Gekeler 1999 March at Taiwan

Arithmetic on Drinfeld Modular Forms

CONTENTS

References

Gekeler 1999 3/11 at 中研院 I

— Arithmetic on Drinfeld Modular Forms —

Review of classical modular forms

Elliptic curves over $\mathbb{C}$ and lattices in $\mathbb{C}$

$\Lambda \subset \mathbb{C}$ lattice, ie discrete subgp $\mathbb{C}/\Lambda$ cpt

$\Leftrightarrow \Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$,

ω_1, ω_2 $\mathbb{R}$-linearly independent

$f: \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ Λ-periodic $\Leftrightarrow f(z+\lambda) = f(z) \quad \forall \lambda \in \Lambda$

construction of functions which is Λ-periodic

$$P_\Lambda(z) = \text{"} \sum_{\lambda \in \Lambda} \frac{1}{z-\lambda} \text{"} \quad \text{not convergent}$$

actually: $$P_\Lambda(z) := \frac{1}{z^2} + \sum_{0 \neq \lambda \in \Lambda} \left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right]$$

converges locally uniformly Get P_Λ well-defined meromorphic function on $\mathbb{C}$, Λ-periodic, obvious poles

$$P_\Lambda'(z)^2 - 4P_\Lambda^3(z) \underset{\uparrow}{=} -g_2(\Lambda) P_\Lambda(z) - g_3(\Lambda)$$

(cancel 6 order pole)
direct computation
shows only remains order 2 pole ; hence ↑

for uniquely determined constant $g_2(\Lambda)$, $g_3(\Lambda)$

This we get the Weierstrass equation

- $g_2(\Lambda)$, $g_3(\Lambda)$ given as lattice sums, eg

$$g_2(\Lambda) = 60 \sum{}' \frac{1}{\lambda^4} =: 60\, E_4(\Lambda)$$

$$g_3(\Lambda) = 140 \sum{}' \frac{1}{\lambda^6} =: 140\, E_6(\Lambda)$$

$E_k(\Lambda)$ Eisenstein series of weight k

- $4x^3 - g_2x - g_3$ has 3 different zeros

$\Rightarrow y^2 = 4x^3 - g_2x - g_3$ nonsingular affine cubic curve

Definition Elliptic curve E/K, K field

- abelian variety of dim one
- algebraic curve of genus one with a distinguished pt
- nonsingular projective cubic curve, ie $E \hookrightarrow \mathbb{P}^2/K$ given by a cubic eq'n

(It is not completely trivial to see the equivalence)

E^{proj} : $zy^2 = 4x^3 \quad g_2xz^2 - g_3z^3$

then $E^{aff} \longleftrightarrow E^{proj}$ by $(x, y) \mapsto (x : y : 1)$

hence $E^{proj}(K) = E^{aff}(K) \cup \{(0:1:0)\}$

$\uparrow$ this is the distinguished K-rat'l point.

$z \mapsto (\wp(z), \wp'(z))$ get

$$(\mathbb{C}-\Lambda)/\Lambda \xrightarrow{\sim} E^{aff}_{\Lambda}(\mathbb{C}) \hookrightarrow \mathbb{A}^2_{\mathbb{C}} = \mathbb{C}\times\mathbb{C}$$

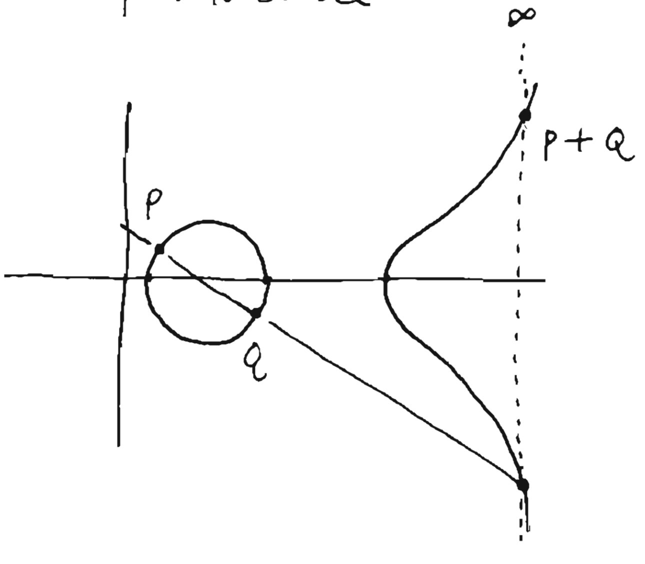
$\Rightarrow \mathbb{C}/\Lambda \xrightarrow{\sim} E^{proj}_{\Lambda}(\mathbb{C})$ with $\bar{\Lambda} \mapsto (0:1:0)$

(defines a group structure on $E^{proj}_{\Lambda}(\mathbb{C})$.)

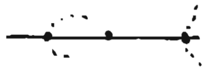
So, in general field K, if char $K \neq 2, 3$, then

E : $y^2 = 4x^3 - g_2x - g_3$ makes sense

Group structure



chordent-tangent rule

get 3 nontrivial pts of order 2 

Theorem (Weierstrass) Bijection

$$\left\{\begin{array}{l}\text{lattices in } \mathbb{C}\\ \text{modulo } \mathbb{C}^\times\end{array}\right\} \cong \left\{\begin{array}{l}\text{isomorphism classes}\\ \text{of elliptic curves } /\mathbb{C}\end{array}\right\}$$

$$\Lambda \longmapsto E_\Lambda$$

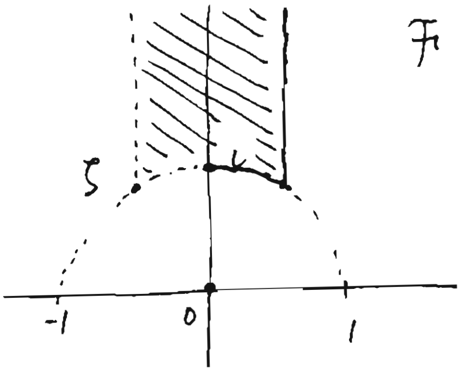
the RHS is a purely algebraic object, the LHS is an analytic object the thm gives the possibility to apply analytic methods

lattice / $\mathbb{C}^\times$-action can be given by, let $\Gamma = SL(2, \mathbb{Z})$

$$\Gamma \backslash \{\mathbb{Z}\omega + \mathbb{Z} \mid \omega \in H, \text{ ie } \operatorname{Im}\omega > 0\}$$

ie. $\Gamma \backslash H \xrightarrow{\sim} \{\text{lattices in } \mathbb{C}\}/\mathbb{C}^\times = \left\{\begin{array}{l}\text{classes of}\\ \text{elliptic curves } /\mathbb{C}\end{array}\right\}$

Γ acts on H through $\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d}$



$\mathcal{F}$ = "fund domain"

$= \{ z \in \mathbb{C} \mid |\operatorname{Re}(z)| \leqq \frac{1}{2}, |z| \geqslant 1 \}$

topologically –

j-invariant

for $E/\mathbb{C}$: $y^2 = 4x^3 - g_2 x \quad g_3$

$\Delta(E) :=$ discriminant $= g_2^3 - 27 g_3^2$

($\neq 0 \iff$ distinct roots of the cubic poly)

$$j(E) := 728 \frac{g_2^3}{\Delta}$$

Theorem : $E \cong E' \iff j(E) = j(E')$

(even true for arbitrary base field K but need to be careful when char = 2, 3)

corollary : $\{E/\mathbb{C}\}/\text{isom} \xrightarrow{\sim} \mathbb{C}$ by $E \mapsto j(E)$

Get rather nontrivial isom (analytic spaces)

$$\Gamma\backslash H \xrightarrow{\sim} \text{lattices}/\mathbb{C}^\times \xrightarrow{\sim} \begin{matrix}\text{elliptic}\\ \text{curves}/\mathbb{C}\\ \text{isom classes}\end{matrix} \xrightarrow[j]{\sim} \mathbb{C}$$

Define functions $g_2, g_3 : H \longrightarrow \mathbb{C}$

eg g_2 :

$$z \mapsto g_2(z) = g_2(E_{\mathbb{Z}z+\mathbb{Z}}) = 60 \sum_{\lambda \in \mathbb{Z}z+\mathbb{Z}}' \frac{1}{\lambda^4}$$

$$= 60 \sum_{a,b\in\mathbb{Z}}' \frac{1}{(az+b)^4}$$

$$g_3(z) = 140 \sum_{a,b\in\mathbb{Z}}' \frac{1}{(az+b)^6}$$

get $H \longrightarrow \mathbb{C}$

$$z \mapsto j(z) = 1728 \frac{g_2^3(z)}{g_2^3(z) - 27\, g_3^2(z)}$$

and induces $\Gamma\backslash H \xrightarrow{\sim} \mathbb{C} \Rightarrow \mathbb{A}^1_{\mathbb{C}} \hookrightarrow \mathbb{P}^1/\mathbb{C}$

Get naturally compatification of $\Gamma\backslash H$ "analytically"

All we really want to know is the <u>Eisenstein series</u>

$$E_k : H \longrightarrow \mathbb{C} \qquad z \mapsto \sum_{a,b\in\mathbb{Z}}' \frac{1}{(az+b)^k}$$

converges if $k \geqslant 3$ and unif defines holo fcn E_k on H

$E_k = 0$ if k is odd $\quad g_2 = 60\, E_4, \quad g_3 = 40\, E_6$

One line computation

$$E_k\left(\frac{az+b}{cz+d}\right) = (cz+d)^k E_k(z)$$

Definition of modular form :

A modular form for Γ of weight $k \in \mathbb{N}$ is a function $f : H \longrightarrow \mathbb{C}$ with

(i) $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \quad \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma = SL(2,\mathbb{Z})$

(ii) f is holomorphic, even at ∞ ($\leftarrow$ called (iii)) explaination about holo at ∞ :

Suppose f satisfies (i) and (ii) :

$$f(z+b) = f(z) \quad \forall\, b \in \mathbb{Z}$$

then f(z) has Fourier expansion

$$f(z) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n z} =: \sum a_n q^n$$

$$(q(z) := e^{2\pi i z})$$

(iii) means $\underline{a_n = 0 \text{ if } n < 0}$

$M_k = M_k(\Gamma) := \{ f \mid f \text{ modular of weight } k \}$

- $f \in M_k,\ g \in M_{k'} \Rightarrow f g \in M_{k+k'}$
- all M_k with different k are linearly independent

$\Rightarrow$ $M := \bigoplus_{k \geq 0} M_k$ is an algebra over $\mathbb{C}$

Theorem $0 \neq f \in M_k$, et $v_z(f)$ be vanishing order at z

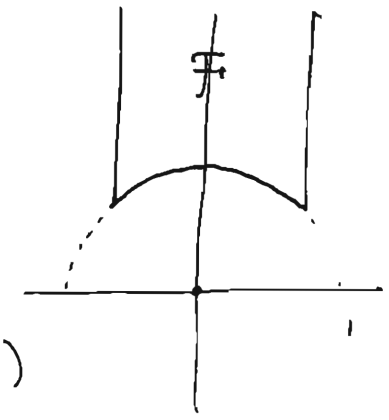
$$\Rightarrow \sum_{\substack{z \in \Gamma \backslash H \\ z \not\sim \zeta, i}} v_z(f) + \frac{1}{3} v_\zeta(f) + \frac{1}{2} v_i(f) + v_\infty(f) = \frac{k}{12}$$

Idea of pf: Contour integral of

$$\int_{\partial \mathcal{F}} \frac{f'(z)}{f(z)}\, dz$$

and then apply the residue thm □

(all of the above, see Serre's book: a course in Arithmetic)



Corollary: $\dim M_k < \infty$ $\forall k$, in fact

$\dim M_k$ is completly determined, even more

$M = \mathbb{C}[E_4, E_6]$ and E_4, E_6 algebraicaly indep

(so $\dim M_k = \{ (i,j) \mid 4i + 6j = k \}$ and

$\dim M_{k+12} = \dim M_k + 1$, by $\cdot \Delta$!

to be contined

Giekeler 1999 3/12 at 中研院 II.

modular form for $SL(2,\mathbb{Z}) =: \Gamma$

$M := \oplus M_k$, $0 \neq E_k \in M_k$ if $k \geq 4$, even

where $E_k(z) := \sum'_{a,b \in \mathbb{Z}} \frac{1}{(az+b)^k}$ Eisenstein series

$= 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!} \sum_{n>0} \sigma_{k-1}(n) q^n$

— the Fourier expansion of $q(z) := e^{2\pi i z}$

$\sigma_{k-1}(n) := \sum_{d | n} d^{k-1}$

$2\zeta(k)$ is called $E_k(\infty)$

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ the Riemann zeta function

$\zeta(k) = \frac{(2\pi)^k}{2 k!} |B_k|$, B_k Bernouli number by

(this only for k even) nothing is know for k odd except that $\zeta(3)$ is irrational

$\frac{t}{e^t - 1} =: \sum_{k \geq 0} B_k \frac{t^k}{k!}$

$B_k \in \mathbb{Q}$: $B_0 = 1$, $B_2 = 1/6$, $B_4 = -1/30$

$B_1 = -1/2$, $B_{2k+1} = 0$ $\forall k \in \mathbb{N}$

$\Rightarrow E_k(z) = 2\zeta(k)\left[1 - \frac{2k}{B_k} \sum_{n>0} \sigma_{k-1}(n) q^n\right]$

inside part are all rarional $=: \tilde{E}_k$

Know that $M = \mathbb{C}[E_4, E_6] = \mathbb{C}[g_2, g_3]$, eq

$M_0 = \mathbb{C}$

$M_2 = 0$

$M_4 = \underline{\mathbb{C} E_4}$

$M_6 = \underline{\mathbb{C} E_6}$

$M_8 = \mathbb{C} E_4^2 = \mathbb{C} E_8$

$M_{10} = \mathbb{C} E_4 E_6 = \mathbb{C} E_{10}$

$M_{12} = \langle E_6^2, E_4^3 \rangle = \langle E_{12}, \Delta \rangle$

where $\underline{\Delta = g_2^3 - 27 g_3^2}$

$\Delta(z) \neq 0$ if $z \in H$

but its Fourier exp $= 0$ at ∞

hence Linear. indep with E_{12}

In general $M_{k+12} = \Delta M_k \oplus \mathbb{C} E_{k+12}$

These lead to nontrivial identities in number theory like

$$\tilde{E}_4^{\,2} = \tilde{E}_8$$

$$\tilde{\hat{E}}_4 \, \tilde{\hat{E}}_6 = \tilde{\hat{E}}_{10}$$

$$\Rightarrow \quad \sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\,\sigma_3(n-m)$$

and σ_9 in terms of σ_3, σ_5

No elementary proof is known for these formulae !!
For higher E_k this method breaks down since may have many choices, but the study of

$$\left(\tilde{\hat{E}}_k \; \tilde{\hat{E}}_\ell \quad \tilde{\hat{E}}_{k+\ell} \right)$$ leads to cusp forms

whose growth for the Fourier coefficients is much slower than other modular forms (analytic number theory)

In general, $E_k = P_k(E_4, E_6)$, $P_k \in \mathbb{Q}[x, y]$
with very complicated coefficients

$\exists$ recursion formula $P_k \leftarrow P_{k-2}, P_{k-4} \quad P_4$
but even with computer, can only compute up to $k \sim 120$
(Deligne has results for $k = p-1$, p:prime)
Will see this problem becomes easier in the char p case

$$\Delta = g_2^3 - 27 g_3^2 = (60 E_4)^3 - 27 (140 E_6)^2$$

$$= (2\pi i)^{12} \left(q - 24 q^2 + 252 q^3 - 1472 q^4 + \cdots \right)$$

this expansion partially explains the choice of previous coeff 60, 140 the inner part is $\mathbb{Z}$ coeff. is not accident

$$\underset{\text{Jacobi Product formula}}{\overset{!}{=\!=}} (2\pi i)^{12} \, q \prod_{n \geq 1} (1 - q^n)^{24} \in (2\pi i)^{12} \mathbb{Z}[[q]]$$

An elementary pf of Jacobi formula is given in Serre's book (only uses ~~Advanced Calculus~~), but does not show any structure of the inside Math

More natural proof use the theory of <u>elliptic curves</u>

$$q\prod(1-q^n)^{24} := \sum_{k>0} \tau(k)\, q^k \quad , \quad \tau(1)=1$$

$\Rightarrow \tau(kl) = \tau(k)\,\tau(l)$ if $(k,l)=1$ (very non obvious)

$\underline{\tau(p) \leq 2\, p^{11/2}}$ p : prime

this is Ramanujan Conj, Proved by Deligne 1973

This is really important, related to motive theory and Galois representations (eq $\nexists$ elliptic curve $/\mathbb{Q}$ everywhere un.ramf)

Now some easier: $\prod(1-q^n)^{-1} = \sum_{n\geq 0} p(n)\, q^n$

p the partition function; so is related to modular forms

Go back:

$$\begin{tikzcd}
 & \text{lattice} \arrow[d] & & & \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{C} \arrow[r] & \mathbb{C}^\times \arrow[r] & 1
\end{tikzcd}$$

$$z \mapsto e^{2\pi i z}$$

$$\mathbb{G}_{m/\mathbb{C}} = \mathbb{C}^\times = \mathbb{C}/\mathbb{Z}$$

analogously: $E/\mathbb{C} = \mathbb{C}/\Lambda$, or for general alg gp

$$\mu_n := \{z \in \mathbb{C} \mid z^n = 1\} = {}_n\mathbb{G}_m$$

$\mathbb{G}_m/\mathbb{Q}$: $\mathbb{Q}({}_n\mathbb{G}_m) = \mathbb{Q}(\mu_n)$

$\mathbb{Q}(\mu_n)/\mathbb{Q}$ is Galois with group $(\mathbb{Z}/n)^\times$

$$\bar{b}(\zeta) = \zeta^b \,, \quad \zeta^n = 1$$

Kronecker-Weber thm

Every finite ab. ext of $\mathbb{Q}$ is contained in some $\mathbb{Q}(\mu_n)$

Alg # theory = study of $\overline{\mathbb{Q}}/\mathbb{Q}$ and its Galois gp

K-W thm study the part

$$\begin{tikzcd} & \overline{\mathbb{Q}} \arrow[d, dash] \\ \text{the maximal ab ext}/\mathbb{Q} \arrow[r] & \mathbb{Q}^{ab} \arrow[d, dash] \\ & \mathbb{Q} \end{tikzcd}$$

Can do analogue using elliptic curves

E/K, K alg number field $\subset \mathbb{C}$

$E = E_\Lambda$, ${}_nE_\Lambda = \{ x \in E(\mathbb{C}) \mid nx = 0 \} \cong \mathbb{Z}/n \times \mathbb{Z}/n$

non canonically (need to choose basis)

this set need to be solved in a finite ext of K called $K({}_nE)$

In fact, $K({}_nE)/K$ is finite Galois, let $G = Gal(K({}_nE)/K)$ then $G \hookrightarrow Aut({}_nE) \cong GL(2, \mathbb{Z}/n)$

ramification of $K({}_nE)/K$ is determined by E/K (not easy to prove)

- Ie Elliptic curves lead to a construction of (maybe) non-abelian Galois extension which can be well-controlled
- For non-CM elliptic curve (which is the general case) Serre got quite precise result about G Produce very large Galois extensions So have a chance to go beyond $\mathbb{Q}^{ab}$ and generate an important part of it, and may be enough to solve questions about $\overline{\mathbb{Q}}/\mathbb{Q}$ for some problems? (even consider (semi-) abelian V)

Same game for ground field $K/\mathbb{Q}$

But much easier if replace K by function fields

(Here the term motive really mean $\mathbb{G}_m$)

START THE GAME

$\mathbb{Z}$	A
$\mathbb{Q}$	K
$\mathbb{R} = \hat{\mathbb{Q}}$ w.r.t "$\lvert\ \rvert$"	$K_\infty = \hat{K}$ w r t $\lvert\ \rvert$
$\mathbb{C}$	$C := \widehat{\overline{K_\infty}}$ since $\overline{K_\infty}$ is not complete fortunately $\widehat{\overline{K_\infty}}$ is complete and also alg closed !
$H^{\pm} = \mathbb{C} - \mathbb{R}$	
$\Gamma = GL(2,\mathbb{Z})$	

$\mathcal{C}$ algebraic curve over $\mathbb{F}_q$ ($\# \mathbb{F}_q = q$)

$\infty \in \mathcal{C}$ place, fixed once for all

$K = \mathbb{F}_q(\mathcal{C})$ = function field of $\mathcal{C}$ over $\mathbb{F}_q$

(Ex. $\mathcal{C} = \mathbb{P}^1$, "$\infty$" = ∞ point at ∞ of $\mathbb{P}^1/\mathbb{F}_q$ with valuation $v_\infty\left(\frac{a}{b}\right) = \deg b - \deg a$, $K = \mathbb{F}_q(T)$)

A = affine ring of the affine curve $\mathcal{C} - \{\infty\}$, a Dedekind ring (eg. $\mathcal{C} = \mathbb{P}^1$ case $A = \mathbb{F}_q[T]$)

(will assume for simplicity that $\mathcal{C}$ is non-singular)

$K = \mathrm{Quot}(A)$, v_∞ = valuation (normalized) on K at ∞

get $\lvert\ \rvert$, (eg. $\mathcal{C} = \mathbb{P}^1$ case, $\left|\frac{a}{b}\right| = q^{-v_\infty\left(\frac{a}{b}\right)} = q^{\deg a - \deg b}$)

$\lvert x \rvert := q^{-v_\infty \cdot d_\infty}$ where $d_\infty := \deg [\mathbb{F}_q(\infty) : \mathbb{F}_q]$

$K_\infty = \hat{K}$ w.r.t $\lvert\ \rvert$, (eg. $\mathcal{C} = \mathbb{P}^1$, $K_\infty = \mathbb{F}_q((\pi_\infty))$, $\pi_\infty = \frac{1}{T}$)

$C := \widehat{\overline{K_\infty}}$ has good function theory like $\mathbb{C}$,

but no simple formula for C is known , so we usually argue indirectly

to be contined .

C is complete wrt $|\cdot| =$ ext of $|\ |$ on K_∞
and algebraically closed (due to Krasner)

- $|$ non-archemedean $\begin{cases} |x+y| \leq \max(|x|, |y|) \\ |x-y| = |x| \text{ if } |y| < x \end{cases}$

$\sum_{i=1}^\infty x_i$ converges $\Leftrightarrow$ $x_i \to 0$
$\prod_{i=1}^\infty x_i$ converges $\Leftrightarrow$ $x_i \to 1$

So the analysis is much easier than in $\mathbb{C}$ or $\mathbb{R}$

- $B_r(x) = \{ y \in C \mid |y-x| \leq r \}$ "closed ball"

$B_r^\circ(x) = \{ y \in C \mid |y-x| < r \}$ "open ball"

(Caution: in fact both B_r, B_r° are open & closed in the top defined here on C)

$(C, |\cdot|)$ is totally disconnected ! ← cause problems !
but we need like "identity thm" or "analytically continuation theorem" so dis-connectedness is BAD

Solution: Find the right notion of holomorphic functions (should including polynomials) so that the identity thm holds

Solution by J Tate : use power series :

- $\sum_{i\geq 0} a_i x^i$ power series over C

$\rho :=$ radius of convergence given by ~~Abel's formula~~
$\overline{\lim} |a_i|^{1/i} = \rho^{-1}$

- $f : C \to C$ entire function $\Leftrightarrow$ f given by an everywhere convergent power series

$Z(f) := \{ z \in C \mid f(z) = 0 \}$

$D \subset C$ discrete $\Leftrightarrow$ $D \cap B_r(x)$ finite $\forall x \in C$, $r > 0$

(Notice that the field C is not loc cpt)

- f entire $\Rightarrow$ $Z(f)$ discrete
- f entire $\Rightarrow$ $f(z) = \text{const} \prod_{\substack{a_i \in Z(f) \\ 0 < |a_i| \leq 1}} (z-a_i)^{v_i} \prod_{\substack{a_i \in Z(f) \\ |a_i| > 1}} \left(1 - \frac{z}{a_i}\right)^{v_i}$

  this is definitely wrong in the $\mathbb{C}$ case, eg e^z

- Conversely, let D be a divisor on C with discrete support (ie $D = \sum v_i \cdot a_i$, $a_i \in C$) then there exists an entire function $f(z)$ with $\operatorname{div} f = D$, f unique up to C^*

(Rmk here a div may be a infinite sum but $\cap B_r$ is always finite : (discrete support))

- f entire, non-constant $\Rightarrow$ f surjective

<u>Drinfeld Upper Half Plane</u> $\Omega := C - K_\infty$

Group $\Gamma := GL(2, A)$

via $\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d}$

Analogue of the imaginary part :

$$|z|_i := \inf_{x \in K_\infty} |z - x| \quad (\text{in fact} = \min$$
since K_∞ is ocally cpt)

(classically $im(z) = \min_{x \in \mathbb{R}} |z-x|$)

Properties

(I) $|z|_i = 0 \iff z \in K_\infty$

(II) $|cz|_i = |c|\,|z|_i$ if $c \in K_\infty$

By definition we have $|K_\infty^\times| = q^{d_\infty \mathbb{Z}}$, now have $|C^\times| = q^{\mathbb{Q}}$ (rmk: an alg closed field can not carry a discrete valuation)

$C \supset O_C := B_1(0) \supset m_C := B_1^\circ(0)$ and

$$\begin{tikzcd} O_C/m_C \arrow[r, phantom, "="] & \overline{\mathbb{F}_q(\infty)} \arrow[r, phantom, "="] & \overline{\mathbb{F}_q} \\ O_\infty/m_\infty \arrow[u] \arrow[r, phantom, "="] & \mathbb{F}_q(\infty) \arrow[u] & \end{tikzcd}$$

: This part has no classical analogue since it's archemedean

(III) $z \in C$, $|z| \notin q^{d_\infty \mathbb{Z}} \Rightarrow |z| = |z|_i$

(IV) $|z| =$, then $|z|_i < 1 \Leftrightarrow \bar{z} \in \mathbb{F}_q(\infty)$

(V) $|\gamma z|_i = \dfrac{|z|_i}{|cz+d|^2}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, K_\infty)$

this is a nice exercise

Analytic Structure on Ω:
("schemes" with analytic structure taken care)
the following is the crude way, but may not be the best

$A \qquad q_\infty := q^{d_\infty}$

K

$K_\infty := \hat{K} \hookleftarrow O_\infty$ ring of integers, $\hat{}$ wrt $|\ |$

$C := \hat{\bar{K}}_\infty$

$\Omega := C - K_\infty$

$\Gamma := GL(2, A)$

$z|_i := \min_{x \in K_\infty} |z - x|$

$U_n := \{ z \in C \mid |z| \leq q_\infty^n, |z|_i \geq q_\infty^{-n} \}$, $n \in \mathbb{N}$

$\Omega = \bigcup_{n \in \mathbb{N}} U_n$

<u>Definition</u>: f holomorphic on U_n if and only if
$f = \lim_{i \to \infty} f_i$ uniform limit of rational functions f_i without poles on U_n

f meromorphic function on $U_n \Longleftrightarrow \exists$ rat'l funct g st fg holomorphic

f holomorphic / meromorphic on $\Omega \Longleftrightarrow$
$f|_{U_n}$ is holo/mero for each n

<u>Examples</u>: (1) any rat'l function f without poles on Ω is holomorphic on Ω
(ie. poles only on $\mathbb{P}^1(K_\infty) = K_\infty \cup \{\infty\}$)

(ii) $E_k(z) = \sum_{a,b\in A}{}' \frac{1}{(az+b)^k}$ "Eisensten Series",

holomorphic on Ω for $k \geqslant 1$

but in fact $E_k \equiv 0$ if $k \not\equiv 0 \ (q-1)$

$E_k(\gamma z) = (cz+d)^k E_k(z)$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

Similarly, analytic structure on

- $C = \bigcup_{r\in\mathbb{N}} B_r(0)$

(Rmk: Ball $B_r(0)$ is like affine scheme in alg geom)

eg funct holo on $B_r(0) \Longleftrightarrow$ given by power series converges on the whole ball

- $\mathbb{P}^1(C) = C \cup \{\infty\} = B_1(0) \cup B_1(\infty)$

$B_1(\infty) := \{z \in C \mid |z| \geq 1\} \cup \{\infty\} \cong B_1(0)$ via $z \mapsto \frac{1}{z}$

Since we got 3 different definition about entire/holo, we need the following facts:

- f on C holomorphic $\Longleftrightarrow$ f entire
- f on $\mathbb{P}^1(C)$ holomorphic $\Longleftrightarrow$ f constant
- f on $\mathbb{P}^1(C)$ meromorphic $\Longleftrightarrow$ f rational

What's the analogue of "elliptic curve" ??

Answer is : Drinfeld Module (of rank 2)

Definition. A lattice Λ in C is a finitely generated A-submodule Λ of C which is discrete

~~Example~~ (i): $A \underset{\text{discrete subring}}{\hookrightarrow} K_\infty \hookrightarrow C$

(ii) $\omega \in \Omega$, $\Lambda := A\omega \oplus A \underset{\text{discrete}}{\hookrightarrow} C$

since $\omega, 1$ are K_∞-linearly independent

Λ f g + torsion free over Dedekind ring A

$\Rightarrow$ Λ is projective (module)

In particular, for each $n \in A$, we have

$$\Lambda / n\Lambda \cong (A/n)^r \qquad r = \text{rk of } \Lambda = \dim_K \Lambda \otimes_H K$$

$$\Lambda \otimes K = K\Lambda \subset C$$

Warning: now we do not have the cpt lattice!

in fact may have arbitrary large rk

the classical case only have 2 cases:

$\mathbb{Z} \subset \mathbb{C}$ rk $= 1$, $\mathbb{Z}\omega + \mathbb{Z} \subset \mathbb{C}$ rk $= 2$

Definition: $e_\Lambda(z) := z \prod\limits_{\lambda \in \Lambda}' \left(1 - \frac{z}{\lambda}\right)$, $z \in C$

(where Λ is a lattice of proj rank $r \in \mathbb{N}$)

- ~~converges locally uniformly~~ to an entire fun e_Λ

(Rmk: unif limit of holo fcn is again holo)

- e_Λ is additive ($e_\Lambda(x+y) = e_\Lambda(x) + e_\Lambda(y)$)

  and even $\mathbb{F}_q$ - linear.

  ($\Lambda \cap B_r(0)$ finite dim $\mathbb{F}_q$ vs Λ_r

  $e_\Lambda = \lim\limits_{r \to \infty} e_{\Lambda_r}$; $e_{\Lambda_r} = z \prod\limits_{\lambda \in \Lambda_r}' \left(1 - \frac{z}{\lambda}\right)$ etc)

- $\mathrm{div}(e_\Lambda) = \Lambda$
- e_Λ is Λ-periodic: $e_\Lambda(z+\lambda) = e_\Lambda(z)$ $\forall \lambda \in \Lambda$
- e_Λ surjective (bec entire)
- $e_{c\Lambda}(cz) = c\, e_\Lambda(z)$

This is the "quasi" analogue of Weierstrass P_Λ

$0 \neq n \in A$:

$$\begin{tikzcd}
0 \arrow[r] & \Lambda \arrow[r] \arrow[d, "n"] & C \arrow[r, "e_\Lambda"] \arrow[d, "n"] & C \arrow[r] \arrow[d, "\phi_n^\Lambda"] & 0 \\
0 \arrow[r] & \Lambda \arrow[r] & C \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

ϕ_n^Λ : gp homo st diagram commutes

Basic properties ~~(to think for weekend~~!)

- $\phi^{\Lambda}_{n}(z)$ has the form :

$$= nz + l_1 z^{q} + l_2 z^{q^2} + \quad + l_N z^{q^N}$$

$N = r - \deg n$; $r = \text{rk of } \Lambda$

$\deg n = \log_q \# (A/n)$

with $l_N \neq 0$

(may use power series expansion of e_Λ to determine some structure of these l_i's)

- $n \mapsto \phi^{\Lambda}_{n}$ is a ring homomorphism from A to

$$\text{End}_{C, \mathbb{F}_q}(\mathbb{G}_a)$$
$$\shortparallel$$
$$\Big\{ \sum_{\text{finite}} l_i z^{q^i} \mid l_i \in C \Big\} = C\{\tau\}$$

ring with composition as multiplication

$\tau := (z \mapsto z^q)$, $\tau^i = (z \mapsto z^{q^i})$

$C\{\tau\}$ is a non commutative polynomial ring in τ :

$\tau x = x^q \tau$, $x \in C$

then $\sum l_i z^{q^i} = \sum l_i \tau^i$

(eg $a\tau^i \, b\tau^j = a b^{q^i} \tau^i \tau^j = a b^{q^i} \tau^{i+j}$) $\square$

(to be continued)

Gickeler 1999 3/16 at 中研院 IV

$A, K, \infty, K_\infty, C, |\ |, \Omega, GL(2,A)$
$\mathbb{Z}, \mathbb{Q}, \infty, \mathbb{R}, \mathbb{C}, |\ |, H, GL(2,\mathbb{Z})$

$\Lambda \subset C$ lattice, $rk(\Lambda) = r$
$e_\Lambda(z) := z \prod'_{\lambda\in\Lambda}\left(1-\frac{z}{\lambda}\right)$, $e_\Lambda : C \to C$ entire, $\mathbb{F}_q$-linear
$= \sum_{i\geq 0} \alpha_i z^{q^i}$
$End_{C,\mathbb{F}_q}(\mathbb{G}_a) = \{\sum \text{finite } a_i X^{q^i} \mid a_i \in C\} = C\{\tau\} \subset C[X]$
since field is infinite, may identify polynomial with polynomial maps.

$End_{C,\mathbb{F}_q}(\mathbb{G}_a)$: noncommutative ring wrt. composition
$\tau := X^q = (z \mapsto z^q)$ hence $\tau a = a^q \tau$
consider $C\{\tau\} \subset C_{ent}\{\{\tau\}\} \subset C\{\{\tau\}\}$
entire $\mathbb{F}_q$-linear functions, eq e_Λ.

Have both left and right Euclidean Algorithm
The right EA is cannical : $f, g \mapsto f = hg + r$
the left EA is not always possible unless C is alg. closed which is true now. In general need to take q-th roots.
So is "not canonical".

let $n \in A$,
$|n| > 1$

$$\begin{tikzcd}
0 \arrow[r] & \Lambda \arrow[r] \arrow[d, "n"] & C \arrow[r, "e_\Lambda"] \arrow[d, "n"] & C \arrow[r] \arrow[d, dashed, "\phi_n^\Lambda"] & 0 \\
0 \arrow[r] & \Lambda \arrow[r] & C \arrow[r, "e_\Lambda"] & C \arrow[r] & 0
\end{tikzcd}$$

· $\phi_n^\Lambda \in C\{\tau\}$, in fact $\phi_n^\Lambda :=$ const $\prod_{v \in e_\Lambda(n^{-1}\Lambda)} (x - v)$

$e_\Lambda(n^{-1}\Lambda) \cong n^{-1}\Lambda/\Lambda \cong \Lambda/n\Lambda \cong$ (non-canonical:) $(A/n)^r$
$\mathbb{F}_q$ vector space of dim $r \cdot \log_q |n|$.
$\Rightarrow$ the product formula is $\mathbb{F}_q$-linear

Now $e_\Lambda \circ \phi^\wedge_n(z)$ and $e_\Lambda(nz)$ are entire functions sharing same zeros. the actual scaling is

$$\phi^\wedge_n = nX \cdot \prod_v \left(1 - \frac{X}{v}\right)$$

- $n \mapsto \phi^\wedge_n$ is a ring homomorphism ($\mathbb{F}_q$-algebra homo)

$\phi^\wedge : A \to C\{\tau\}$. pf is straight-forward.

Via $\phi^\wedge$, C acquires a new structure as an A-module

$$a * z := \phi^\wedge_a(z)$$

This is called the Drinfeld Module structure

Definition. A Drinfeld module structure of rank $r \in \mathbb{N}$ on C is a module structure described through a morphism of $\mathbb{F}_q$-algebras

$$\phi : A \to C\{\tau\} \; ; \; \phi : n \mapsto \phi_n$$

satisfies (i) $\forall n \in A$, $\phi_n = nX$ + higher terms $= n \cdot \tau^0$ + higher
(the identity is not 0, is $X^{q^0} = X$)

(ii) $\deg_\tau \phi_n = r \log_q |n| = r \deg n$

First consequences.

If rk $r = 0$, then just get $\phi_n = nX$, ie. the original trivial A-module structure. And may say that a Drinfeld module structure is infinitesimal the same as the trivial one since for $nX + \cdot\cdot X^q + \cdots + \cdot\cdot X^{q^i}$, $\frac{d}{dX} = n$.

May think.

$E/\mathbb{C} = \mathbb{C}/\Lambda$ elliptic curve$/\mathbb{C}$, the Lie algebra simply goes back to $\mathbb{C}$ again (1st order diff op).

So Drinfeld module =
deformation of an additive group scheme which infinitesimally is the trivial one (diff. op)

In fact, for any subfield $L \subset C$ st $A \hookrightarrow L$
the same def gives Drinfeld module defined over L.
Even more let $\gamma : A \to L$, L any ring ($\mathbb{F}_q$-algebra)
then can define D-M / L st. $\forall n \in A$,

(I) $\phi_n = \gamma(n) X +$

(II) $\deg_\tau \phi_n = r \cdot \log_q |n| = r \deg n$

A homomorphism $u : \phi \to \psi$ of D modules is some $u \in L\{\tau\}$ st $\forall n \in A$: $u \circ \phi_n = \psi_n \circ u$.

Notice : the general notion for D modules contains like

$L = A/p$ finite field

$L = C$

$L = K, K_\infty$

only C, K_∞ have analytical meanning ! It is useful however to consider D modules over any field L.

For C, much more is known :

THEOREM (I) The association $\Lambda \mapsto \phi^\Lambda$ defines an equivalence of "small" categories between

$$\begin{pmatrix} \text{Cat. of lattices of} \\ \text{rk } r \text{ in } C \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{Drinfeld modules} \\ \text{of rk } r \text{ over } C \end{pmatrix}$$

Bijectivity of objects and morphisms !

sketch of proof :

(I) $\Lambda \mapsto \phi^\Lambda$ well-defined, ok.

(II) injective since e_Λ determines the lattice Λ, trivial

(III) surjectivity (hard part) :

let ϕ be a D module of rk $= r$: want to solve Λ.

(a) Firstly, Fix $n \in A$, $|n| > 1$

want Λ st. $e_\Lambda(nz) = \phi_n(e_\Lambda(z))$, so consider $e = \sum_{i \geq 0} \alpha_i \tau^i$ and solve $e\,n = \phi_n \cdot e$ in $C\{\{\tau\}\}$; ie.

$$\sum_{k \geq 0} \alpha_k \tau^k \cdot n = \left(\sum n_i \tau^i\right)\left(\sum \alpha_j \tau^j\right) = \sum_k \left(\sum_{i+j=k} n_i \alpha_j^{q^i}\right) \tau^k$$

get $\alpha_k = \frac{1}{n^{q^k} - n} \sum_{j<k} n_{k-j} \alpha_j^{q^{k-j}}$

get well-defined solution e in $C\{\{\tau\}\}$.

(b) Use estimate to show $e \in C_{ent}\{\{\tau\}\}$

i.e. α_k decays very fast to 0.

(c) The $e = e^{(n)}$ by construction depends on n, but in fact it is independent : (easy)

since $e^{(n)}$ is the unique solution of

$$e \cdot n = \phi_n \cdot e$$

raise powers get $e^{(n)} = e^{(n^2)} = e^{(n^3)} =$ $\Rightarrow$

$e^{(n)}$ only depends on $\mathbb{F}_q[n] \subset A$, not n itself

If $n, m \in A$ satisfies $|\ | > 1$ then

$\mathbb{F}_q[n] \cap \mathbb{F}_q[m] \supsetneq \mathbb{F}_q$ (Dedekind Domain, dim $= 1$)

$\Rightarrow$ independence of $e^{(n)}$ of n.

(d) Now let $\Lambda =$ zeros of e, then by construction, Λ is discrete $\mathbb{F}_q$ v.s. need to show is an A-module

$$e(nz) = \phi_n(e(z)) \quad n \in A$$

implies that $x \in \Lambda \Rightarrow ax \in \Lambda$

(e) Λ is discrete and torsion free over $A \Rightarrow \Lambda$ is A-projective.

(f) $\Lambda / n\Lambda \xrightarrow{\sim} \ker \phi_n$, need the deg estimate.

rank $\Lambda = r :=$ rk of $\phi \rightsquigarrow \phi^{\Lambda} = \phi$. end the proof $\square$

We are now approaching the main business — modular forms

Ex. $A = \mathbb{F}_q[T]$, $r = 1$, all rk 1 lattices have the form

Ax, $0 \neq x \in C$ (bec. it is a PID, rk 1 $\Rightarrow$ free)

$\phi : A \to C\{\tau\}$

$T \mapsto \phi_T = T + c\tau$

case $c = 1$, is called the Carlitz module (1960's)

ϕ Carlitz module $\leftrightarrow A\bar{\pi}$, $\bar{\pi}$ analogy $\leftrightarrow 2\pi i$.

Call $\Lambda = A\bar{\pi}$, ($\bar{\pi}$ is defined up to $\mathbb{F}_q$ = units in A
just like $2\pi i$ is unique up to ± 1 = units of $\mathbb{Z}$

All formulas for π like $\sum \frac{1}{n^2} = \pi^2/6$ have analogue for $\bar{\pi}$.

$e_\Lambda = e = \sum \alpha_i T^i$

$e \cdot T = \phi_T e = Te + e^q \longrightarrow \alpha_i = \frac{1}{D_i}$

$[K] := T^{q^k} - T \in A \qquad = \prod n$
monic primes deg $n|k$

$D_K := [K]\,[K-1]^q \quad [1]^{q^{k-1}} \qquad = \prod n \ldots \deg = k$

$L_K := [K]\,[K-1] \cdots [1] \qquad = \text{l.c.m of } (\cdots \deg = k)$

These formulas will be useful.

Next time: Torison Points. (to be continued).

Giekeler 1999 3/19 at Academia Sinica V

$A, K, K_\infty, C = \hat{\overline{K_\infty}}, \tau = X^q = (z \mapsto z^q)$

$C\{\tau\} = \{\sum_{\text{finite}} a_i X^{q^i} \mid a_i \in C\} \subset C_{ent}\{\{\tau\}\} \subset C\{\{\tau\}\}$

Drinfeld module $\phi : A \longrightarrow C\{\tau\}, \ n \longmapsto \phi_n$

D modules $\longleftrightarrow$ lattices in C ; $e_\Lambda(z) := z \prod'(1 - \frac{z}{\lambda})$

$\phi^\Lambda \longleftarrow\!\shortmid \Lambda$ $\in C_{ent}\{\{\tau\}\}$

uniquely determined by the functional eq'n

$$\phi^\Lambda_n(e_\Lambda(z)) = e_\Lambda(nz)$$

$u_c \ \phi^\Lambda \longrightarrow \phi^{\Lambda'}$; $(c\Lambda \hookrightarrow \Lambda')$, $c \in C^*$, then $\exists\, u_c$ st

homomorphism $e_{\Lambda'} = u_c\, e_\Lambda$, u_c is a polynomial

in $C_{ent}\{\{\tau\}\}$ $= c +$ higher terms in τ

$\deg u_c = \dim_{\mathbb{F}_q}(\Lambda'/c\Lambda)$

In general, u $\phi \longrightarrow \phi'$, $u \in C\{\tau\}$ is called a homomorphism if $u\,\phi_n = \phi'_n\, u$ $\forall\, n \in A$

Examples

(1) $A = \mathbb{F}_q[T]$, $rk = r = 1$ (Carlitz module, 1930~40)

$\phi_T = T + \tau =: TX + X^q$; $\phi \longleftrightarrow \bar\pi A$, $\bar\pi \dashleftarrow\dashrightarrow 2\pi i$

$\bar\pi A \dashleftarrow\dashrightarrow 2\pi i \mathbb{Z}$

$e_\Lambda = \sum_{i \geq 0} \frac{1}{D_i} \tau^i$; $D_i := [i][i-1]^q \quad [1]^{q^{i-1}}$

$[i] := T^{q^i} - T$

then can also define like zeta function

$$\zeta(s) = \sum_{n \in A,\, \text{monic}} \frac{1}{n^s} \in K_\infty, \ s \in \mathbb{N}$$ cf Goss' book

this case is not interesting since all $rk = 1$ lattice are free and isomorphic, so modular variety = one pt

$\phi_{T^2} := \phi_T \phi_T = (T+\tau)(T+\tau) = T^2 + (T+T^q)\tau + \tau^2$.

(2) $A = \mathbb{F}_q[T]$, $r = 2$

Drinfeld	elliptic
g	g_2, g_3
Δ	Δ

$\phi_T = T + g\tau + \Delta\tau^2$

ϕ, ϕ' isomorphic $\Leftrightarrow \exists\, c \in \mathbb{C}^\times$ st

$$\begin{matrix} c\phi_T & = & \phi'_T c \\ \parallel & & \parallel \\ cT + cg\tau + c\Delta\tau^2 & & cT + c^q g'\tau + c^{q^2}\Delta'\tau^2 \end{matrix}$$

$\Leftrightarrow \exists\, c \in \mathbb{C}^*$ st $g' = c^{1-q} g$, $\Delta' = c^{1-q^2}\Delta$

$\Leftrightarrow j(\phi) = j(\phi')$ where $j(\phi) := \dfrac{g^{q+1}}{\Delta}$

$$\begin{tikzcd} \left\{\begin{matrix}\text{isom classes}\\ \text{of rk 2}\\ \text{Drinfeld modules}\\ \text{over } \mathbb{C}\end{matrix}\right\} \arrow[d, "\wr"', "\phi \mapsto j(\phi)"] & \left\{\begin{matrix}\text{rk 2}\\ \text{lattices}\\ \text{in } \mathbb{C}\end{matrix}\right\}/\mathbb{C}^* \arrow[l, Rightarrow, "\sim"'] & G\backslash\left\{\begin{matrix}(\omega_1, \omega_2) \in \mathbb{C}\times\mathbb{C}\\ \omega_i \ K_\infty\text{-indep}\end{matrix}\right\}/\mathbb{C}^\times \arrow[l, "\sim"'] \\ \mathbb{C} \ni j(\phi) & & \end{tikzcd}$$

$G = GL(2, A)$

in the elliptic case the surjectivity is not completely trivial but now it's very obvious!

But the very RHS set is in fact $GL(2, A)\backslash\Omega$ via $(\omega_1, \omega_2) \longmapsto \omega_1/\omega_2 \in \Omega$

This equipped "smooth Riemann Surface str" on $G\backslash\Omega$

Rmk: this chain of isomorphisms works for general A except the j-inv part in fact, $G\backslash\Omega$ would be of higher genus !!

A general, r = rank arbitrary.

Eisenstein series

$$E_k(\Lambda) := {\sum}'_{\lambda\in\Lambda} \frac{1}{\lambda^k} \quad (\neq 0 \Leftrightarrow k \equiv 0 \ (q-1))$$

$e_\Lambda = \sum \alpha_i \tau^i \in \mathbb{C}_{\text{ent}}\{\{\tau\}\}$

$\log_\Lambda :=$ inverse of e_Λ in $\mathbb{C}\{\{\tau\}\}$, ie

$e_\Lambda \log_\Lambda = \log_\Lambda e_\Lambda =$

$$\sum_{i+j=k} \alpha_i \beta_j^{q^i} = \begin{Bmatrix} 1 & k=0 \\ 0 & k>0 \end{Bmatrix} = \sum \alpha_i^{q^j} \beta_j$$

converges on $\{z \in C \mid |z| < \text{diam}(\Lambda)\} =: \sum \beta_i T^i$

Def of γ_i: $\frac{z}{e_\Lambda(z)} =: \sum_{i \geq 0} \gamma_i z^i$ (since $e_\Lambda(z) = z + \alpha_1 z^q + \dots$)

$$= z \sum_{\lambda \in \Lambda} \frac{1}{z-\lambda}$$

(via ("log" e_Λ)$'$ = bec

$$= 1 - \sum_\lambda{}' \frac{z/\lambda}{1-z/\lambda}$$

$$= 1 - \sum_\lambda{}' \left(z/\lambda + (z/\lambda)^2 + \dots \right)$$

$\frac{d}{dz} e_\Lambda(z) \equiv 1$ so $\frac{e_\Lambda'(z)}{e_\Lambda(z)} = \frac{1}{e_\Lambda(z)}$ and notice that the log-derivative behaves like homomorphism (even for ∞ products))

$$= 1 - \sum_{k \geq 1} \left(\sum_{\lambda \in \Lambda}{}' \frac{1}{\lambda^k} \right) z^k = 1 - \sum_{k \geq 1} E_k(\Lambda) z^k$$

Can rearrange double sum since know convergence

This formula $\frac{z}{e_\Lambda(z)} = 1 - \sum_{k \geq 1} E_k(\Lambda) z^k$

also implies $E_k \neq 0 \Rightarrow k \equiv 0 \ (q-1)$

$E_{pk} = (E_k)^p$

$E_0 = -$

Proposition: $n \in A$, $\phi^\Lambda_n = \sum n_i T^i$, then

$$n E_{q^k-1} = \sum_{i+j=k} E_{q^i-1} n_j^{q^i}$$

pf (1) let $e = e_\Lambda$, $\log = \log_\Lambda$, $\phi = \phi^\Lambda$

$e n = \phi_n e$

$\Rightarrow n = \log \phi_n e \Rightarrow n \log = \log \phi_n$

ie $n \beta_k = \sum_{i+j=k} \beta_i n_j^{q^i}$

(2) $\underline{j = q^k - q^i \Rightarrow \gamma_j = \beta_{k-i}^{q^i}}$

pf In view of $\gamma_{qj} = \gamma_j^{\,q}$, may assume $i = 0$

induction on k

$k = 0$ trivial

$k > 0$ from $\frac{z}{e(z)}\, e(z) = z$, have

$$\sum_{0 \le i \le k} \gamma_{q^k - q^i}\, \alpha_i = 0 \text{ , therefore}$$

$$\gamma_{q^k - 1} = -\sum_{1 \le i \le k} \gamma_{q^k - q^i}\, \alpha_i$$

$$= -\sum_{1 \le i \le k} \beta_{k-i}^{q^i}\, \alpha_i \quad \text{(induction hyp)}$$

$$= \beta_k \quad \text{(by def of log)} \quad \text{OK}$$

(3) $\underline{\beta_k \overset{(2)}{=} \gamma_{q^k-1} = -E_{q^k-1}}$ (entire), results follows from (1)

pf is complete □

- The data $\{\alpha_i\}$, $\{\underset{E_{q^i-1}}{\underset{\parallel}{\beta_i}}\}$ or $\{n_i\}$ where $n \in A - \mathbb{F}_q$ determine each other
- other Eisenstein Series also have arith meaning but not in an obvious way like above

Example 1: $A = \mathbb{F}_q[T]$, $r = 1$, $\Lambda = A$, $\phi_T^A = T + n_1\tau$

$$n = T, \quad T E_{q-1} = E_0\, n_1 + E_{q-1}\, n_0^{\,q}$$

$$= -n_1 + T^q E_{q-1}$$

$$\Rightarrow n_1 = (T^q - T)\, E_{q-1}(A) = (T^q - T) \sum'_{n \in A} \frac{1}{n^{q-1}} \in K_\infty = \mathbb{F}_q((\tfrac{1}{T}))$$

Now want to ck $c\Lambda$: $\alpha_i(c\Lambda) = c^{1-q^i}\alpha_i(\Lambda)$

$\beta_i(c\Lambda) = c^{1-q^i}\beta_i(\Lambda)$

$E_k(c\Lambda) = c^{-k} E_k(\Lambda)$, and

$\phi_n^\Lambda =: \sum n_i(\Lambda)\tau^i \Rightarrow n_k(c\Lambda) = c^{-k} n_i(\Lambda)$

ie all behaves like modular forms

$\phi^{\bar\pi A}_{T} = T + \tau$

$\phi^{A}_{T} = T + n_1 \tau$ $\Rightarrow n_1 = \bar\pi^{q-1}$, combine together get

" $\underline{\bar\pi^{q-1} = (T^q - T) \sum'_{n\in A} \frac{1}{n^{q-1}}}$ " formula for periods

This is the analogue of $\frac{\pi^2}{6} = \zeta(2) = \sum_{n\geqslant 1} \frac{1}{n^2}$

or even more precisely: $(2\pi i)^2 = -12 \sum'_{n\in\mathbb{Z}} \frac{1}{n^2}$

Notice that $\sum' \frac{1}{n^{q-1}}$ has $| \ | = 1$, hence

$$\log_q |\bar\pi| = \frac{q}{q-1}$$

Also $\zeta_A(s) := \sum_{n \text{ monic} \in A} |n|^{-s} = \frac{1}{1-q^{1-s}}$, $s \in \mathbb{C}$

then $\zeta_A(2) = \frac{1}{1-q^{-1}} = \frac{q}{q-1}$

i.e. $\log_q |\bar\pi)$ is a special ζ_A value

(to be contime)

Griekeler 1999 3/19 at Academia Sinica VI

$\Gamma\backslash\Omega \xrightarrow{\sim} \mathbb{C}$ via $j(z) = g^{q+1}(z)/\Delta(z)$:

$z \mapsto \Lambda_z = Az + A \mapsto \phi^{\Lambda_z} \mapsto j(\phi^{\Lambda_z})$

$\Gamma = GL(2,A)$, $A = \mathbb{F}_q[T]$, $\Omega = C - K_\infty$, $\phi_T = T + g\tau + \Delta\tau^2$

k=1 : $T E_{q-1}(z) = E_0(z) g + E_{q-1} T^q$

$\Rightarrow g(z) = (T^q - T) E_{q-1}(z)$

k=2 $T E_{q^2-1} = -\Delta + E_{q-1} g^q + E_{q^2-1} T^{q^2}$

$\Rightarrow \Delta = (T^{q^2} - T) E_{q^2-1} + (T^q - T)^q E_{q-1}^{q+1}$

ie g, Δ are all expressible in terms of Eisenstein series

Definition A modular form of wt k for Γ is a function $f: \Omega \to \mathbb{C}$ st

(i) $f(\gamma z) = (cz+d)^k f(z)$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

(ii) f holomorphic

(iii) f "holo at ∞" ← need $\underset{\shortparallel}{|z|_i}$ further explaination

let $c \in \mathbb{Q}$, $c > 1$, $\Omega_c = \{ z \in \Omega \mid im(z) \geq c \}$, called admissible open subset of Ω ($c \in \mathbb{Q}$ is important from the point of view in rigid analytic geom in defining G-top)

lemma $\gamma \in \Gamma$ st

$\gamma(\Omega_c) \cap \Omega_c \neq \emptyset \Rightarrow \gamma \in \Gamma_\infty = \{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Gamma \}$, and

$$|\gamma z|_i = \frac{|\det \gamma|}{|cz+d|^2} |z|_i$$

$\Rightarrow \Gamma_\infty \backslash \Omega_c \longleftrightarrow \Gamma \backslash \Omega$ as open embedding of analytic spaces

lemma $t \quad \Omega \to C$, $z \mapsto \frac{1}{e_A(z)} \bar{\pi}^{-1} \equiv \frac{1}{e_L(\bar{\pi} z)}$ has

(i) $|t(z)|$ depends only on $|z|_i$

(ii) $\exists\, c_0 > 1$ st $|z|_i > 1 \Rightarrow |z|_i \leq \log_q |t(z)| \leq c_0 |z|_i$

Meanning for rescaling by π

$L := \bar{\pi} A$ (Standard notation for Carlitz module)

$e_L(\bar{\pi} z) = e_{\bar{\pi}A}(\bar{\pi} z) = \bar{\pi}\, e_A(z)$, hence $\frac{1}{e_L(\bar{\pi} z)} = \bar{\pi}^{-1} e_A^{-1}(z)$

So the "f holomorphic at ∞" really means some rapidly decay condition as described below:

Consequence: $A\backslash\Omega_c \xrightarrow[\sim]{t} B_{r(c)}(0) - \{0\}$
for some small radius $r(c)$

Γ acts on Ω via $\Gamma = GL(2,A) \to PGL(2,A) = GL(2,A)/\underset{\text{Scalar matrices}}{\mathbb{F}_q^\times}$

$\Gamma_\infty/\mathbb{F}_q^\times = \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} \ltimes \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| \ a \in \mathbb{F}_q^\times,\ b \in A \right\}$

$$\begin{tikzcd} \Gamma_\infty\backslash\Omega_c = \mathbb{F}_q^\times\backslash(A\backslash\Omega_c) = \mathbb{F}_q^\times \backslash B_{r(c)}(0)-\{0\} \arrow[d, "\wr\ \text{via } t \mapsto t^{q-1}"] \\ B_{r(c)^{q-1}}(0)-\{0\} \end{tikzcd}$$

$t(az) = a^{-1} t(z)$, $a \in \mathbb{F}_q^\times$

$\Rightarrow \Gamma_\infty\backslash\Omega_c \xrightarrow[t^{q-1}]{\sim} B_{r(c)^{q-1}}(0) - \{0\}$ $(*)$

this gives the uniformizer at ∞, and be completed by adding the j-value

"(iii) f holo at ∞" means $f(z)$ has a convergent power series expansion: $f(z) = \sum_{i \geq 0} a_i t^{(q-1)i}(z)$

the $t^{(q-1)}$ corresponds to the above isom $(*)$

Rmk: for rk $= r$ case, $\Lambda \cong A^r$

$\{\Lambda \cong A^r\}/\mathbb{C}^\times = GL(n,A)\backslash\{(\omega_1, \dots, \omega_r)\ K_\infty \text{ linear indep}\}/\mathbb{C}^\times$

$\Omega^r = \{\underline{\omega} \in \mathbb{P}^{r-1}(\mathbb{C}) \mid \underline{\omega} \notin K_\infty \text{ hyperplane}\}$

$GL(r,A)\backslash\Omega^r$ $(r-1)$ dim'l V manifold

admits compatification of Satake or Baily-Borel

The difficulties lies at

Elliptic curves ↔ $GL(2)$
Abelian variety with polarization ↔ $Sp(2n)$

Drinfeld modules ↔ $GL(n)$

In the language of motives, Drinfeld module really corresponds to $GL(n)$ motives, hence $\nexists$ the notion of dual oby, or like polarization But one thing better is that to compactify $GL(r,A)\backslash\Omega^r$ we add only "divisors" unlike classical Satake compatification one add lower dim'l varieties

Proposition $\Lambda \subset C$ any discrete $\mathbb{F}_q$ sub module (eg Λ lattice), $e_\Lambda(z) = \sum \alpha_i z^{q^i}$; $t_\Lambda(z) := \frac{1}{e_\Lambda(z)}$ $= \sum_{\lambda\in\Lambda} \frac{1}{z-\lambda} = s_1(z)$, $s_k(z) := \sum_{\lambda\in\Lambda} \frac{1}{(z-\lambda)^k}$ meromorphic functions on C, $\exists$ a unique series of polynomials $G_k(X) \in \overline{\mathbb{F}_q(\Lambda)}[X]$ such that

(I) $S_{k,\Lambda}(z) = G_{k,\Lambda}(t_\Lambda(z))$

(II) $G_k = 0$ for $k \le 0$, $G_1 = X$, and

$$G_k(X) = X(G_{k-1} + \alpha_1 G_{k-q} + \alpha_2 G_{k-q^2} + \quad)$$

(III) G_k monic of degree k

(IV) $G_k(0) = 0$

(V) $k \le q \Rightarrow G_k(X) = X^k$

(VI) $G_{pk} = (G_k)^p$

(VII) $X^2 G_k' = k G_{k+1}$

(VIII) $\sum_{k\ge 1} G_k(X) u^k = \frac{Xu}{1 - X e_\Lambda(u)}$

(IX) $k = q^j - 1 \Rightarrow G_k(X) = \sum_{0\le i<j} \beta_i X^{q^j - q^i}$

These are simply analogue of :

$$\sum_{m\in\mathbb{Z}} \frac{1}{(z-m)^k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{n\in\mathbb{N}} n^{k-1} q^n$$

$$q = e^{2\pi i z}$$

(I),(II) ⇒ others easily except (IX)

for proofs of (I), (II), we use a classical "Newton's formula" : (cf Jacobson's book)

$$f(x) = X^n + a_1 X^{n-1} + \quad + a_n$$

$x_1, \cdots, x_n$ roots of $f(x)$, $S_k := \sum_{i=1}^{n} x_i^k$

then: $S_k + a_1 S_{k-1} + \quad + a_{k-1} S_1 + a_k k = 0 \quad k \leqslant n$

$$a_{k-1} S_{n-k+1} + a_k S_{n-k} = 0 \quad k \geqslant n$$

sketch for pf of (I), (II) : may assume $\dim_{\mathbb{F}_q} \Lambda = m < \infty$

$\left(\Lambda_i \subset \Lambda_{i+1} \subset \quad \Lambda \right.$ exhausion system of finite dim $\mathbb{F}_q$ submodules of Λ and do approx $\left. \right)$

$\lambda \in \Lambda$, $\frac{1}{z-\lambda}$ root of $e(X^{-1} z) X^{q^m} =: \tilde{f}(X) \in C(z)[X]$

$\deg \tilde{f}(x) = q^m$, leading coeff $= -e(z)$

Apply Newton's relation to $f(X) = \frac{\tilde{f}(x)}{-e(z)}$ ⇒ (I) and (2)

(IX) needs only further computations

(to be continued)

For $\Lambda = \mathbb{F}_q \subset C$, $e_\Lambda(z) = z - z^q$, $\log_\Lambda(z) = \sum_{i \geqslant 0} z^{q^i}$

$$G_k(x) = \sum_{0 \leqslant i \leqslant \left[\frac{k-1}{q}\right]} \binom{k-1-i(q-1)}{i} \cdot (-1)^i X^{k-i(q-1)}$$

$$\sum_{n \in \mathbb{F}_q} \frac{1}{(z-n)^k} = G_k\left(\frac{1}{z-z^q}\right), \quad G_{q^k-1}(X) = \sum_{0 \leqslant i < k} X^{q^k - q^i}$$

but this is the very trivial case

- Challenge problem I: Find Galois representation

$\Delta_{\text{elliptic curve}} \longleftrightarrow \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ (Deligne 1968, Sem. Bourbaki)

Δ Drinfeld (rk = 2) $\longleftrightarrow \mathrm{Gal}(\bar{K}/K)$ even for $A = \mathbb{F}_q[T]$

what's the Galois repr (rk = 2) ??

- Challenge Prob II

  Compute cohomology of modular variety of Drinfeld modules (cf Laumon's book)

- III Description of the algebra of modular forms

  $$M_A = \bigoplus M_k \qquad \dim M_k = ?$$

  for $A = \mathbb{F}_q[T]$, $M_A = A[g, \Delta]$ for rk 2 case

  How about general A, general rk ?

Giekeler 1999 3/22 at 清大/理論中心 VII No. 31

$\Lambda \subset C$ discrete $\mathbb{F}_q$-submodule

$S_k(z) := \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^k}$, $k \in \mathbb{N}$

$S_k(z) = G_k(S_1(z))$ for $G_k(x) \in \overline{\mathbb{F}_q(\Lambda)}[x]$

fix $A \longrightarrow C\{\tau\}$ Carlitz module $A = \mathbb{F}_q[T]$

$T \longmapsto T + \tau = TX + X^q = S_T$

$S_{T^2} = T + (T + T^q)\tau + \tau^2$, .

For $n \in A$, put $f_n(x) := S_n(x^{-1})\, X^{q^{\deg n}}$

$S_n = nX + * X^q + \cdots + (\underset{\uparrow}{\quad}) X^{q^{\deg n}}$

leading coeff of n in T

$\deg f_n(x) = q^{\deg n} - 1$

leading coeff of $f_n(x) = n$, if n monic:

$f_n(x) = 1 + \cdots + X^{q^{\deg n} - q^{\deg n - 1}} + \cdots n X^{q^{\deg n} - 1}$

Let k_n be the splitting field of $\frac{S_n(x)}{(f_n(x))}$ over k.

Theorem

(I) K_n/K is abelian with gp $(A/n)^\times$

$\bar{b} \in (A/n)^\times$ acts on λ (a root of $S_n(x)$) by

$\bar{b}(\lambda) := S_b(\lambda)$

(II) n monic, $n = \prod f_i^{e_i}$ prime decomposition, then

$$K_n = \bigotimes_K K_{f_i^{e_i}}$$

(III) $n = f^e$, f prime, then K_n/K is totally ramified at (f), unramified at all other finite primes of K.

(IV) $\mathbb{F}_q^\times \hookrightarrow (A/n)^\times$ get decomp and inertial gp of the place ∞ of K.

$$\overbrace{\underbrace{K_n \;—\; }_{\mathbb{F}_q^{\times}} \underbrace{K_n^{\times} \;—\; K}_{(A/n)^{\times}/\mathbb{F}_q^{\times}}}^{(A/n)^{\times}},$$

The case for $r=1$:

$S_n(X) \longleftrightarrow X^n - 1$

$f_n(X) \longleftrightarrow 1 - X^n$

Example of $S_k(z) = G_k(S_1(z))$.

$\Lambda = \bar{\pi} A =: L$ asso. to Carlitz module

$e_L(nz) = S_n(e_L(z))$

$e_L(z) = \sum_{i \geq 0} \frac{1}{D_i} z^{q^i}$

$\log_L(z) = \sum \frac{(-1)^i}{L_i} z^{q^i}$, $L_i := [i][i-1] \cdots [1]$

$[i] := T^{q^i} - T$

$G_{q-1} = X^{q-1}$

$G_{q^2-1} = X^{q^2-1} - \frac{1}{[1]} X^{q^2-q}$

$G_{q^3-1} = X^{q^3-1} - \frac{1}{[1]} X^{q^3-q} + \frac{1}{[1][2]} X^{q^3-q^2} \cdots$

And for Einstenstein Series:

$$\sum\nolimits'_{a,b \in A} \frac{1}{(az+b)^k} = \sum_{b \in A}{}' \frac{1}{b^k} + \sum_{a}{}' \sum_{b \in A} \frac{1}{(az+b)^k}$$

$$= \bar{\pi}^k \underbrace{\sum_{\lambda \in L}{}' \frac{1}{\lambda^k}}_{E_k(L) \in K} - \bar{\pi}^k \sum_{a \text{ monic}}{}' \sum_{b \in A} \frac{1}{(\bar{\pi} a z + \bar{\pi} b)^k}$$

Remark: (1) $E_k(L) \longleftrightarrow \frac{2\zeta(k)}{(2\pi i)^k} \sim B_k$ (k even)

(B_k determines class number of $\mathbb{Q}(\mu_p)$

$E_k(L)$ determines divisor properties of K_n)

cf. Goss' book.

(2) $\bar{\pi} \longleftrightarrow 2\pi i$

$\bar{\pi}$ is transcental over K (Yu, Wade)

Definition. $t_a(z) := t(az) = e_L^{-1}(\bar{\pi} a z)$, $a \in A$

$$t_a^{-1}(z) = e_L^{-1}(\bar{\pi} a z)$$
$$= \frac{1}{S_a(t^{-1}(z))} = \frac{t^{q \deg a}}{S_a(t^{-1}(z))\, t^{\deg a}}$$
$$= \frac{1}{f_a(t)} t^{\deg a} \in K(t) n \cap t^{q \deg a} A[\![t]\!]$$

So
$$\sum_{a,b \in A}{}' \frac{1}{(az+b)^k} = \bar{\pi}^k \Big[E_k(L) - \sum_{a \text{ monic}} G_k(t(az)) \Big]$$
$$= \sum_{i \geq 0} a_i t^i$$

If $a \in A$ contributes to the coeff a_i of t^i, then $q \deg a \leq i$ (ie. only finite number of such a)

Corollary 1 :

All the a_i's are rational (ie. $\in K$)

Corollary 2 .

E_k is holomorphic at ∞, ie.

$\sum a_i t^i$ converges for $|t|$ small enough.

Corollary 3 :

$$\bar{\pi}^{1-q} E_{q-1}(z) = \frac{1}{[1]} - \sum_{a \text{ monic}} t_a^{q-1}$$

$$\bar{\pi}^{1-q^2} E_{q^2-1}(z) = -\frac{1}{[1][2]} - \sum_{a \text{ monic}} t_a^{q^2-1} - \frac{1}{[1]} t_a^{q^2-q}$$

Corollary 4 .

$$\bar{\pi}^{-k} E_k \in n_k^{-1} A[\![t]\!] \ , \ n_k \in A$$

General Case .

$$\Gamma = GL(2, A) \xrightarrow{\det} A^{\times} = \mathbb{F}_q^{\times}$$

Definition of Modular Forms:

f modular form of weight $k \geq 0$ and type $m \in \mathbb{Z}/(q-1) \Leftrightarrow$

(i) f is holomorphic in Ω

(ii) $f\left(\frac{az+b}{cz+d}\right) = \frac{(cz+d)^k}{(ad-bc)^m} f(z)$; $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

(iii) f holo at ∞ (ie. $f = \sum a_i t^i$ somewhere converges)

let $M_{k,m} := \{ f$ of weight k, type $m \}$, $\mathbb{C}$-v.s.

Know: $E_k(z) \in M_{k,0}$

$g \in M_{q-1,0}$, $\Delta \in M_{q^2-1,0}$

let $k, m \in \mathbb{N}$, $m \leq \left[\frac{k}{q+1}\right]$.

$$P_{k,m}(z) := \sum_{\gamma \in H \backslash \Gamma} \frac{(\det \gamma)^m}{(cz+d)^k} t(\gamma z)$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $H := \left\{ \begin{pmatrix} \underset{\mathbb{F}_q^\times}{*} & \underset{A}{*} \\ 0 & 1 \end{pmatrix} \right\} \subset \Gamma$

$\Rightarrow P_{k,m} \in M_{k,m}$

let $h := P_{q+1,1}$

Theorem.

(i) $\bigoplus_{k \geq 0} M_{k,0} = \mathbb{C}[g, \Delta]$

(ii) $\bigoplus_{\substack{k \geq 0 \\ m \in \mathbb{Z}/(q-1)}} M_{k,m} = \mathbb{C}[g, h]$

(iii) $h^{q-1} = -\Delta$

Now $\Omega \to \Gamma \backslash \Omega$ unramified for $x \in \mathbb{F}_{q^2} - \mathbb{F}_q \subset \Omega$
ramified for degree $q+1$ in $\mathbb{F}_{q^2} - \mathbb{F}_q$.
hence a similar work in the classical case shows.

Theorem. for $0 \neq f \in M_{k,m}$, have :

$$\sum_{\substack{z \in \Gamma\backslash\Omega \\ z \not\sim \mathbb{F}_{q^2} - \mathbb{F}_q}} \nu_z(f) + \frac{\nu_e(f)}{q+1} + \frac{\nu_\infty(f)}{q-1} = \frac{k}{q^2-1}$$

Basic table.

	e	Ie	∞
g	1	0	0
h	0	0	0
Δ	0	0	$q-1$

$M_{k,m} \neq 0 \Rightarrow k \equiv 2m \, (q-1)$

$\nu_\infty(f) \equiv m(q-1)$ for $0 \neq f \in M_{k,m}$.

(to be continued.)

Gekeler 1999 3/26 at 清大/理論中心 VIII.

$A = \mathbb{F}_q[T]$

$E_k(z) = \sum'_{a,b\in A} \frac{1}{(az+b)^k}$ Einsenstein Series

$0 \neq E_k \in M_{k,0}$ if $0 < k \equiv 0 \ (q-1)$

$$\bar{\pi}^{1-q^k} E_{q^k-1}(z) = \sum a_i t^i(z) \ , \quad a_i \in K$$

denominators bounded

$$g_k := (-1)^{k+1} \bar{\pi}^{1-q^k} L_k E_{q^k-1} \in A[[T]] \ !$$

$(L_k := [k][k-1]\cdots[1] , \quad [i] = T^{q^i} - T)$

Proposition.

(I) $g_k \in A[[T]]$, $g_k = 1 + o(t^{q-1})$

(II) $g_0 = 1$, $g_1 = (\bar{\pi}^{1-q})g$ (see below to get $g_1 = g$)

$$g_k = g_{k-1} g^{q^{k-1}} - [k-1] g_{k-2} \Delta^{q^{k-2}} , \quad k \geq 2$$

Use "g_{new}" $:= \bar{\pi}^{1-q} g_{old}$ "Δ_{new}" $= \bar{\pi}^{1-q^2} \Delta_{old}$ to get rid of the transcental factor. (from now on)

Recall: $t_a(z) = t(az) = e_L^{-1}(\bar{\pi} a z)$ ... (analogue of $q = e^{2\pi i z}$)

$f_a(X) = \rho_a(X^{-1}) X^{q^{\deg a}}$ for ρ = Carlitz module,

$M_{A,k,m} := M_{k,m} \cap A[[t]]$,

$M_A := \bigoplus_{\substack{k \geq 0 \\ m \in \mathbb{Z}/(q-1)}} M_{A,k,m}$ A-algebra

$h = P_{q+1,1} \in M_{A,q+1,1}$; $h^{q-1} = -\Delta$

Theorem. (not very difficult)

(I) $M = M_A \otimes_A C$, $M_0 = M_{A,0} \otimes_A C$

(II) $M_A = A[g, h]$

$M_{A,0} := \bigoplus_{k \geq 0} M_{A,k,0} = A[g, \Delta]$.

Theorem

$$\Delta(z) = -t^{q-1} \prod_{a \in A,\ \text{monic}} f_a(t)^{(q^2-1)(q-1)}$$

(this is the analogue of $\Delta = q \prod_{n \geq 1} (1-q^n)^{24}$.
$24 = 12 \cdot 2$; $2 \longleftrightarrow q-1$ and $12 \longleftrightarrow q^2-1$ is the special value of zeta function, $\zeta(2)$?)

$$h(z) = -t \prod f_a(t)^{(q^2-1)}$$

(t as a function on Ω has no roots
classical case H is simply connected. so can take roots, but now Ω is not, so can't take further roots to get Dedekind η function. However can consider $\Delta(z)/\Delta(az)$ then still "partially inv." ← Yu's thesis)

$A_k(x,y) \in A[x,y]$
$A_k(g,\Delta) = g_k$, get simple recursion formula:
$$A_k = A_{k-1}(x,y)\, X^{q^{k-1}} - [k-1]\, A_{k-2}(x,y)\, Y^{q^{k-2}}$$

Proposition: let $p \subset A$ a prime of degree $d \in \mathbb{N}$
$$g_{k+d}(t) \equiv g_k(t^{q^d}) \pmod{p}$$
($\Rightarrow g_d \equiv 1 \pmod{p}$).
$$A_{k+d} \equiv A_k^{q^d} A_d \pmod{p}.$$
Proofs are direct but lengthy computations. No new ideas

Now some New Ideas. let $\mathbb{F}_p := A/p$

$$\begin{tikzcd} g \arrow[d, mapsto] & \Delta \arrow[d, mapsto] & \in M_{A,0} \arrow[r, leftrightarrow] \arrow[d, "\text{red. mop } p"] & A[[t]] \arrow[d] \\ x & y & \in \mathbb{F}_p[x,y] \arrow[r, "\varepsilon_0"'] & \mathbb{F}_p[[t]] \end{tikzcd}$$

(better write $\mathfrak{p}$ for p)

let $\tilde{?} = ?$ red mod $\mathfrak{p}$

Theorem. (Congruence Relations)

(a). $\mathrm{Ker}(\varepsilon_0) = (\tilde{A}_d(x,y) - 1)$.

If use the full algebra then (b).

$\mathrm{Ker}(\varepsilon) = (\tilde{A}_d(x, -Z^{q-1}) - 1)$

$$\begin{tikzcd} M_A \arrow[r, hook] \arrow[d] & A[\![t]\!] \arrow[d] \\ \mathbb{F}_p[X,Z] \arrow[r, "\varepsilon"] & \mathbb{F}_p[\![t]\!] \end{tikzcd}$$

$a \mapsto X$, $h \mapsto Z$

Rmk.

the classical case the congruence relations are due to Swinnerton-Dyer and Serre.

(c) $\tilde{A}_d$ = supersingular polynomial in characteristic p.

(i.e. for elliptic curves $/\mathbb{F}_q$, then End(E) generically is rk 2 = scaling and Frobenius, but for special exceptions, End(E) has higher rk, rk = 4. This $\leftrightarrow$ no p division pts at all. These j value are called supersingular values.

Supersingular polynomial (in char p) :=

$$ss_p(x) := \prod_{j:\,s.s} (x - j), \quad \deg \sim p/12$$

So $\tilde{A}_d$ is the analogue of this in Drinfeld module case.)

$\tilde{A}_d$ is charaterized by the property:

$\tilde{A}_d(g, \Delta) = 0 \iff \phi$ defined by $\phi_T = \tilde{T} + g\tau + \Delta\tau^2$ is supersingular (i.e. End ϕ has rk 4)

In classical case, Deligne showed the polynomial for $E_k = P_k(g_2, g_3)$

P_k are $\mathbb{Q}$-coeff and no p in denominators, hence can mod p, and get supersingular information.

Applications:

- $E/\overline{\mathbb{F}}_p$ supersingular $\Rightarrow$ $\mathrm{End}_{\overline{\mathbb{F}}_p}(E)$ maximal order in $H(p,\infty)$ = quaternion algebra $/\mathbb{Q}$ ramified in p and ∞.

$$\begin{tikzcd} \mathcal{O} \arrow[r, hook] \arrow[d, dash] & H(p,\infty) \arrow[d, dash, "\}\,4"] \\ \mathbb{Z} \arrow[r, hook] & \mathbb{Q} \end{tikzcd}$$

total quotient field of $\mathcal{O}$ is the non-commutative $H(p,\infty)$.

but can still defines class numbers..

Now for the Drinfeld case also have the same:

- $\phi/\overline{\mathbb{F}}_{\mathfrak{p}}$:

$$\begin{tikzcd} \mathrm{End}_{\overline{\mathbb{F}}_{\mathfrak{p}}}(\phi) \arrow[r, hook, "\text{max order}"'] & H(\mathfrak{p},\infty) \arrow[d, dash] \\ & K \end{tikzcd}$$

So some arithmetic problem can be solved only after Drinfeld modules..

Conclusion of the talk (recent status):

Lopez, Wack, Trautwein

$$\Delta = -t^{q-1}\prod f_a(t)^{(q^2-1)\cdot(q-1)} \in A[\![t^{q-1}]\!] \quad \text{let } s := t^{q-1}$$

$$h = -t\prod f_a(t)^{(q^2-1)} \ ; \ h/t \in A[\![s]\!]$$

$$u := \prod f_a(t)$$

u has analogue property of Euler's partition function formula:

$\prod(1-X^n)^{-1} = \sum p(n)X^n$, but $\prod(1-X^n) \underset{*}{=} \sum_{n\in\mathbb{Z}}(-1)^n X^{\frac{(3n+1)\cdot n}{2}}$

$= 1+X+X^2-X^5-X^7+X^{12}+X^{15}$.

very easy to be inverted and get recursive relations of $p(n)$ which no elementary proofs is known.

Pf of $*$ is only "Calculus".

Theorem. let $f(s) = \sum c_i s^i$ be one of u, $u^{q^2-1} = -h/t$, $u^{(q^2-1)(q-1)} = -\Delta/s$. then $\deg c_i \leq i$ with equality ∞-ly often. Expeicitly: let $l(c_i) \in \mathbb{F}_q$ be the coeff of T^i in c_i, then

(I) u, $l(c_i) = 1$ if i is a sum of different odd powers of q, and 0 otherwise.

(II) u^{q^2-1}, $l(c_i) = (-1)^{i/q}$ if $i \equiv 0 \ (q)$, 0 otherwise

(III) $u^{(q^2-1)(q-1)}$:

$$l(c_i) = \begin{cases} (-1)^j & \text{if } i = j\, q^2 \text{ or } i = j q^2 + q, \\ 0 & \text{otherwise}. \end{cases}$$

Corollary. $\rho(f)$ in i is $q^{-\frac{1}{q-1}}$

(radius of convergence)

$\Rightarrow$ power series converges on all $\mathcal{F}$ (fund. domain).

(classically, modular form converges on all H not just $\Gamma \backslash H$)

$\rightsquigarrow$ many different congruence properties

See the paper for details.

eg. for $-h/t = u^{(q^2-1)} = \sum c_i s^i$

$c_i = -[d]$ if $i = q^d$ and $q > 2$

$1 + [d]$ " $q = 2$

(related to Hecke operator theory)

$c_{i-q^d} \equiv c_i \pmod{[d]}$

(to be continued.)

Gekeler 1999 3/26 at 清華/理論中心 IX

No 41

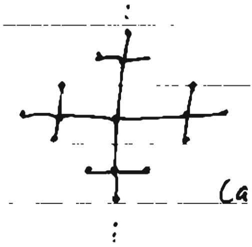
(Last lecture of this series)

$\mathcal{T}$ Bruhat – Tits tree of $GL(2, K_\infty)$, $A = \mathbb{F}_q[T]$ case

I. $(q+1)$ - regular infinite tree

$V(\mathcal{T})$, $E(\mathcal{T})$ vertices, edges of $\mathcal{T}$ (oriented)

general case are more complicated


case $q=3$

II. $V(\mathcal{T}) = G(K_\infty)/Z(K_\infty)\mathcal{K} \ni o(e)$ origin

$\qquad\qquad\qquad\quad \uparrow \qquad\qquad\quad \uparrow$

$E(\mathcal{T}) = G(K_\infty)/Z(K_\infty)\mathcal{J} \ni e$

here $e : \underset{o(e)}{\bullet} \longrightarrow \underset{t(e)}{}$, let $\bar{e} = \longleftarrow \bullet$.

$E(\mathcal{T}) \longrightarrow E(\mathcal{T})$, $e \mapsto \bar{e}$ via in fact

$$[X] \longmapsto \left[X \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}\right]$$

$\Gamma \hookrightarrow G(K_\infty)$ acts on $\mathcal{T}$.

$G = GL(2)$

$\mathcal{K} := G(O_\infty)$

$Z :=$ center of G = scalar mtrixes

$\mathcal{J} :=$ Iwolon sub gp $= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{K} \mid c \equiv 0 \pmod{\pi_\infty} \right\}$

III. Yet another description via lattices:

$V(\mathcal{T}) = \{ [L] \mid L \subset K_\infty \times K_\infty ; O_\infty\text{-lattice}, \; L \sim L' \Leftrightarrow \exists t \text{ st. } tL = L' \}$

Theorem. (Goldman - Iwalion)

$\mathcal{J}(\mathbb{R}) \xrightarrow{\sim} \left\{ \begin{array}{l} \text{non-archemedean norms } v \text{ on} \\ K_\infty \times K_\infty \text{ modulo scalars} \end{array} \right\}$

the top space ass. to gph $\mathcal{T}$.

$V(\mathcal{T}) = \mathcal{J}(\mathbb{Z}) \ni v \leftrightarrow [L] \mapsto v_L$ norm with unit ball L compatible with $G(K_\infty)$-action

GL(n) case is similar, get the Bruhat - Tits building.

$\lambda : \Omega \longrightarrow \mathcal{J}(\mathbb{R})$

$\qquad z \longmapsto \lambda(z) =$ norm v_z

$\qquad v_z(u, v) := |uz + v|$. is a norm since $z \in \Omega$

$\Rightarrow$ well-defined, $G(K_\infty)$-equivarient map.

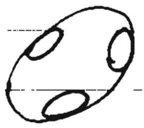
onto $\mathcal{J}(\mathbb{Q})$ since $|\Omega| = q^{\mathbb{Q}} \cup \{0\}$.

Good reference: Deligne - Husemöller. Contemp. Math 1987

Theorem (Drinfeld, D-H)

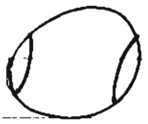
$\lambda^{-1}(v) \underset{\uparrow}{\cong} \mathbb{P}^1(G) - \bigcup$ disjoint "open" balls $(\# = q+1)$

rigid analytically isomorphic to



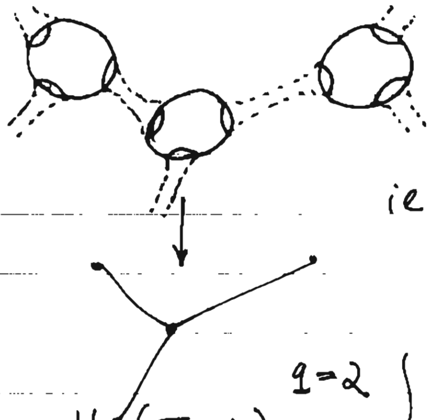
balls labeled by $x \in \mathbb{P}(L/\pi L)$.

$\lambda^{-1}(e^\circ(\mathbb{R})) \cong \mathbb{P}^1(G) - 2$ disjoint balls



eg.

$$\begin{tikzcd} \Omega \arrow[d] \\ \mathcal{J}(\mathbb{R}) \end{tikzcd}$$



ie. Ω is like the tubular nbd of $\mathcal{J}(\mathbb{R})$,

$q=2$

$H_1(\Omega^\times, \mathbb{Z}) = \underline{H}(\mathcal{J},\mathbb{Z}) = \left\{ \begin{array}{ll} f: E(\mathcal{J}) \longrightarrow \mathbb{Z} & \\ f(e) + f(\bar{e}) = 0 & \text{with cpt} \\ \sum_{o(e)=v} f(e) = 0 & \text{support} \end{array} \right\}$

for "real surface Ω"

But for our actual Ω (analytic space), have

$$H^1_{\acute{e}t}(\Omega, \mathbb{Q}_\ell) = \underline{H}(\quad, \mathbb{Q}_\ell)$$

Since we use coh. can drop cpt support cond.

Apply this to Modular forms:

Theorem: We have a canonical exact sequence compatible with $G(K_\infty)$-actions.

$$1 \to \mathbb{C}^\times \to \underset{\substack{\text{invertible} \\ \text{holo. fcn on } \Omega}}{\mathcal{O}_\Omega(\Omega)^\times} \xrightarrow[r]{} \underset{\substack{\text{without} \\ \text{cpt supp}}}{\underline{H}(\mathcal{J}, \mathbb{Z})} \to 0$$

$r: f \mapsto f'/f$ is the logarithmic derivation $r \text{ "}=\text{" } \frac{d}{dz}\log$".

How important is this?

eg. Pick $\Delta, h \in \mathcal{O}_\Omega(\Omega)^\times$, get pictures in $\underline{H}(\mathcal{J}, \mathbb{Z})$.

Idea of pf. f invertible $\Rightarrow |f|$ factors over λ

$r(f)(e) = \log_q \left(\frac{|f|t(e)}{|f|o(e)} \right) \in \mathbb{Q}$, not all obvious $\in \mathbb{Z}$. but is true.

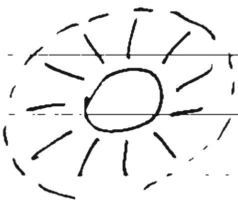
$$\begin{tikzcd} \Omega \arrow[d, "\lambda"] \\ \mathcal{T}(\mathbb{R}) \end{tikzcd}$$

Non-archemedean residue thm + contour integration

eg. consider annulus. Laurent expansion

$f(z) = \sum_{i \in \mathbb{Z}} a_i z^i$

$\mathrm{res}_c (f(z)dz) = a_{-1}$



inverse image of an edge

Briefly:

$$\begin{tikzcd} 1 \arrow[r] & C^\times \arrow[r, "\text{Liouville thm}"] & \mathcal{O}_\Omega(\Omega)^\times \arrow[r, "\text{residue thm}"] \arrow[d, "f \mapsto f'/f"] & \underline{H}(\mathcal{T}, \mathbb{Z}) \arrow[r] \arrow[d, "\bmod p"] & 0 \\ & & \mathcal{O}_\Omega(\Omega) \arrow[r, "\mathrm{res}"'] & \underline{H}(\mathcal{T}, \mathbb{F}_p) & \end{tikzcd}$$

$f \mapsto \mathrm{res}\, f$ defined as above

for Δ, $\Delta(\gamma z) = (cz+d)^{q^2-1} \Delta(z)$

$\Rightarrow r(\gamma\Delta) = (q^2-1)s(\gamma, c) + \gamma(\Delta)$, $s(\gamma, \ ) = r(cz+d)$.

To proceed, need further definitions:

$\Gamma = GL(2, A) \underset{\text{discrete}}{\hookrightarrow} G(K_\infty)$

$\mathrm{Stab}_{G(K_\infty)}(v) \cong \mathcal{K} = G(\mathcal{O}_\infty) \cdot Z(K_\infty)$

$Z(K_\infty)$

$\mathrm{stab}_\Gamma(v) = \Gamma \cap \mathcal{K}^\times =$ discrete $\cap$ cpt $=$ finite

Can define quotient graph:

$V(\Gamma\backslash\mathcal{T}) = \Gamma\backslash V(\mathcal{T})$

$E(\Gamma\backslash\mathcal{T}) = \Gamma\backslash E(\mathcal{T})$

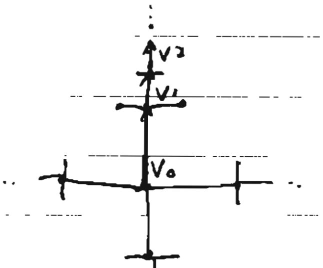
Fact. $\Gamma\backslash\mathcal{T} = \mathcal{H} =$ half line  $= (v_0, v_1, v_2, \cdots$

$v_i := [\pi^{-i}\mathcal{O}_\infty \oplus \mathcal{O}_\infty]$, $i \in \mathbb{Z}$

$e_i = (v_i, v_{i+1})$ get isom.

$$\begin{tikzcd} \mathcal{H} \arrow[r, hook] \arrow[rr, bend right, "\cong"'] & \mathcal{T} \arrow[r] & \Gamma\backslash\mathcal{T} \end{tikzcd}$$

$\lambda^{-1}(\mathcal{H}) = \mathcal{F}$ is the fund. domain for Γ on $\mathcal{T}$.



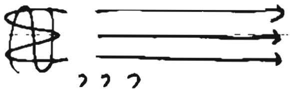
Consider congruent sub gp

$\Gamma(n) \subset \Gamma' \subset \Gamma$

$\overset{\shortparallel}{\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv 1 \pmod{n} \right\}}$ have similar theory for Γ'.

but $\Gamma' \backslash \mathcal{T}$ is usually not a tree, have cycles

$\Gamma' \backslash \mathcal{T} =$



??? $\gamma_1(K)$ finite # of half lines

$\gamma_i \leftrightarrow \Gamma/\Gamma'$

philosophy:

analysis on Ω transform to combinatorial type argument in the graph

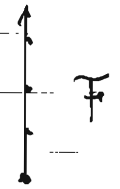
eg.

$$\begin{tikzcd} \Gamma \backslash \Omega \arrow[r, "\tilde{j}"] \arrow[d, "\lambda_\Gamma"] & \mathbb{C} \arrow[d] \\ \Gamma \backslash \mathcal{T}(\mathbb{R}) \arrow[r, equal] & \mathcal{H}(\mathbb{R}) \end{tikzcd}$$ half line

Theorem:

$z \in \lambda^{-1}(v_i), \ i > 0 \Rightarrow |j(z)| = q^{i+1}$

ie. the j-inv depends only on how far it is from the starting pt in fund. domain since j is inv under transformation to $\mathcal{F}$.



Same method leads to formulas (eg. zeros) of g, Δ, and "$g_k = \cdots E_{q^k-1}$", a_i

zeros of g_k if $z \in \mathcal{F}$ is a zero of g_k then $|z - z_0| < 1$ for a unique $z_0 \in \mathbb{F}_{q^{k+1}} - \mathbb{F}_q \subset \mathcal{F} \subset \Omega$

The End

Some references

Basics on Drinfeld modules:

- P Deligne - D Husemöller: Survey of Drinfeld modules Contemp Math 67 (1987), 25-91
- D Goss: Basic structures of function field arithmetic Springer 1996
- Drinfeld modules, modular schemes and applications (E-U Gekeler, M van der Put, M Reversat, J Van Geel eds.). World Scientific 1997 (= AB)

Rigid analysis and geometry:

- L. Gerritzen - M van der Put: Schottky groups and Mumford curves Lecture Notes in Math 817, 1980
- J Fresnel - M van der Put: Géométrie ~~rigide~~ analytique rigide et applications Birkhäuser 1981
- AB

Bruhat-Tits trees:

- E-U Gekeler - M Reversat: Jacobians of Drinfeld modular curves J reine angew. Math 476 (1996), 27-93
- AB

Modular forms:

- E-U Gekeler: On the coefficients of Drinfeld modular forms Invent. Math 93 (1988) 667-700
- AB, Gekeler-Reversat (see above), recent articles

Explicit results on Drinfeld modular forms for $\Gamma = GL(2, \mathbb{F}_q[T])$

G Cornelissen: C. R Acad Sc Paris 321, 1995, 817-820 (in french)

" Ph. D Thesis Gent, Belgium 1997

(presently at Max-Planck-Institut für Mathematik, Bonn, Germany)

E-U Gekeler: J N Th 21, 1985, 35-140

Invent Math 93, 1988, 667-700

manuscripta math 86, 1995 367-391 (harmonic analysis on Bruhat-Tits tree)

Comp. Math 106, 997, 181-202

Contemp Math 224, 1999, 11-141

"Growth order and congruences of coefficients of the Drinfeld discriminant function", J N Th, to appear

B Lopez - B Wack: J N Th ?, February 1999

M Trautwein: Diploma thesis Saarbrücken, Germany, 1998 (in german)