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CTS Summer School Lect I.

Example : $f : X \to \mathbb{P}^1$
un-isotrivial family of g-dim abelian variety with s.s singular fiber over $S \subset \mathbb{P}^1$

$\Rightarrow \quad s := \# S \geq 4$

$s = 4 \Leftrightarrow (f : X \to \mathbb{P}^1) \simeq E \times_{\mathbb{P}^1} \times \cdots \times_{\mathbb{P}^1} E \to \mathbb{P}^1$
of a univ. family of elliptic curves over a modular curve $\mathbb{P}^1 \setminus S$.

Interpretation : $\mathbb{P}^1 \setminus \{y_1, \cdots, y_4\}$ is "minimal" subvariety in A_g.

Candidate for "minimal" univ. family over shimura varieties.

More precisely, Shimura variety of Hodge type = Moduli scheme of ab. v. with a fixed Mumford-Tate gp.

Programme: 1) define "minimal" $U \subset A_g$ for example by numerical condition and by Alg cond. eg "non-rigid family of C-Y 3fold"; "Yukawa coupling".

2) Extend this to other moduli spaces, we like to show:

$\nexists$ Shimura curve of Mumford type
$\hookrightarrow$ moduli of canonically polarised mfds with birat'l $|K|$. (eg. curves $g \geq 2$)

This is related to Oort conj, but he claims this even in Zariski closure of Jacobian locus.

p. 2 Set up: $\mathcal{Y}$ complete proj mfd, $S \subset \mathcal{Y}$ normal crossing div, $U = \mathcal{Y} \setminus S$

$\varphi : U \to \mathcal{A}_g$ submfd induced by family $f : V \to U$ of abelian var. $\varphi \rightsquigarrow$ VHS $R^1 f_* \mathbb{Q}_V$

choose a compactification $f : X \to Y = U \cup S$ with $f^{-1}(S)$ NCD. (Simply want to have unipotent monodromy)

Problem: Are there numerical condition of f or of $R^1 f_* \mathbb{Q}_V$ which imply U is Shimura?

(i.e. $f : X \to U$ is a univ. family of ab. v. defined by Mumford-Tate gp, i.e. by Hodge cycle on the generic fiber)

$GL(\mathbb{Q}) \curvearrowright H_{\mathbb{Q}} = \bigoplus H^{p,q}$ — induces

$GL(\mathbb{Q}) \curvearrowright H_{\mathbb{Q}}^{\otimes n}$

consider $MT \subset GL(\mathbb{Q})$

maximal subgp fixing Hodge cycles on $H^{\otimes n}$.

2nd example, $\dim U = 1$

$f : X \to Y = U \setminus S$ family of g-dim ab. v.

$\mathbb{V} = R^1 f_* \mathbb{Q}_V$

Higgs bundle $E = f_* \omega_{X/Y} \oplus R^1 f_* \mathcal{O}_X$ wrt Hodge decomp.

Kodaira-Spencer map

$\theta : f_* \omega_{X/Y} \longrightarrow R^1 f_* \mathcal{O}_X \otimes \Omega^1_Y (\log S)$

1) Faltings:

$\deg f_* \omega_{X/Y} \leq \frac{g}{2} \deg \Omega^1_Y (\log S)$.

Alg-geom viewpoint : height inequality.

diff-geom viewpoint : Yau-Schwartz ineq.

2) Viehweg-Zuo :

"$=$" $\Longleftrightarrow$ U rigid Schimura curve of Mumford type arising from the corestriction of a quaternion algebra, defined over a totally real number field, which are ramified at all ∞ places except one.

- corestriction := $\sigma \in Gal(K/\mathbb{Q})$

$$A^{\sigma_1} \otimes A^{\sigma_2} \otimes \cdots \otimes A^{\sigma_d} / \mathbb{Q}$$

$$\Updownarrow$$

units , $\Gamma \subset Sl_2(\mathbb{R}) \otimes \overbrace{U_2 \otimes \cdots \otimes U_d}^{\text{unitary}}$

$\otimes \mathbb{R} \hookrightarrow$ arithmetic gp

but over $\mathbb{Q}$ it's complicate.

$G/K = Sl_2(\mathbb{R})/SO(2) = \mathbb{H}$

Mumford construct $Y \setminus S = \mathbb{H}/\Gamma$.

(notice that S could be ϕ.)

Remark : let $f : X \to Y$ a s.s family of curves of genus g. by Faltings $\deg f_* \omega_{X/Y} \leq \frac{g}{2} \deg \Omega^1_Y(\log S)$.

Tan : $<$ for $g \geqslant 2$.

(ie. for Jacobian varieties, no $=$).

Viehweg-Z : Arakelov inequality holds strictly for families of mfd with birational canonical linear system.

(will define the Arakelov inequality later) for high dim base.

P. 4   dim $Y = n \geq 1$

Assumption : * $\omega_Y(S)$ nef and ample wrt $U = Y \setminus S$.

Def'n : Slope of coherent sheaf F on Y with $rk = r$ as

$$\mu_{\omega_Y(S)}(F) = \frac{1}{r} c_1(F).c_1(\omega_Y(S))^{n-1}$$

discriminent $\delta(F) = \left(2r\, c_2(F) - (r-1)\, c_1(F)^2\right) c_1(\omega_Y(S))^{n-2}$.

Thm (Viehweg-Zuo). Let V be an irreducible $\mathbb{C}$ sub VHS in $R^1 f_* \mathbb{C}_V$ with Higgs bundle $E = (E^{1,0} \oplus E^{0,1}, \theta)$, $\theta : E^{1,0} \to E^{0,1} \otimes \Omega^1_Y(\log S)$, then the following Arakelov ineq holds:

① $\mu_{\omega_Y(S)}(E^{1,0}) - \mu_{\omega_Y(S)}(E^{0,1}) \leq \mu(\Omega^1_Y(\log S))$.

Moreover, "=" $\nRightarrow$ $E^{1,0}$, $E^{0,1}$ semi-stable

cor 2: (Bogomolov inequality)

② $\delta(E^{1,0}), \delta(E^{0,1}) \geq 0$.

Hope : If ① and ② are equalities then U Shimura.

2 examples Y an algebraic surface

Candidate

a) U is a generalized Hilbert modular surface i.e. $\mathbb{H} \times \mathbb{H} \circlearrowleft \Gamma$ arithmetic subgroup

Hilbert's original consideration $SL_2(\mathcal{O})$ real quadratic ext.

b) U 2-dim cpx ball quotient

$\Gamma \backslash U(2,1)/K$.

Q : find numerical condi for ab. v. over a), b) ?!

$f: X \to Y$ family of ab. v. $f: X_0 \to U$ sm

① $0 \leq \deg f_* \omega_{X/Y} \leq \frac{g}{2} \deg \Omega^1(\log S)$

eq ⟺ isotrivial Faltings

② 2nd "=" holds ⟺ $f: X \to Y$ is a Shimura family of Mumford Hodge type.

poly stability of VHS : (Simpson)

$$\mathbb{V} = R^1 f_* \mathbb{C}_{X_0} \rightsquigarrow f_* \Omega^1_{X/Y}(\log \Delta) \xrightarrow{\theta} R^1 f_* \mathcal{O}_X \otimes \Omega^1_Y(\log S)$$

Deligne's canonical extension $E^{1,0} \oplus E^{0,1} = E$

Sub-Higgs bundle $(F, \theta) \subset (E, \theta)$

⇒ $\deg F \leq 0$ and if $\deg F = 0$

⟺ $(E, \theta) = (F, \theta) \oplus (Q, \theta)$

$\mathbb{V} \overset{?}{=} \mathbb{V}_1 \oplus \mathbb{V}_2$ polarised VHS.

pf of 1) constructed a Higgs subbundle

$(F^{1,0} \oplus F^{0,1}, \theta) < (E^{1,0} \oplus E^{0,1}, \theta)$ as follows

$F^{1,0} = E^{1,0}$, $F^{0,1} = \theta(F^{1,0}) \otimes \Omega^1_Y(\log S)^{-1}$

$$\begin{tikzcd}
& E^{1,0} \arrow[r, "\theta"] \arrow[d, phantom, "\parallel"] & E^{0,1} \otimes \Omega^1_Y(\log S) \arrow[d, phantom, "\cup"] \\
0 \arrow[r] & \operatorname{Ker}\theta \arrow[r] & F^{1,0} \arrow[r, "\theta"] & F^{0,1} \otimes \Omega^1_Y(\log S)
\end{tikzcd}$$

a Higgs sub-bundle

$$\deg F^{0,1} = \deg(F^{0,1} \otimes \Omega) - \operatorname{rk} F^{0,1} \deg \Omega$$
$$\geq \deg(F^{0,1} \otimes \Omega) - \operatorname{rk} E^{1,0} \deg \Omega$$
$$\geq \deg E^{1,0} - \operatorname{rk} E^{1,0} \otimes \deg \Omega$$

$$0 \geq \deg F = \deg E^{10} + \deg F^{10}$$

$$\geq 2\deg E^{10} - \mathrm{rk}\, E^{10} \deg \Omega_Y^1(\log S)$$

$$\Rightarrow \deg f_*\omega_{X/Y} - \deg E^{10} \leq \frac{g}{2} \deg \Omega_Y^1(\log S).$$

preparation for 2) :

"=" $\Leftrightarrow \mathbb{V} = \mathbb{L} \otimes \mathbb{U}$, where $\mathbb{U}$ is unitary

$\mathbb{L}$ rk 2 VHS $\rightsquigarrow \mathcal{L} \oplus \mathcal{L}^{-1}$, $\theta : \mathcal{L} \xrightarrow{\sim} \mathcal{L}^{-1} \otimes \Omega_Y^1(\log S)$

$\mathcal{L} = \Omega_Y^1(\log S)^{1/2}$.

Pf of "=" $\Rightarrow E^{10}, E^{01}$ poly stable, $E^{10} \overset{\theta}{\xrightarrow{\sim}} E^{01} \otimes \Omega_Y^1(\log S)$.

Uhlenbeck-Yau-Donaldson correspondence

$\mathcal{L} := \Omega_Y^1(\log S)^{1/2}$

$E^{10} \otimes \mathcal{L}^{-1}$, $\deg(E^{10} \otimes \mathcal{L}^{-1}) = 0$ (since "=" holds)

$E^{10} \otimes \mathcal{L}^{-1}$ coming from a unitary local system $\mathbb{U}$

$$\begin{tikzcd} E^{10} = \mathcal{L} \otimes \mathbb{U}, & E^{10} \arrow[r, "\theta", "\sim"'] \arrow[d, phantom, "\shortparallel"] & E^{10} \otimes \Omega_Y^1(\log S) \arrow[d, phantom, "\shortparallel"] \\ & \mathcal{L} \otimes \mathbb{U} \arrow[r, phantom, "\simeq"] & E^{01} \otimes \mathcal{L}^2 \end{tikzcd}$$

$$E^{01} = \mathcal{L}^{-1} \otimes \mathbb{U}, \quad E^{10} \xrightarrow{\sim} E^{01} \otimes \Omega_Y^1(\log S)$$

$$\wr$$

$$(\mathcal{L} \xrightarrow{\sim} \mathcal{L}^{-1} \otimes \Omega_Y^1(\log S)) \otimes \mathbb{U}$$

$\Rightarrow \mathbb{V} = \mathbb{L} \otimes \mathbb{U}$.

Recall : Special Mumford-Tate group.

Given an ab. v. B with polarized $\mathbb{Q}$-Hodge str $H^1(B, \mathbb{Q})$ and polarization Q.

$Hg(Q)$ is the largest $\mathbb{Q}$-algebraic subgp of $Sp(H^1(B, \mathbb{Q}), Q)$ which leaves all the Hodge cycles of $B \times B \times \dots \times B$ invariant.

$\eta \in [H^{2p}(B \times \cdots \times B, \mathbb{Q})]^{p,p}$

$= [\Lambda^{2p}(H^1(B,\mathbb{Q}) \oplus \cdots \oplus H^1(B,\mathbb{Q}))]^{p,p}$.

For a smooth family of alg varieties $f: X_0 \to U$ which with $B = f^{-1}(y)$ for some $y \in U$, and for the corresponding $\mathbb{Q}$-polarized VHS $R^1 f_* \mathbb{Q}_{X_0}$, we consider Hodge cycles η on B, which remains Hodge cycle under the monodromy action of $\pi_1(U,y) \curvearrowright H^1(B,\mathbb{Q})$ (Imaging these are Hodge cycles from global)

$Hg(R^1 f_* \mathbb{Q}_{X_0})$ is the largest $\mathbb{Q}$-alg subgroup which leaves all those Hodge cycle invariant. by def, the Zariski closure of monodromy gp is inside this gp.

Notice that, in arith case the monodromy gp is replace by the Galois gp then its conj that $Hg = Gal$. but in geom case monodromy gp could be much smaller (eg. trivial family)

But would like to formulate condition st Hg = Monodromy gp. (turns out $\equiv$ Shimura family)

Lemma: a) $\forall y \in U$, $Hg(f^{-1}(y)) \subseteq Hg(R^1 f_* \mathbb{C}_{X_0})$, $\forall y$ in the complement U' of union of countable many closed subset, have "=" (like behavior of Pic)

b) G^{mon} denotes the smallest reductive $\mathbb{Q}$-alg subgp of $Sp(H^1(B,\mathbb{Q}),\mathbb{Q})$ containing the image

$\pi_1(U,y) \xrightarrow{\rho} Sp(H^1(B,\mathbb{Q}),\mathbb{Q})$,

Then the connected comp G_0^{mon} of id in G^{mon} is a subgp of $Hg(R^1 f_* \mathbb{Q}_{X_0})$ (Deligne).

P8. c). If $\deg \omega_{X/Y} = \frac{g}{2} \deg \Omega_Y^1(\log S)$, then

$G_0^{mon} = Hg(R^1 f_* \mathbb{Q}_{X_0})$ (Viehweg-Zuo).

proof of c): The pf of 3.1c) in (Deligne's paper Hodge cycles on ab. v. LNM 100) can be carried over to show that

$G_0^{mon} = Hg(R^1 f_* \mathbb{Q}_{X_0})$ if they have the same tensor $\eta \in \Lambda^{2p}(H^1(B,\mathbb{Q}) \oplus \cdots \oplus H^1(B,\mathbb{Q}))$

invariant

Since $G_0^{mon} \subset Hg(R^1 f_* \mathbb{Q}_{X_0})$, only need to show any G_0^{mon}-inv tensor $\eta \in \Lambda^{2p}(H^1(B,\mathbb{Q}) \oplus \cdots \oplus H^1(B,\mathbb{Q}))$ is also $Hg(R^1 f_* \mathbb{Q}_{X_0})$ inv.

a flat section

$\eta \in \Lambda^{2p}(R^1 f_* \mathbb{Q}_{X_0}) \oplus \cdots \oplus R^1 f_* \mathbb{Q}_{X_0})$

Now $R^1 f_* \mathbb{Q}_X = \mathbb{L} \otimes \mathbb{U}$,

$\Lambda^{2p}(\dots) = \bigoplus S^i(\mathbb{L}) \otimes S_i(\mathbb{U})$ — e.g. Fulton-Harris's book

irreducible since Zariski dense in SL.

$\supset (\det(\mathbb{L})^c \otimes S_{\lambda_c}(\mathbb{U}))$

Computation $\Rightarrow$ exactly (p,p) type in standard rep theory.

hence $\exists\ \eta$

$\Rightarrow \eta$ is flat section of (p,p) type

$\Rightarrow \eta$ is $Hg(R^1 f_* \mathbb{C}_{X_0})$-inv //

Studying moduli problem (Mumford):

Given a special Mumford-Tate gp, $Hg \subset Sp(2g, \mathbb{Q}; \mathbb{Q})$, Mumford consider the moduli functor $\mathcal{M}(Hg)$ of isomorphism classes of polarized abelian variety B with

$Hg(B) \subseteq Hg$.

He shows that $M(Hg)$ admits a quasi-proj coarse moduli space $M(Hg) \subset \mathcal{A}_g$ and admit a fine moduli space by taking suitable étale cover

$$M(Hg) = \underset{\text{arithmetic subgp.}}{\Gamma} \backslash Hg(\mathbb{R}) / K$$

//

we will show that $f : X \to Y$ is a universal family with given Hg over Y.

pf: c) $\Rightarrow$ $G_0^{mon} = \underset{\text{let } Hg \text{ be this}}{Hg(R^1 f_* \mathbb{Q}_{X_0})}$

$$\rho : \pi_1(U,u) \to Sp(\underset{\uparrow}{\mathbb{L}} \otimes \mathbb{U})$$

Zariski dense in this part

The corresponding sym space is simple

$Hg/K = G_0^{mon}/K =$ upper half plane

$\Rightarrow \dim M(Hg) = 1$.

$f : X_0 \to U \quad \forall y \quad Hg(f^{-1}(y)) \subseteq Hg$

$\varphi : U \longrightarrow M(Hg)$ hence $Y \twoheadrightarrow \overline{M}(Hg)$.

Finally need to show φ is $\overset{\text{proper}}{\text{étale}}$ on U :

i.e. unbranched.

$$d\varphi : T_Y(\log S) \xrightarrow[\sim]{?} \varphi^* T_{\overline{M}(Hg)}(\log S)$$

but this is $\overset{\shortparallel}{\mathcal{L}}^{-2} \xleftarrow[\theta]{\sim} T_Y(\log S))$

by the very definition $\square$

How about family of other varieties ?

$f : X \to Y$ family of polarized proj mfd over Y with semi-stable normal crossing of dim $= n$ singular fiber $\Delta \to S \subset Y$.

Deligne : $R^n f_* \mathbb{Q}_{X_0} \rightsquigarrow \bigoplus_{p+q=n} \underset{\shortparallel \, E^{p,q}}{R^q f_* \Omega^p_{X/Y}(\log \Delta)}$.

1.9 Viehweg - Zuo :

$\deg E^{n,0} \leq \frac{n}{2} \operatorname{rk}(E^{n,0}) \deg \Omega^1_Y(\log S)$

$\deg E^{n-1,1} \leq \frac{n-2}{2} \operatorname{rk}(E^{n-1,1}) \deg \Omega^1_Y(\log S)$

... etc

"=" holds $\Leftrightarrow$ $R^n f_* \mathbb{Q}_{X_0} \subset \bigotimes R^1 g_* \mathbb{Q}_{A_0}$

$g : A \to Y$ is a universal family of ab. v. with a given Hg.

Thm (next time to show) :

let $f : X \to Y$ be a family of proj mfds with birational canonical linear system of generic fiber of f. Then

$\deg E^{n,0} = \deg f_* \omega_{X/Y} < \frac{n}{2} p_g \cdot \deg{}^1_Y(\log S)$ //

$J_g \hookrightarrow A_g$

moduli of Jacobians — moduli of abelians

from $f : X \to U$ family of curves of genus g

$\rightsquigarrow J(X/U) \to U$

$\rightsquigarrow \varphi : U \to J_g \subset A_g$.

holomorphic totally geodesic embedding : if $\exists$ lifting:

$$\begin{tikzcd} U \arrow[r, hook] & A_g \\ SL_2(\mathbb{R})/K' \arrow[r, hook] \arrow[u] & Sp(2g,\mathbb{R})/K \arrow[u] \\ SL_2(\mathbb{R}) \arrow[r, hook] \arrow[u] & Sp(2g,\mathbb{R}) \arrow[u] \end{tikzcd}$$

Oort's conj : arithmetic holomorphic totally geod embedding can not be contained in $\overline{J}_g \subset A_g$, for $g \gg 0$.

Arithmetic $\Leftrightarrow$ $U = \underset{\text{sub gp}}{\text{arith}} \backslash SL_2(\mathbb{R})/K$ $\Leftrightarrow$ U is Shimura.

Th: (Tan) $f : X \to Y$ semi-stable family of curves $g \geq 2$, then $\deg \omega_{X/Y} < \frac{g}{2} \deg \Omega^1_Y(\log S)$. ($\Rightarrow$ Shimura curve of Mumford type $\not\subset J_g \subset A_g$).

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A stronger version of Tan's thm :

Thm (Viehweg – Z) : Family of ab. v $f: X \to Y$ st.
" $R^1 f_* \mathbb{Q}_{X_0}$ contains a sub VHS of the form $(\mathcal{L} \oplus \mathcal{L}^{-1}, \theta) \otimes \mathbb{U}$
$\theta : \mathcal{L} \simeq \mathcal{L}^{-1} \otimes \Omega^1_Y(\log S)$) with $\operatorname{rk} \mathbb{U} \geq 2$ " $\not\subset J_g$.

Thm (Mc Mullen) $\exists$ family of ab. v. $f: X \to Y$ st
$R^1 f_* \mathbb{Q}_{X_0} \supset (\mathcal{L} \oplus \mathcal{L}^{-1}, \theta)$ and $\varphi(Y \setminus S) \subset J_g$ for any $g \geq 1$.

Generalized Oort's conj :

Moduli of polarized mfds F
M_h with fixed Hilb poly h.

Moduli of Hodge structures

$$M_h \xrightarrow[\text{period map}]{\varphi} \mathcal{D}/\Gamma$$

Assume φ is algebraic (this is still conjectural)
φ is also assumed to be a local embedding .
Conj is : $\overline{\varphi(M_h)} \not\ni$ with hol tot geod embeddings
if $\kappa(F) = \dim F$ and $p_g = h^0(\Omega^n_F) >> 1$

Thm 1 (Viehweg – Z) : Let $f: X \to Y$ be s.s over curve Y, not iso-trivial , of n-folds . Assume $|\omega_F|$ is bi-rat'l ,
then $\deg f_* \omega_{X/Y} < \frac{n}{2} p_g \cdot \deg \Omega^1_Y(\log S)$.

(The pf of Tan's thm uses Miyaoka-Yau's inequality for surfaces and Parsin's trick , But the higher dim case can't do it directly , hence use alg geom.)

Lemma 2. " = " $\Rightarrow$ $f_* \omega_{X/Y} = \mathcal{L}^{\otimes n} \otimes \mathbb{U}_{\text{unitary}}$, $\mathcal{L}^2 = \Omega^1_Y(\log S)$.
The pf uses stability of Higgs bundle of VHS $R^n f_* \mathbb{C}_{X_0}$.
But this is not enough , it makes not much difference with the ab. v. case .

※ Idea : $H^p(\wedge^p T_F)$, $\bigoplus_{p=0}^n H^p(\wedge^p T_F)$ not VHS exc for C-Y case
but may try to imbed $\hookrightarrow$ Hodge structures .

P.11 Review of: How to construct VHS algebraically.

$$0\to\Omega^{p-1}_{X/Y}(\log\Delta)\otimes f^*\Omega^1_Y(\log S)\to\Omega^p_X(\log\Delta)\to\Omega^p_{X/Y}(\log\Delta)\to 0$$

Apply $R^q f_*(\cdot)$ get alg. construction of GM connection:

$$R^q f_*\,\Omega^p_{X/Y}(\log\Delta)\xrightarrow[\uparrow\ \text{conn. map}]{\theta}R^{q+1}f_*(\Omega^p_{X/Y}(\log\Delta))\otimes\Omega^1_Y(\log S)$$

This is Deligne's extension of VHS $R^n f_*\mathbb{C}_{X_0}$

let M be a line bundle on X st. $|M^k|\neq\emptyset$, $\otimes M^{-1}$:

$$0\to\Omega^{p-1}_{X/Y}(\log\Delta)\otimes M^{-1}\otimes f^*\Omega^1_Y(\log S)\to\Omega^p_X(\log\Delta)\otimes M^{-1}$$
$$\to\Omega^p_{X/Y}(\log\Delta)\otimes M^{-1}\to 0$$

get

claim: $s\in|M^k|\Rightarrow$

$$\begin{tikzcd} R^q f_*(\Omega^p_{X/Y}(\log\Delta)\otimes M^{-1}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & (R^{q+1}f_*\Omega^{p-1}_{X/Y}(\log\Delta)\otimes M^{-1})\otimes\Omega^1_Y(\log S) \arrow[d] \\ G^{p,q} \arrow[r, "\theta'"] & G^{p-1,q+1}\otimes\Omega^1_Y(\log S') \end{tikzcd}$$

$\rightsquigarrow(\bigoplus_{p+q}G^{p,q},\theta)\rightsquigarrow$ VHS on Y/S'.

To construct VHS, usually need the negative curvature condition, here no such, will do algebraically

Standard construction: (Main idea only)

X proj mfd, $M\to X$ line bd. $|M^k|\neq\emptyset$,
$s\in H^0(M^k)$, $\neq 0\rightsquigarrow K(X)(\sqrt[k]{s})$ leads to k-th cyclic cover $Z\xrightarrow{\sigma}X$ ramified along $D=(s)$,
$\mathrm{Gal}(Z/X)\curvearrowright H^n(Z,\mathbb{C})$

rmk: Z would be very singular, need to resolve it.

$\rightsquigarrow H^n(Z,\mathbb{C})=H^n(X,\mathbb{C})\oplus H^n(Z,\mathbb{C})_1\oplus\cdots$ eigen space decomp.

$$H^n(Z,\mathbb{C})_1=\bigoplus_{p+q=n}H^q(X,\Omega^p_X(\log D)\otimes M^{-1})$$

↑ if this is i

↑ then this is i.

Apply same construction for $f: \overset{M}{X} \to Y$ p.
relative case,

$$\begin{tikzcd} Z \arrow[r, "\sigma"] \arrow[dr, "g"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

the singularity will not m

One obtains $\mathbb{C}$-VHS $R^n g_*(\mathbb{C}_Z)$, $/ Y \setminus S'$ $\rightsquigarrow$

$$R^q f_* \Omega^p_{X/Y}(\log \Delta') \otimes M^{-1} \xrightarrow[\theta]{} R^{q+1} f_* \Omega^{p-1}_{X/Y}(\log\Delta') \otimes M^{-1}) \otimes \Omega^1_Y$$

the original one (with $\otimes M$, not M^{-1}) is then obviously an embedding into this one. notice we have to enlarge Δ, S to Δ', S'.

In our situation, $M := \omega_{X/Y} \otimes f^*\mathcal{L}^{-n}$, $\mathcal{L}^2 = \Omega^1_Y(\log S)$.
by claim 1, $0 \neq s \in H^0(X, M^0) \rightsquigarrow$

$$\begin{tikzcd} R^q f_*(\Omega^p_{X/Y}(\log\Delta) \otimes M^{-1}) \arrow[r, "\theta"] \arrow[d, phantom, "\shortparallel"] & & R^{q+1} f_* \Omega^{p-1}_{X/Y}(\log\Delta) \otimes M^{-1}) \otimes \Omega^1_Y(\log S) \arrow[d, phantom, "\cup"] \\ E^{p,q} & {} \arrow[d] & E^{p-1,q+1} \\ & G^{p,q} \arrow[r, "\theta"] & G^{p-1,q+1} \otimes \Omega^1_Y(\log S') \end{tikzcd}$$

$\bigoplus_{p+q=n} G^{p,q} \rightsquigarrow$ VHS $/ Y \setminus S'$.

considering sub Hodge bundle $\underset{(F,\theta)}{\bigoplus_{p+q=n}(F^{p,q}, \theta)}$
generated by $F^{n,0} = E^{n,0} = \mathcal{L}^n$ and

$$F^{p-q-1,q+1} = \theta(F^{n-q,q}) \otimes \Omega^1_Y(\log S)^{-1}$$

A simple calculation $\Rightarrow$ $\deg F^{n-q,q} \geq (n-2q)\deg L$
Sum up, $\deg F = \sum_{q=0}^n \deg F^{n-q,q} \geq \underset{\shortparallel\; 0}{\sum_{q=0}^n (n-2q)\deg L}$.

On the other hand, by our construction,

$(F,\theta) \subset (\bigoplus E^{p,q}, \theta) \subset (\bigoplus G^{p,q}, \theta) \rightsquigarrow$ VHS.

By Simpson's poly stability then $\deg F \leq 0$; one get almost a contradiction if we show $\deg F > 0$.

If can show $H^0(X, (M \otimes f^*\mathcal{O}_Y(-\varepsilon))^\nu) \neq 0$ for some $\varepsilon \in \mathbb{Q}$, $\nu \in \mathbb{N}$, then done. ($\varepsilon \in \mathbb{Q}$ makes sense in cone.:

P.13 let $M' = M \otimes f^*\mathcal{O}_Y(-\varepsilon)^{\otimes \nu}$; then do again get $H^0 \neq 0$

so $\Rightarrow$ $\deg F \geq (n+1)\varepsilon > 0$ $\times$ claim: can

This $H^0 \neq 0$ needs the geometry of the fiber.

The multiplication map :

Prop. "Arakelov equality" $\Rightarrow$ $\exists$ finite cover $\hat Y \to Y$ etale over U such that for the pull back family $\hat f : \hat X \to \hat Y$ $\exists$ a section in

$$\begin{tikzcd} \hat X \arrow[r] \arrow[d] & \hat Y \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \qquad \omega^{\nu}_{\hat X/\hat Y} \otimes \hat f^*\left(\mathcal{L}^{-n} \overset{\nu}{\otimes} \mathcal{O}_{\hat Y}(-\nu y)\right).$$

Pf (simple application of the mult. map).

"=" $\Rightarrow$ $f_*\omega_{X/Y} = \mathcal{L}^n \otimes \mathbb{U}$, $\operatorname{rk}\mathbb{U} = P_g(F)$, $\mathcal{L}^2 = \Omega^1_Y(\log S)$,

$\rho : f^*\mathbb{U} \xrightarrow[ev]{} \omega_{X/Y} \otimes f^*\mathcal{L}^{-n}$, $\rightsquigarrow$

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[dr, "f"'] & \mathbb{P}(\mathbb{U}) \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

which is bi-rational on the generic fiber F of $f : X \to Y$.

consider the diagram, $W = \sigma(X)$, $\pi' = \pi|_W \to Y$

multiplication map, study its image

$$S^\nu(\mathbb{U}) \xrightarrow{m_\nu} f_*\omega^{\nu}_{X/Y} \otimes \mathcal{L}^{-n\nu}$$

$\mathbb{U}_\nu := \operatorname{Im}(m_\nu) \subset f_*\omega^{\nu}_{X/Y} \otimes \mathcal{L}^{-n\nu}$

claim2: this $\mathbb{U}_\nu$ has positive degree for some ν.

Pf: If $\forall \nu$ $\deg \mathbb{U}_\nu \leq 0$, then

$$\underset{\text{poly stable}}{S^\nu(\mathbb{U})} \twoheadrightarrow \mathbb{U}_\nu \quad \Rightarrow \quad \deg \mathbb{U}_\nu = 0 \ \forall \nu$$

must have s.p quotient

$\Rightarrow$ splits $S^\nu(\mathbb{U}) = \mathbb{U}_\nu \oplus \operatorname{Ker}(m_\nu)$

in particular, $\operatorname{Ker}(m_\nu)$ is also unitary local system.

Note that $\operatorname{Ker}(m_\nu)$ on the fiber F is exactly the <u>poly normal</u> of deg ν vanishing on $\sigma(F) \subset \mathbb{P}$.

$\Rightarrow$ All fibers $\sigma(F)$ are constant $\times$.

Arakelov inequalities in higher dim base

Andrey-Oort conj:

$\overline{\text{CM points}} \subseteq Y \subset \bar{A}_g \Rightarrow Y$ Shimura.

CM pts ⟺ Ab. v. with maximal number of Hodge cycles (i.e. zero-dim'l bounded sym. domain)

Weak version: $\overline{\left\{\begin{matrix}\text{Shimura} \\ \text{sub varities}\end{matrix}\right\}} = Y \subset \bar{A}_g \Rightarrow Y$ Shimura

$\underset{\overset{\|}{D}}{\text{Div}} \subset Y \subset \bar{A}_g$

$D = \Sigma$ shimura subvarieties st. D is big & nef (for exmple $D \in |\omega_Y(S)^{\otimes m}|$)

$\Rightarrow Y$ is shimura.

Consider $f: X \to Y$ family of ab. v. over proj surface Y.

Assume 1). $C_i \subset Y$ st. $f: X \to C_i$ shimura,

$\Rightarrow f_*\omega_{X/C_i} = \frac{g}{2}\deg \Omega^1_{C_i}(S)$

2). $\Sigma_i C_i \in |\omega_Y(S)^{\otimes m}|$, Sum up Arakelov ineq over C_i, get $\deg f_*\omega_{X/Y}(\underset{\overset{\|}{\omega_Y(S)}}{\Sigma_i C_i}) = \underset{\overset{\|}{\frac{g}{2}(\omega_Y(S)+C_i).C_i}}{\frac{g}{2}\Sigma_i \deg\Omega^1_{C_i}(S)}$

3) Hirzebruch's relative proportionality:

$-2\omega_Y(S).C_i = C_i^2 \Rightarrow$ the above $= \frac{g}{4}\omega(S)^2$,

$\Rightarrow (\deg f_*\omega_{X/Y}).\omega_Y(S) = \frac{g}{4}\omega_Y(S)^2$.

Point: want to say any surface satisfies "these" should be Shimura.

turns out to be Arakelov inequalities.

Assumption: $\omega_Y(S)$ nef and ample on $Y\setminus S$, slope $\underset{\text{coherent sheaf}}{\mu(F)} := \frac{c_1(F)\, c_1(\omega_Y(S))^{n-1}}{\mathrm{rk}\, F = r}$

P.15 discriminant : $\delta(F) = \left(2r\,c_2(F) - (r-1)\,c_1(F)^2\right) \cdot c_1(\omega_Y(S))^{n-2}$.

Theorem 1 (Viehweg - Z)

let $f : X \to Y$ s.s family of g-dim ab. v. Let $\mathbb{V}$ be an irreducible $\mathbb{C}$ sub VHS of $R^1 f_* \mathbb{C}_X$ with Higgs bundle

$$E^{1,0} \xrightarrow{\theta} E^{0,1} \otimes \Omega^1_Y(\log S), \text{ Then}$$

1) $\mu(E^{10}) - \mu(E^{01}) \leq \mu(\Omega^1_Y(\log S))$.

Moreover, "=" in 1) ⇒ E^{10}, E^{01} are semi-stable.

cor. (Bogomolov)

2) $\delta(E^{1,0}), \delta(E^{0,1}) \geq 0$.

In the pf, we need

① Simpson's poly-stability of Higgs bundle

② Yau's thm on the existence of K-E metric on $Y \setminus S$ and the poly-stability of $\Omega^1_Y(\log S)$.

Remark : If $\varphi : Y \to \bar{A}_g$ is an imbedding, then $\omega_Y(S)$ is nef and ample on $Y \setminus S$.

Example 1: let $f : X \to Y$ be a family of abelian surfaces over a proj surface Y, with s.s singular fiber $\Delta \to S$,

$\mathbb{V} = R^1 f_* \mathbb{C}_{X_0}$ with Higgs $\underset{E^{1,0}}{\underset{\parallel}{f_* \Omega^1_{X/Y}(\log \Delta)}} \xrightarrow[\theta]{} \underset{E^{0,1}}{\underset{\parallel}{R^1 f_* \mathcal{O}_X \otimes \Omega^1_Y(\log S)}}$

$\Lambda^2 \mathbb{V} = \Lambda^2_{\text{primitive}}(\mathbb{V}) \oplus \underset{\mathbb{Q}}{\underset{\parallel}{\Lambda^0(\mathbb{V})}}$

$\underset{E^{20}}{\underset{\parallel}{\Lambda^2 E^{10}}} \oplus \underset{E^{1,1}}{\underset{\parallel}{(S^2(E^{01}) \otimes \Lambda^2 E^{10})}} \oplus \underset{E^{0,2}}{\underset{\parallel}{\Lambda^2 E^{01}}}$ $\qquad E^{0,1} = T_{\bar{A}_g}(\log S)|_Y$

$\Lambda^2 E^{10} \otimes T_Y(\log S) \xrightarrow{\theta} E^{1,1}$; $\qquad \varphi : Y \to \bar{A}_g$

Considering the sub-Higgs bundle

$$(F,\theta) = (F^{2,0} \oplus F^{1,1} \oplus F^{0,2}; \theta) \subset (E^{2,0} \oplus E^{1,1} \oplus E^{0,2}, \theta)$$

$$\uparrow S \qquad \uparrow \theta$$

$$E^{2,0} \oplus T_Y(\log S) \otimes E^{2,0} \oplus S^2(T_Y(\log S)) \otimes E^{2,0}$$

i.e the smallest Higgs containing $E^{2,0}$.

Simpson's stability $0 \geq c_1(F) . \omega_Y(S)$.

$$= \big(\underset{\substack{\parallel \\ c_1(E^{20}) . \omega_Y(S)}}{c_1(F^{00})\, \omega_Y(S)} + \underset{\substack{\vee\!/ \\ (2c_1(E^{20}) - c_1(\omega_Y(S))) c_1(\omega_Y(S))}}{c_1(F^{11})\, \omega_Y(S)} + \underset{\substack{\vee\!/ \\ c_1(E^{0,2}) \\ \cdot c_1(\omega_Y(S))}}{c_1(F^{02})\, \omega_Y(S)} \big)$$

$$\Longleftrightarrow \quad \underset{\substack{\shortparallel \\ 2 \deg f_* \omega_{X/Y} \,.\, c_1(\omega_Y(S))}}{2 c_1(E^{20}) . c_1(\omega_Y(S))} \leq c_1^2(\omega_Y(S))$$

$\Rightarrow$ Arakelov ineq. for family of ab. v over surfaces, namely $\deg f_* \omega_{X/Y} \cdot c_1(\omega_Y(S)) \leq \frac{2}{4} c_1(\omega(Y))^2$ //

For $f : X \to Y$ g-dim ab. v. fibration, change to $g/4$.

equality $\Rightarrow$ Higgs bundle flat & split (Simpson).

for surface case, $\Rightarrow \mu(E^{02}) = \mu(S^2 T_Y(\log S) \otimes E^{2,0})$

$\Rightarrow (E^{0,2})^{\otimes 2} \leftarrow S^2(T_Y(\log S))$ with the same slop

$\Rightarrow$ by Yau's thm, $T_Y(\log S) = L_1 \oplus L_2$ with the same slop.

uniformization $\Rightarrow \widetilde{Y \setminus S} = H \times H$.

But we may go further to unifomation of fiber.

$\Rightarrow (F, \theta) \underset{\text{split}}{\hookrightarrow} \Lambda^2_{\text{prim}}(R^1 f_* \mathbb{C}_X)$

$\Lambda^2(R^1 f_* \mathbb{C}_X) = \underset{\text{irred. repr.}}{W} \oplus \overset{\shortparallel}{\text{Rest}}$ rk 1 local system, (1,1) part

the Galois conjugation fixes this

hence define / $\mathbb{Q}$

$\Rightarrow$ After base change, $\Rightarrow$ Rest $= \mathbb{Q}$,

under this identification,

$$\mathbb{Q} \subset \underset{\substack{\cup \\ \Lambda^2(R^1 f_* \mathbb{C}_X)}}{\text{End}(R^1 f_* \mathbb{Q}_X)} \text{ trace free,}$$

Want to use this $\mathbb{Q}$ to split the original Hodge str.

$$R^1 f_* \mathbb{Q}_X \xrightarrow[\eta]{} R^1 f_* \mathbb{Q}_X \qquad \text{(rk 2, need to take quadratic extension possibly)}$$

$$\rightsquigarrow (R^1 f_* \mathbb{Q}_X)_{\otimes \mathbb{C}} = \mathbb{V}_1 \oplus \mathbb{V}_1$$

There are 2 cases:

p.17 ① $R^1 f_* \mathbb{Q}_X = \mathbb{V}_1 \oplus \mathbb{V}_2$

② $(R^1 f_* \mathbb{Q}_X) \otimes \mathbb{Q}(\sqrt{a}) = \mathbb{V}_1 \oplus \mathbb{V}_2$

① ⇒ $X \xrightarrow{f} Y = \begin{array}{c} X_1 \times X_2 \\ \downarrow \\ Y_1 \times Y_2 \end{array}$ universal family of elliptic curves.

② ⇒ $f: X \to Y$ univ family of Hilbert modular surface (by definition).

- Theorem 2.

Let $f: X \to Y$ be s.s. family of ab.v over proj surface with big nef $\omega_Y(S)$, then

1) $c_1(f_* \omega_{X/Y})\, c_1(\omega_Y(S)) \le \frac{9}{4} c_1^2(\omega_Y(S))$.

2) if "=" and the Griffiths-Yukawa coupling $\neq 0$, Then $f: X \to Y$ is a Shimura family over a generalized Hilbert modular surface.

- Thm 3:

let $f: X \to Y$ be a family of 3-dim ab.v. over proj surface, assume the Griffiths-Yukawa coupling $= 0$, then

(1) $c_1(f_* \omega_{X/Y})\, c_1(\omega_Y(S)) \le \frac{2}{3} c_1(\omega_Y(S))^2$.

(2) "=" in 1) ⇒ $f: X \to Y$ is 1 Shimura family over a Picard modular surface $\mathbb{B}^2/\Gamma$, $\Gamma \subset SU(2,1) \cap SL_2(3, \mathbb{Q}(e^{2\pi i/3}))$

On Hirzebruch proportionality:

$$\begin{tikzcd} & X \arrow[d, "f"] & \\ C \arrow[r, hook] & Y \arrow[r] \arrow[dr] & H^2/\Gamma \\ & & \mathbb{B}^2/\Gamma \end{tikzcd}$$

⇝ Higgs bundle in 2 cases

$\Omega^1_Y(\log S) = \mathcal{L}_1^2 \oplus \mathcal{L}_2^2$

$(\mathcal{L}_1 \oplus \mathcal{L}_1^{-1}, \theta_1) \oplus (\mathcal{L}_2 \oplus \mathcal{L}_2^{-1}, \theta_2)$

$(\omega_Y(S)^{1/3} \oplus T_Y(S) \otimes \omega_Y^{\frac{1}{3}}, \theta)$

$\oplus\ \Omega^1_Y(\log S) \otimes \omega_Y(S)^{1/3}$

$\oplus\ \cancel{\mathcal{L}}\, \omega_Y(S)^{1/3}, \theta)$

pull back Higgs $|_C$

⇒ Arakelov inequality for $f: X \to C$, get

$$\left\{ \begin{array}{l} -\omega_Y(S).C \le 2C^2 \\ -\omega_Y(S).C \le 3C^2 \end{array} \right.$$

Hirzebruch had remarked "=" ⇒ $f: X \to C$ Shimura, but now it's proved.

Final topic : (worked out just yesterday 8/16 !) p.18

use curve in surface to characterize the surface to be Shimura.

<u>claim</u>: Let $f: X \to Y$ family of g-dim ab. v. st. induced map $\varphi: Y \to \bar{A}_g$ is an imbedding.

Assume i) $\exists$ some Shimura curve $C_i \subset Y$ st $\sum C_i$ big & nef

ii) $d\varphi : T_Y(\log S)|_{C_i} \to T_{A_g}(\log S)|_{C_i}$ splits as holo sub bundle

(This is a necessarily condition since for Shimura it really splits globally, not just on C_i)

(this condition is equivalent to $-\omega_Y(S).C_i = 2C_i^2$ Hirzebruch's proportionality .)

$\Rightarrow$ $f: X \to Y$ is a Shimura family over a generalized Hilbert modular surface.

Idea of Pf :

$\wedge^g_{prim} R^1 f_* \mathbb{C}_X \rightsquigarrow$ Higgs bundle$(\bigoplus_{p+q=n} E^{p,q}, \theta)$

Try to show $E^{g,0} \xrightarrow[\substack{\| \\ d\varphi}]{\theta} E^{g-1,1}$ splits orthogonally.

use Margulis's super rigidity $\Rightarrow$ arithmetic everything.

Just as before many times,

use $(F,\theta) \subset (\bigoplus E^{p,q}, \theta)$ gen by $E^{g,0}$ and θ

To show $\deg(F,\theta) = 0$. by assumption, get

$(F,\theta)|_{C_i} = VHS|_{C_i}$

in particular, $\det F . C_i = 0$. Sum up, get

$(\det F)(\underbrace{\Sigma_i C_i}_{\text{big \& nef}}) = 0$

Simpson's poly-stability $\Rightarrow$ $(F,\theta) \xhookrightarrow[\text{splits}]{} (\bigoplus E^{p,q}, \theta)$ $\Rightarrow$ $d\varphi$ splits orthogonality □

since splits also the Hodge metric $\uparrow$