V+X)) \twoheadrightarrow H^0(mK_X)$.

(2) $K_V + X$ p-eff wrt (V, X), H ample
$\Rightarrow H^0(m(K_V+X)+H) \twoheadrightarrow H^0(mK_X+H_X)$.

means $H^0(kE) \to H^0(kE_X)$ non-zero
$k>0$.
(ample as usual)

Example: Francia Flip

$$\begin{tikzcd} X \subset V^3 \arrow[dr, "\phi"'] & & V^+ \arrow[dl, "\phi+"] \\ & Z \subset W & \end{tikzcd}$$

$p \in X$, $p = \frac{1}{2}(1,1)$

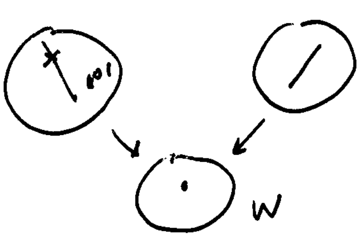
$V \ni p = \frac{1}{2}(1,1,1)$

V^+ Smooth.

$Q \in W$ is very singular,

$Q \in Z$, $Z \subset W$ hyp section

but $Q = \frac{1}{3}(1,1)$ in Z (simpler).



$$\begin{array}{ccc} H^0(3k(K_V+X)) & \longrightarrow & H^0(3kK_X) \quad \forall k>0 \\ \wr\wr & & \wr\wr \\ H^0(3k(\underline{K_W+Z})) & \longrightarrow & H^0(3kK_Z) \end{array}$$

invertible

not $\mathbb{Q}$-Cartier

not surj

So log-term sing leads to wrong thm.
Thm still valid (extended) to the canonical case.

Applications (1) inv. of pluri-genera for variety of general type (Siu). (This is now extended to general case, non-general type, so far no alg. pf. He uses analytic multiplier ideal sh. & pos. metric,

nef < pos. metric < semi-ample.)

(2) deformation of canonical singularities.

$V \to V' \supset X'$ need to ext forms on V' to V

cano $\xleftarrow{?}$ cano first rest to X', then ext to X

(resl. then to V).

log resl.

Also for terminal singularities.

log-version: simple generalisation will leads to inv of log-ter. sing. But this is WRONG.

vanishing theorem: dlt + ample

$(X, B) \xrightarrow{f} S$, L Cartier, $L-(K_X+B)$ ample,

(X,B) dlt $\Rightarrow R^p f_* L = 0$, $p>0$.

Prop: $f: X \to S$ proj, $\mathcal{F}$ coh on X, H v.a. $n = \dim X$,

"$R^p f_* (\mathcal{F}(mH)) = 0$ $\forall p>0$, $m \gg 0$." ← assumption

$\Rightarrow \mathcal{F}(\underline{n}H)$ generated by global sections!

proof: induction on n: $\underset{\text{closed pt}}{x \in X}$, $\mathcal{F}_x = \mathcal{H}^0_{\{x\}}(\mathcal{F})$

$0 \to \mathcal{F}_x \to \mathcal{F} \to \bar{\mathcal{F}} \to 0$ in general no such things!

$\bar{\mathcal{F}}$ generated $H^0(\mathcal{F}) \to H^0(\bar{\mathcal{F}}) \to \underset{0}{\overset{\shortparallel}{H^1(\mathcal{F}_x)}}$

$\Rightarrow \mathcal{F}$ gen.

May assume $\mathcal{F}_x = 0$, $X' \in |H|$ general through x

$0 \to \mathcal{F} \xrightarrow{\Delta} \mathcal{F}(H) \to \mathcal{F}' \to 0$

'supported on X''

Notice the assumption $\Rightarrow$ also holds for $\mathcal{F}'$.

$\mathcal{F}'((n-1)H)$ gen $\Rightarrow \mathcal{F}(nH)$ gen. #

(This kind of procedure does not produce anything but it turns out the prop is useful.)

Thm:

$$\left\{\begin{array}{l} f: \underset{\overset{\cup}{X}}{(V, B)} \to S \text{ proj}, \quad B = X + B' \text{ NC support} \\ \qquad\qquad\qquad\qquad 0 < \text{coeff of } B \le 1 \text{ in } \mathbb{Q} \\ K_V + B|_X = K_X + \Delta, \quad k(K_V+B) \text{ Cartier}. \end{array}\right.$$

p.10 ① $K_V + B$ $\mathbb{Q}$-eff wrt $(V, \ulcorner B \urcorner)$

② B' big wrt (V, X)

$\Rightarrow H^0(m(K_V + B)) \twoheadrightarrow H^0(m(K_X + \Delta))$ if $k | m$.

Can we extend in the way : (today not yet proved)

①' $K_V + B$ big wrt $(V, \ulcorner B \urcorner)$? instead of ① + ②.

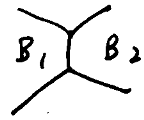
– one condition only, instead of 2.

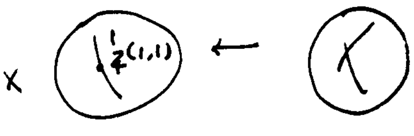
$\ulcorner B \urcorner$ round up. $B = \sum b_i B_i$, $\ulcorner B \urcorner = \sum B_i$.

D : $\mathbb{Q}$-eff wrt. $(V, \ulcorner B \urcorner)$.

Z any intersection of B_i's

$H^0(mD) \longrightarrow H^0(mD|_Z)$ non-zero





The pf is very similar, but need to be careful.

$\tilde{D}$ Cartier on V, $\mathbb{Q}$-eff wrt $(V, \ulcorner B \urcorner)$, $D = \tilde{D}|_X$

mD eff wrt. $(V, \ulcorner B \urcorner)$

log resol $\mu : \overset{}{\underset{\cup Y}{W}} \longrightarrow \underset{\cup X}{V}$, $\mu^*(m\tilde{D}) = \underset{\text{free}}{Q_m} + \underset{\text{fixed}}{N_m}$

m is fixed.

Y = Strict transform of X.

~~$I = \mu_* \mathcal{O}_W(K_W$~~

$\mu_Y^*(K_X + \Delta) = K_Y + \Gamma$, $\Gamma^+ = \max(\ulcorner \Gamma \urcorner, 0)$.

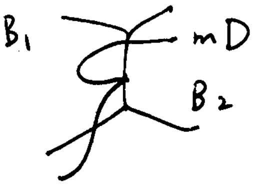
$\mu_Y^*(mD) = P_m + M_m$, $M_m \leq N_m|_Y$

$$I_{\ulcorner \Delta \urcorner, \frac{1}{m} M_m} = \mu_{Y*} \mathcal{O}_Y\left(K_Y + \Gamma^+ - \mu_Y^*(K_X + \ulcorner \Delta \urcorner) - \left[\frac{M_m}{m}\right]\right),$$

$$I_{\ulcorner \Delta \urcorner, \frac{1}{m} N_m|_Y} = \mu_{Y*} \mathcal{O}_Y\left(K_Y + \Gamma^+ - \mu_Y^*(K_X + \ulcorner \Delta \urcorner) - \left[\frac{N_m|_Y}{m}\right]\right).$$

Important remark: can take μ st.

μ is isom at generic pt of Z (any such Z)



$\Rightarrow \Gamma^+$ = strict transform of $\ulcorner \Delta \urcorner$

no exceptional components.

notice that can't allow to blow up Z

(leads to counterexample, so need mD eff...)

Prop: $D = D_1 + D_2$, D_1 ample, D_2 $\mathbb{Q}$-ef wrt $(X, \lceil \Delta \rceil)$,

$$\mu^* D_1 \le P, \quad \mu^* D = \underset{\text{free}}{P} + \underset{\text{fixed}}{M}$$

$$I_{\lceil\Delta\rceil, M} := \mu_* \mathcal{O}_Y(K_Y + \Gamma^+ - \mu^*(K_X + \Delta) - [M])$$

$$\Rightarrow H^p(I_{\lceil\Delta\rceil, M}(K_X + \lceil\Delta\rceil + D)) = 0. \quad p > 0$$

(log extension of vanishing thm for multi ideal sh.)

(∵) E exceptional for μ, ef.

$\mu^* D_1 - E$ ample,

$$\mu^* D = ((1-\varepsilon)P + \underset{\substack{\uparrow \\ \text{ample}}}{\varepsilon}(\mu^* D_1 - E)) + (M + \varepsilon(P - \underset{\substack{\uparrow \\ \text{ef and small}}}{\mu^* D_1} + E))$$

$[\cdots] = [M]$.

$$H^p(\mu_* \mathcal{O}_Y(K_Y + \Gamma^+ + \underset{\text{ample.}}{\lceil P' \rceil})) = 0 \quad \forall p > 0$$

This also implies higher direct image $= 0$ ✲

$$J^0_{\lceil\Delta\rceil, D} = \bigcup_{m>0} I_{\lceil\Delta\rceil, \frac{1}{m} M_m}$$

$$J^1_{\lceil\Delta\rceil, D} = \bigcup_{m>0} I_{\lceil\Delta\rceil, \frac{1}{m} N_m|_Y}$$

$$J^0 \supset J^1 \supset J^1_{\Delta'} \supset J^1_{\lceil\Delta\rceil}, \quad \Delta' \le \lceil\Delta\rceil.$$

Lemma: 1) $J^i_{\lceil\Delta\rceil, D}(D + A_X + H_X)$ gen $\begin{cases} A = (\dim X + 1)H \\ H \text{ v.a.} \end{cases}$

2) $J^i_{\lceil\Delta\rceil, \alpha D} \subset J^i_{\lceil\Delta\rceil, D}$, $\alpha > 1$.

3) L free $\Rightarrow$ $J^i_{\lceil\Delta\rceil, D} \subset J^i_{\lceil\Delta\rceil, D+L}$

4) $\mathrm{Im}(H^0(V, \tilde{D}) \to H^0(V, D)) \subset H^0(X, J^1_{\lceil\Delta\rceil, D}(D))$. This is just def.

($\mu^* \tilde{D} - N_1|_Y \le \mu^* \tilde{D} - \frac{1}{m} N_m|_Y$

$$\le \underbrace{K_Y + \Gamma^+ - \mu^*(K_X + \lceil\Delta\rceil)}_{\text{ef by the important remark}} + \mu^* D - \left[\frac{M_m}{m}\right])$$

Now the main idea:

5) D big wrt $(V, \lceil B \rceil)$ $\Rightarrow$ $H^0(J'_{\lceil\Delta\rceil, D}(D + K_X + \lceil\Delta\rceil))$

$$\underset{\text{reverse inclusion}}{\hookleftarrow} \mathrm{Im}(H^0(\tilde{D} + K_V + \lceil B \rceil)) \to H^0(D + K_X + \lceil\Delta\rceil)).$$

P.12 recall $\mu^*(K_V+B) = K_W + C$

$$0 \to \mathcal{O}_W(\lceil \tfrac{1}{m} Q_m \rceil + K_W + C^+ - Y) \quad \leftarrow H^1 = 0$$

$$\to \mathcal{O}_W(\lceil \tfrac{1}{m} Q_m \rceil \underset{\substack{\uparrow \\ \text{call it } \mathcal{F}}}{+} K_W + C^+) \to \mathcal{O}_Y(\cdots|_Y) \to 0$$

$$H^0(\mathcal{F}) \twoheadrightarrow H^0(\mathcal{F}_Y) = H^0(J^1_{\lceil\Delta\rceil, D}(D + K_X + \lceil\Delta\rceil)) \text{ done.}$$

$$\overset{\cap}{H^0(\hat{D} + K_V + \lceil B \rceil)}$$

The Next Step is to combine these statements into useful statement :

if H is taken suitably. indep of m.

Main Lemma : $J^0_{[\frac{m}{k}]k(K_X+\Delta)+H_X} \subset J^1_{\lceil\Delta\rceil, [m(K_X+\Delta)]+H_X+A_X}$

assume only ①,

pf : induction on m, $m=0$ done, both $=$ str. sh.

assume m,

$$H^0(J^0_{[\frac{m}{k}]k(K_X+\Delta)+H}([(m+1)(K+\Delta)] + H + A))$$

gen by ①

Rmk : in the non-log case H is any v.a. but now is only for some.

$$\Big([(m+1)(K+\Delta)] - \underset{\substack{\uparrow \\ \text{this is periodic in } m.}}{[\tfrac{m}{k}]} k(K+\Delta) - K + H \text{ ample} \Big)$$

$$\hookrightarrow H^0(J^1_{\lceil\Delta\rceil, [m(K+\Delta)]+H+A}([(m+1)(K+\Delta)] + H + A))$$

$$[(m+1)(K+\Delta)] - [m(K+\Delta)] = K + \textcircled{H}$$

$$0 \le \textcircled{H} \le \lceil\Delta\rceil, \quad J^1_{\lceil\Delta\rceil}\ldots \subset J^1_{\textcircled{H}}\ldots$$

$$\underset{⑤}{\hookrightarrow} \mathrm{Im}\big(H^0([(m+1)(K_V+B)] + H + A) \to H^0((m+1)(K_X+\Delta) + H_X + A_X)\big)$$

$$\underset{④}{\hookrightarrow} H^0(J^1_{\lceil\Delta\rceil, [(m+1)(K+\Delta)]+A+H}([(m+1)(K+\Delta)] + H + A))$$

since the sh. is gen . So $H^0 \hookrightarrow H^0 \Rightarrow$ sh $\hookrightarrow$ sh. *

Cor : $H^0(X, m(K+\Delta))$ $\quad k|m \quad$ $A' = A + K_V$ ~~$+ X$~~ .

$$\subset H^0(X, m(K+\Delta) + H)$$

$$\subset \mathrm{Im}\big(H^0(V, m(K_V+B) + H + A') \to H^0(X, m(K_X+\Delta) + H_X + A'_X)\big).$$

(∵) $H^0(\cdots) = H^0(J^0_{m(K+\Delta)+H}(m(K+\Delta)+H))$

$\subset H^0(J^1_{m(K+\Delta)+H+A}(m(K+\Delta)+H+A+K_V))$

still big when remove boundary term $\lceil\Delta\rceil$ ..

so (5) . notice that $H' = H + A'$ is indep of m.

Final Part :

Need assumption ② B' = big wrt (V, X) (can't just $\lceil B \rceil$) for applications

$B' = A + B''$ — $\mathbb{Q}$-eff wrt (V, X) and ③ coeff $B' < 1$

ample and small

st. $(V, X + B'')$ plt (ie dlt & $[B''] = 0$) $B = B' + X$

$s_0 \in H^0(m(K_X+\Delta))$ $k|m$,

$D_0 = \mathrm{div}(s_0)$.

$\exists\, l > 0$, $lD_0 + H_X = \exists D$, $D \in |ml(K_V + B) + H|$.

$\bar{B} = \frac{m-1}{ml} D + B''$

$\mu : W \to V$ log-resol for $(V, X + \bar{B})$ this pair is not lT or whatsoever

$\mu^*(K_V + X + \bar{B}) = K_W + Y + C$

$0 \to \mathcal{O}_W(m\mu^*(K_V+B) - [Y+C]) \to \mathcal{O}_W(m\mu^*(K_V+B) - [C])$

$\to \mathcal{O}_Y(m\mu^*(K_V+B) - [C]) \to 0$

$m(K+B) - (K_V + X + \bar{B}) = A - \frac{m-1}{ml} H$ by def of $\bar{B}$

Take $l \gg 0$ st this is ample. so H^1(1st term) $= 0$

$$\begin{tikzcd} H^0(\text{2nd term}) \arrow[r, hook] \arrow[d] & H^0(m(K_V+B)) \\ H^0(\text{3rd term}) & \end{tikzcd}$$

if $[C]|_Y \le \mu^* D_0$ then $\mu^* s_0 \in H^0(\text{3rd term})$ then done.

p.14

$$
\begin{aligned}
Cl_Y &= \mu^* \bar{B} + \mu^* K_X - K_Y \\
&= \mu^*\left(\frac{m-1}{m} D_0 + \frac{m-1}{\ell m} H_X\right) \quad \leftarrow \text{this part } \le \mu^* D_0 \\
&\quad + \mu^* \underbrace{(K_X + B''_X)}_{\text{coeff} < 1} - K_Y
\end{aligned}
$$

(X, B'') klt $\Rightarrow$ coeff < 1 $\square$.

~~DLAK~~: Work of Hacon - McKernann, X : $\mathbb{Q}$-fact, B - $\mathbb{Q}$-div.

Th : $f : (X, B) \to Z$ det projective
$K_X + B$ f-negative ; $\rho(X/Z) = 1$ } called a flipping contraction

i) $\lfloor B \rfloor = S$ irred
ii) S f-neg. } called plt flipping contr.

may assume Z affine. Assume MMP in dim $n-1$ $\Rightarrow$ $\exists$ flip.

- Need to show $R = \bigoplus_{m \geq 0} H^0(X, \lfloor m(K_X + B) \rfloor)$ f.g.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\because R_m$

$\rightsquigarrow X^+ = \mathrm{Proj}\, R$ gives flip.

equiv to show $R_{(m_0)} = \bigoplus R_{m_0 m}$ f.g. $\leftrightarrow$ $R' = \bigoplus H^0(X, mS)$ f.g.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ bec. pic # 1.

$s \in H^0(X, S)$ totalogical section

$$0 \to \mathcal{O}_X((m-1)S) \xrightarrow{s} \mathcal{O}_X(mS) \to \mathcal{O}_S(mS) \to 0$$

$\tilde{e}_i \mapsto e_i$ ‵ reflexive sh of rank 1 on S.

$R'_S = \bigoplus_{m \geq 0}$ ~~$H^0(S, \mathcal{O}_S(mS))$~~ $\mathrm{Im}(H^0(X, mS) \overset{\Delta}{\to} H^0(S, mS))$

$e_1, \cdots, e_t$ generator then need only R'_S f.g.

since R' given by $s, \tilde{e}_1, \cdots, \tilde{e}_t$.

Notice Δ not surj since $H^1((m-1)S) \neq 0$ (S is neg !)
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ this is the difficulty.

$R'_S = \bigoplus H^0(S, mS)$ f.g

$\Rightarrow R'' = \bigoplus H^0(S, \lfloor m(K_S + \Delta) \rfloor)$ f.g.

$K_S + \Delta = K_X + B|_S$.

However this is not the standard flip case in dim $n-1$.

$$\begin{tikzcd} S \arrow[r, phantom, "\subset"] \arrow[d] & X \arrow[d] \\ f(S) \arrow[r, phantom, "\subset"] & Z \end{tikzcd}$$

need to find a subspace to perform ext thm.

Lemma : $k(K_X + B)$ Cartier div. $\exists$ very log-resl.

$\mu_m : Y_m \to (X, B)$, $\mu_m^*(K_X + B) = K_{Y_m} + C_m$

comp of C_m^+ w coeff < 1 are disjoint !
$\qquad\qquad\qquad\qquad$ $^\frown [0,1]$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ "called very-log".

p.16 $\quad 0 \le \exists B_m \le C_m^+$ }, mB_m Cartier on Y_m

$$\begin{tikzcd} H^0(m(K_X+B)) \ni \mathcal{D} & H^0(m(K_{Y_m}+B_m)) \arrow[l, "\sim"'] \arrow[d] \\ & \vartheta \in H^0(m(K_{T_m}+\Theta_m)) \end{tikzcd}$$

$K_{Y_m}+B_m|_{T_m} = K_{T_m}+\Theta_m$

$T_m = \mu_m^{-1}{}_* S$.

($\mathcal{D}$ is changed ↔).

note : $T_m \simeq T_{m'}$ for $m \ne m'$, call it T.

This is crucial, since to resolve linear system $m(K_X+B)$ say, no unique resol. but when restr to S, we can have unif. one.

$m\Theta_m + m'\Theta_{m'} = (m+m')\Theta_{m+m'}$

$\lim \Theta_m = \Theta$, (T, Θ) klt.

This is the key, but which is an easy consequence of the ext. thm.

16 :
$$\begin{tikzcd} \mu : Y \arrow[r] & X \\ T \arrow[u, phantom, "\cup" description] \arrow[r] & S \arrow[u, phantom, "\cup" description] \end{tikzcd}$$
log resol.

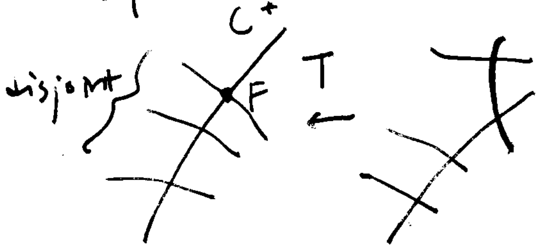
$\mu^*(K_X+B) = K_Y + C$, $D_X \in |m(K+B)|$ general

$D_Y \in |m(K_Y+C^+)|$

fixed comp $\cap$ C^+ ⟶ decrease coeff of C^+ then replace C^+ by it

D_Y contains a component $\underline{F}$ of $(C^+ - T) \cap T$ (divisor on T)

disjoint



blow up along F

Exc. div is a common comp of D_Y & C^+.

decrease C^+

repeat this process, it stops and get

D_Y is effective wrt $(Y_m, \lceil B_m^+ \rceil)$

← the C^+ after replacement

← omit m anymore.

To apply extension theorem ; need ① $K_{Y_m}+B_m$ is $\mathbb{Q}$-eff wrt $(Y, \lceil B \rceil)$, ② $B-T$ big wrt (Y, T).

$$\begin{tikzcd} T \arrow[r, hook] \arrow[d] & Y \arrow[d] \\ S \arrow[r, hook] \arrow[d] & X \arrow[d] \\ f(S) \arrow[r, hook] & Z \end{tikzcd}$$

~~birat'l~~ ⇒ any div is big

~~notice~~ that ① is not true wrt the whole pair $(Y, |B|)$.

some ~~[illegible]~~ can be thrown away, this proves lemma *

Cor. ~~$\exists p: T$~~ $\to (S, B_\Delta)$ log resol. $\exists\, \ell > 0$ $\forall m$ $k | m$

~~$p^*(K+B) = K_Y + C$, $K_Y + C|_T = K_T$~~

s.t. $\mathrm{Mov}|\ell m(K_T + \Theta_m)|$ free! (This T is not the previous T, need further blow-up)

usually this kind of thing is hopeless,

Pf: (T, Θ) klt, $|\Theta_m - \Theta| < \varepsilon$

finitely MMP processes for (T, Θ_m), $m \geq m_0$ (as in the 1st lecture)

$$\begin{tikzcd} T = T_0 \arrow[r, dashed] \arrow[dr, dashed] \arrow[ddr, dashed] & T_1 \\ & T_2 \\ & T_t \end{tikzcd}$$

take new T

V_i dominant all.

K_{T_i} + Θ_m nef (hence eventually free)

$m(K_{T_i} + \Theta_m)$ nef Cartier

"effective base point free thm" (☆)

L Cartier, $f: (X, B) \to Z$, L nef, $L - (K+B)$ nef & big

$\exists$ ℓL free for some ℓ depends only on $\dim X$.

$\Rightarrow$ $\ell m(K_{T_i} + \Theta_m)$ free $\forall m$.

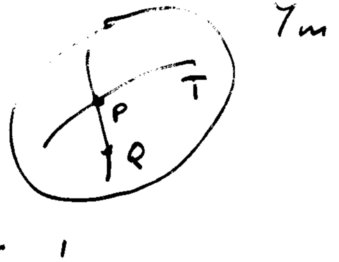
$k | m$, $P_{\ell m} = \mathrm{Mov}|\ell m(K_T + \Theta_m)|$ free

$Q_{\ell m} = \mathrm{Mov}|\ell m(K_{Y_m} + B_m)|$ $Q_{\ell m}|_T = P_{\ell m}$ free ← blow-up further Y_m

$\tilde{Q}_m = \mathrm{Mov}|m(K_{Y_m} + C_m^+)|$

$\tilde{P}_m = \tilde{Q}_m|_T$ $H^0(m(K_X+B)) \to H^0(m(K_S + \Delta))$ image

$\tilde{P}_m$ is what we want to investigate.



$\ell \tilde{P}_m \leq P_{\ell m} \leq \tilde{P}_{\ell m}$

$\lim \frac{\tilde{P}_m}{m} = \lim \frac{P_{\ell m}}{\ell m} = D$ = $\mathbb{R}$-divisor !

p.18 Want to prove ⊛ $D = \frac{\tilde{P}_m}{m}$ for some m.

$\Rightarrow$ D semi-ample.

base pt free thm for ℝ-div: since by the sequence of MMP ⊛

$D := K_T + \Theta$ nef & big.

$D = r_i^*(K_{T_i} + \Theta) = \lim r_i^*(K_{T_i} + B_m)$ ← s.a.

$\shortparallel$

P_m/m

once get ⊛ then

$\tilde{P}_{m'} = m'D$ if $m | m'$,

$\Rightarrow \bigoplus_{m \geq 0} H^0(T, \tilde{P}_m)$ f.g. which is basically R'_S.

Final part: pf of ⊛ (idea due to Shokurov,

$\mu_m^*(K_X + B) = K_{Y_m} + C_m$; $C_m = T + C'_m$

$C'_m|_T = \Gamma$ fixed, $\lfloor \Gamma \rfloor \leq 0$, (S, Δ) klt.

Lemma: $\forall i, j > 0$,

$$\mathrm{Mov}\left(\left\lceil \frac{jP_{il}}{il} - \Gamma \right\rceil\right) \leq \tilde{P}_j$$

Cor: $\forall j > 0$, $\mathrm{Mov}(\lceil jD - \Gamma \rceil) \leq \tilde{P}_j$

∵ easy application of vanishing thm:

$$0 \to \underset{H^1 = 0}{\mathcal{O}_{Y_i}\left(\left\lceil \frac{jQ_{il}}{il} - C_i \right\rceil\right)} \to \mathcal{O}_{Y_i}\left(\left\lceil \frac{jQ_{il}}{il} - C'_i \right\rceil\right) \to \mathcal{O}_T\left(\left\lceil \frac{jP_{il}}{il} - \Gamma \right\rceil\right) \to 0$$

$$\left\lceil \frac{jQ_{il}}{il} - C'_i \right\rceil \leq j(K_{Y_j} + C'_j)$$

this is at most $\tilde{Q}_m$, restr. get $\tilde{P}_j$. *

From cor:

case 1: D ℚ-div, ∃ m, $L = mD$ free

$L = \mathrm{Mov}(\lceil L - \Gamma \rceil) \leq \tilde{P}_m \leq mD$ here $\tilde{P}_m = mD$.

by cor.

case 2: D not ℚ-div, $D = \sum d_i L_i$, L_i free

$d_i \in \mathbb{R}$, $d_i > 0$, $\varepsilon \Sigma$: reduced div contains all supp of L_i

$(\varepsilon > 0)$

$\lceil + \varepsilon \Sigma \rceil \leq 0$

m, $\exists L$ free st. $mD \not\geq L$

$\lfloor mD - L \rfloor < \varepsilon \Sigma$ by Diophantine approximation.

$\leq \mathrm{Mov}\lceil mD - P \rceil \leq \tilde{P}_m \leq mD$ $\ast \leftarrow$ to $mD \not\geq L$.

(nowhere that since D is s.a. and fixed T, so can apply the base point free thm, in Shokurov's case he has saturation thm, but since he only use B-div so have no base point free thm.

nope: cone thm $\leftarrow$ bpf thm already
flip $\leftarrow$ bpf almost
termination $\leftarrow$?? bpf ?)

final step: (Fujino's paper is short and good & easy to read, better than Hacon-McKernan)

so sketch:

1) plt flip } $\Rightarrow$ gen flip
sp term }

2) $(n-1)$ MMP $\Rightarrow$ sp term

$f: (X, B) \to Z$ gen flipping contr.

Cartier div H' on Z, $f^*H' = H$

(i) $H \supset \mathrm{Exc}(f)$, $^*H \supset \mathrm{Sing}(\mathrm{Supp}\, B)$, $H \not\supset$ comp of B

(ii) H' reduced, $H' \supset \mathrm{Sing}\, Z$.

(iii) $\exists$ $Z' \to Z$ resl. $\exists \{C_i\}$ generator of $N^1(Z'/Z)$ st. $f(C_i \cdot K_{H'}$

$$\begin{tikzcd}
(X, B+H) \arrow[d] & Y \arrow[l, "g"', "\cancel{\text{log resl}}"] \arrow[dl, "h"] \arrow[d] \\
Z & Z' \arrow[l]
\end{tikzcd}$$

[$N^1(Y/Z)$ generated by components of g^*H]
bec. of $*$

Now run MMP of $(Y, g_*^{-1}\underbrace{(B+H)}_{\frac{1}{B}} + E_{exc})$ MMP/Z

Every curve is on h^*H'.

~~components~~ R extremal ray in this process.

p.20 $\exists$ boundary comp F, $F.R < 0$
with coeff 1.

$(h^*H'.R = 0)$ $\Rightarrow$ plt flip. if flipping contr:

notice $\bar{B} = F + \bar{B}'$

$F + (1-\varepsilon)\bar{B}'$, $((K_Y + \bar{B}).R) < 0$ if flipping

How about termination?

$R \subset F$ (in fact only need $R \cap F \neq \phi$)

then just require special termination.

log MMP, $(Y, S + C + H)$, $H = h_*^{-1}H'$

S = sum of coeff $1 = \sum S_j$, C other

$K_Y + S + C + H$ nef. $\leftarrow$ has something excess
need to reduce

Δ $h^*H' = H + \sum b_j S_j$

notice $H \xrightarrow{MMP} H = h_*^{-1}H'$ $B = B_1 + B_{<1}$

$B_{<1} + E \longmapsto S$

$B_1 \longmapsto C$

technique due to Shokurov:

$t = \min\{ t \in [0,1] \mid K_Y + S + C + tH \text{ nef} \} \leq 1$

Suppose $t > 0$, $\exists$ $K_Y + S + C + (t - \varepsilon)H$ extr ray R

$H.R > 0$. by Δ, and $h^*H'.R = 0$

$\Rightarrow \exists_j$ st. $S_j.R < 0$.

so a plt flip...

By sp ter $\rightsquigarrow$ $t = 0$.

ie. $\underbrace{K_Y + S + C}$ is nef.
no exceptional component
(by Hodge index thm)

$\Rightarrow$ Y is the flip.

Recall: the pf is from (X, B), and: H, $(X, B + H)$,
may no 1 comp, even $B = ..$

the log MMP, reducing ...

ie. even no-log prob. $(B = 0)$ need to go to log world.
(Shokurov).

ecial Termination:

$$\begin{tikzcd}
(\ldots 0, B^0) \arrow[dr, "f_0"] & & (X_1, B^1) \arrow[dl, "\phi_1"'] \arrow[dr, "\phi_1"] & & (X_2, B^2) \arrow[dl] \arrow[dr, "\phi_2"] & \cdots \\
& Z_0 & & Z_1 & & Z_2
\end{tikzcd}$$

ant $[B_m] \cap \operatorname{Exc} \phi_m = \phi$ $\quad m \geq m_0$.

$B^0 = B = \sum b_i B_i$ $\qquad (b) = \underset{\cup\{1\}}{\{b_i\}} \subset [0,1]$.

$$\underset{\underset{(b)}{\cup}}{S(b)} = \left\{1 - \frac{1}{m} + \sum \frac{r_i b_i}{m} \,\middle|\, m > 0,\ r_i \geq 0,\ b_i \in (b)\right\} \cap [0,1]$$

Lemma (easy exercise): $S(S(b)) = S(b)$. Such set are closely related to ter problem.

i.e. S = some kind of closure operation.

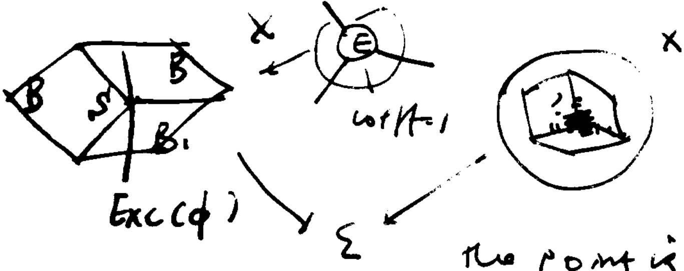
$\varepsilon > 0$, $S(b) \cap (0, 1-\varepsilon]$ = finite if (b) finite. so called DCC.

S = intersection of components of $[B]$

$k = \dim S$.

$S_m] \cap \operatorname{Exc}(\phi_m) = \phi$ $\forall m \geq m_0$ by induction on k.

$k = 0$:



don't know what happen in the intersection

The intersection disappears. The point is: discrepancy ↗ or coeff ↘

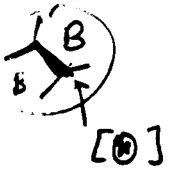
if $k-1$ true, prove k: assume $\operatorname{Exc}(\phi_m) \not\supset S_m$ otherwise just by the $k=0$ step.

$$\begin{tikzcd}
X_m \arrow[ddr] & & X_{m+1} \arrow[ddl] \\
S_m \arrow[u, phantom, "\cup"] \arrow[rr, dashed, "\text{birat'l}"] \arrow[dr, "\text{birat'l}"', "\text{isom in codim 1}"'] & & S_{m+1} \arrow[u, phantom, "\cup"] \arrow[dl] \\
& Z &
\end{tikzcd}$$

By induction,

$K_{X_m} + B_m|_{S_m} = K_{S_m} + \Theta_m$

$[\Theta_m] \cap \operatorname{Exc}(\phi_m) = \phi$.



Adjunction: coeff $B \subset (b)$

$\Rightarrow$ coeff of $\Theta_m \subset S(b)$ $\quad \dim S = \dim X - 1$

$\Rightarrow$ coeff of $\Theta \subset SS(b) = S(b)$, $\dim S = \dim X - 2$.

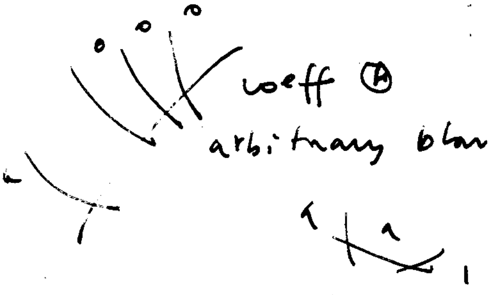
p. 22

$$\begin{tikzcd}
(S_m, \textcircled{B}_m) \arrow[rr, dashed] \arrow[dr] & & (S_{m+1}, \textcircled{B}_{m+1}) \arrow[dl] \\
& t(S_m) = T &
\end{tikzcd}$$

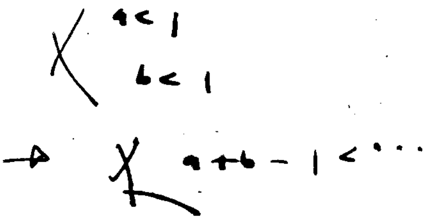
α isom on $[\textcircled{B}_m]$,

define the difficulty : $d_S = \sum_{v \in S(b)} \# \{ E \mid \text{coeff} > v, \text{centre}(E) \not\subset [\textcircled{B}_i] \}$

$< \infty$



but



① finiteness of d

② flip decreases d, in broader sense (maybe ~~constant~~,

③ d decreases eventually.

$\dim H_{2k-2}(S_m, \mathbb{Q}) =: \delta_m$

if div $C \subset (S_m, \textcircled{B}_m)$ which is contracted.

then $\sum$ divisor $\overset{\text{coeff } c}{C} \subset \textcircled{B}_{m+1}$, coeff $= c^+ \subset S(b)$

when div $= E$ is exceptional, $c > c^+$.

hence $\underset{\smile}{\overset{}{\text{coeff}}} \; \underset{c^+}{\overset{}{>}} v$ so $d \searrow$.

a) $\exists$ prime div on S_{m+1} contracted $\Rightarrow d \searrow$

b) $\exists$ prime div on S_m contracted and no contr div on S_{m+1}, then $\delta_m \searrow$.

c) from a), b) may assume $\begin{tikzcd}[column sep=small] S_m \arrow[dr] & & S_{m+1} \arrow[dl] \\ & T & \end{tikzcd}$ isom in codim 1.

but this is flip in dim = 3. (not ~~exactly~~, but almost)

$$\begin{tikzcd}
\tilde{S}_m \arrow[r, dashed] \arrow[d] & \tilde{S}_{m+1} \arrow[d] \\
S_m \arrow[dr] & S_{m+1} \arrow[d] \\
& T
\end{tikzcd}$$

→ read Fujino.

End