The First

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NCTS Summer School in Alg.-Geom.

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Chern-Simon / Cheeger-Simon Theory

0. Motivation:

		E : vector bundle	$c_n(E)$
X mfd	(topological)	$H^{2n}(X, \mathbb{Z})$	Betti
	C^∞	$H^{2n}_{DR}(X)$	de Rham
	analytic	$H^{2n}_{\mathcal{D}}(X, n)$	Deligne (-Beilinson)
	algebraic	$CH^n(X)$	Chow groups

Supplementary structure on E:

$\to$ richer char. class of E

connection $\nabla : E \xrightarrow[k\text{-linear}]{} \Omega^1 \otimes E$

Ω^1 1-forms (C^∞, analytic, alg)

st. Leibnitz rule.

curvature : $\nabla^2 : E \to \Omega^2 \otimes E$: $\mathcal{O}$-linear

In particular, ∇ flat $\longleftrightarrow$ $\nabla^2 = 0$

topologically / analytically (E, ∇) flat $\longleftrightarrow$ local system

$\mathcal{E} = \{ e \in E,\ \nabla e = 0 \}$

$\longleftrightarrow H^1(X, GL_r(\mathbb{C}))$ pointed set

$r = \mathrm{rk}\, E$:

$r=1$: example of richer invariant

$$\underset{\mathbb{C}^\times}{H^1(X, \underset{\shortparallel}{GL_1(\mathbb{C})})} = \text{gp} = \text{Hom}(H_1(X,\mathbb{Z}), \mathbb{C}^\times)$$

$$= \text{Hom}(\pi_1(X), \mathbb{C}^\times)$$

$=$ characters of $\pi_1(X, x)$

$=$ set of chern-Simon invariants

classical chern-Simon theory:

$$r=1: \quad \begin{tikzcd} H^1(X, \mathbb{C}^\times) \arrow[r] \arrow[d, "\substack{\text{boundary} \\ \text{via exp.}}"'] & H^{2n-j}(X, \mathbb{C}/\mathbb{Z}(n)) \arrow[d] \\ H^2(X, \mathbb{Z}(1)) & H^{2n}(X, \mathbb{Z}(n)) \end{tikzcd}$$

$\mathbb{Z}(n)$: $(2\pi i)^n$

$\mathbb{C}/\mathbb{Z}(2\pi i) \xrightarrow{\exp,\ \wr} H^1(X,\mathbb{C}^\times)$

$2\pi i$: call it 1.

$H^2(X,\mathbb{Z}(1))$: top chern class

$H^{2n}(X,\mathbb{Z}(n))$: torsion classes.

Want good behavior in families, compatible with inverse images (so. coh. theory).

defect of compatible with direct images is the

— Riemann-Roch Theorem —.

Setup: $f: X \to S$ morphism, proper

$E \to X$ v.b.

in order to take direct image, need to know still in the same category.

f: proper, Grauert $\Rightarrow R^if$ coherent

S, X smooth (Hilbert) $\Rightarrow \exists$ resol. via v.b.'s.

Grothendieck Riemann-Roch :

$$ch(Rf_* E) = F(f, ch\, E)$$

$$\bigoplus CH^n(S) \otimes \mathbb{Q} \xrightarrow[/\mathbb{C}]{} \bigoplus H^{2n}_{\mathcal{D}}(X, n) \otimes \mathbb{Q} \to \bigoplus H^{2n}(X, \mathbb{Z}) \otimes \mathbb{Q}$$

More precisely :

$F(f, ch\, E)$ = 2 pieces of information

$f_* (Td(f) \odot ch(E))$

Todd class $\in CH(X) \otimes \mathbb{Q}$

direct image (trace)

For (E, ∇), like classes which behave "correctly" in alg. morphism.

1st analogy (Deligne)

X smooth curve $\xrightarrow{f} S = \mathrm{Spec}\, \mathbb{F}_q$

$\rho : \pi_1(X) \longrightarrow GL(1, \mathbb{Q}_\ell)$, rk 1 local system

$Rf_*(\rho) = H(\bar{X}, \rho) \in K_0(\mathbb{Q}_\ell) = \{ \text{v.s.} / \mathbb{Q}_\ell \} / \text{exact seq.}$

base ext. to $\bar{\mathbb{F}}_q$

$\det H(X, \rho)$

1-dim'l v.s / $\mathbb{Q}_\ell$.

becomes a $\pi_1(\mathbb{F}_q)$ module

$$\pi_1(\mathbb{F}_q) = \mathrm{Gal}(\bar{\mathbb{F}}_q / \mathbb{F}_q) \cong \hat{\mathbb{Z}} \ni \mathrm{Frob}.$$

give $\rho_{Im} : \pi_1(\mathbb{F}_q) \longrightarrow GL(1, \mathbb{Q}_\ell)$

$(\det H(\bar{X}, \rho))$

Theorem: (Deligne)

(motivation) $\rho'_{Im} = F(f, \rho)$ (of shape of Grothendieck R-R.)

$= -\mathrm{Tr}\,\rho \int$ divisor of a meromorphic section of ω = sheaf of 1-forms on X.

$$= -f_*\big(\underbrace{c_1(\Omega^1_{X/\mathbb{F}_q})}_{CH^1(X) \,=\, \mathrm{Pic}(X)} \cup \underbrace{\rho}_{\in \mathrm{Hom}(\pi_1(X),\, \mathrm{Aut}\, \mathbb{Q}_\ell)}\big)$$

$\downarrow \mathrm{Tr}$

geometry if one starts with ∇.

$$H^1(\bar{X}, \rho) \sim \sum (-1)^i R^i f_* (\Omega^{\cdot}_{X/S} \otimes E)$$

relative de Rham coh.

E: defined by connection

$H^1(\bar{X}, \rho)$: comes a $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ action $\xrightarrow[\nabla \text{ flat}]{}$ coh. sheaf with a connection: Gauß-Manin conn.

Wants:

$$ch(\textstyle\sum (-1)^i \text{"GM"}, \text{Gauß-Manin connection}) = F(f, ch(E, \nabla)).$$

Another motivation from topology:

$\mathcal{D}$-modules: X smooth$/k$

ex. (E, ∇) flat connection, $E \xrightarrow{\nabla} \Omega^1_X \otimes E$

$\nabla \leftrightarrow A$ = matrix of 1-forms $= \sum A_i \, dx_i$

(alg.) choice of coor. x_i

T locally given by ∂x_i, $\langle \partial x_i, dx_j \rangle = \delta_{ij}$

∂x_i acts on $e_j \in E$ by $\partial x_i (e_j) = A$

action of $\mathcal{D}$: $\partial_{x_i} \partial x_j = \partial x_j \partial x_i$.

∇ flat $\longrightarrow$ action of $\mathcal{D}$ on E.

$\exists$ more complicated examples:

① $U \underset{j}{\hookrightarrow} X$ open smooth

(E, ∇) flat on $U \rightarrow j_* (E, \nabla) = \mathcal{D}$-mod on X.

② $Z \underset{i}{\hookrightarrow} X$ closed imbedding

(E, ∇) conn. on $Z \longrightarrow i_* (E, \nabla)$.

$f: X \rightarrow S$ proper morphism

M: $\mathcal{D}$-module on $X \longrightarrow Rf_* M$ defined as $\mathcal{D}$-modules on X

$\rightarrow$ R.R. Question

$ch(Rf_* M) = F(f, ch M)$.

Problem: One needs a good theory of char classes of $\mathcal{D}$-modules. Don't have this yet!

If $\mathcal{D}$-module connection (E, ∇) do have this / also for $j_*(E, \nabla)$.

in particular, on $U(\mathbb{C})$; $(E, \nabla)|_U$

$$\left\{\begin{array}{l} \text{top} \\ C^\infty \\ \text{an} \end{array}\right. \quad \xleftrightarrow{1-1} \quad \begin{array}{l} \text{local system} \\ \text{Chern-Simons} \\ \text{Cheeger } H^{2n-1} \end{array}$$

2nd Motivation (C^∞):

Theorem (Bismut-Lott, Bismut)

$f: X \to S$ proj / $\mathbb{C}$ smooth

$\uparrow$

$\mathcal{V}$ local system

$ch(\mathcal{V}) \in H^{2n-j}(X, \mathbb{C}/\mathbb{Z}(i))$

$ch(Rf\,\mathcal{V}) = F(f, ch\,\mathcal{V})$.

Want thm $f: X \to S$ proj smooth, (E, ∇) flat conn which should have the fun of Deligne in # theory and

$$\swarrow \qquad \searrow$$

G.R.R $\qquad\qquad$ Bismut.

Classes: * $\exists$ good classes in group of "alg. differential characters"

$$\xswarrow{\text{alg. cycles}} \qquad \xsearrow{\text{top}}$$

$\oplus\, CH(X)$ $\qquad\qquad$ Chern-Simons, Cheegers

* explicit understanding of those classes at generic point of X (w. S. Bloch).

RR: general RR thm:

— for regular connection ("half" flat)
$r=1$, piece of classes detected at generic point of var.
$r \geq 2$: loose some information
— for irregular connections
for rk 1 irr. conn. $\to$ formula

$$ch(Rf\nabla) = F(f, ch\nabla)$$
plus extra information
(should give swan conductor in # theory)
Higher rk : $\to$ new invariant $\to$ conjecture.

Recall of Chern classes (after Beilinson / Kazhdan, unpublished)

Chern-Weil homomorphism:
$G = GL(r, \mathbb{C})$, $\mathfrak{g} = M(r\times r, \mathbb{C})$
$$\omega = \oplus\, \omega_n : \oplus\, S^n(\mathfrak{g}^*)^G \longrightarrow \oplus\, H^{2n}_{DR}(BG) = H^{2n}(BG, \mathbb{C})$$
G acts on $\mathfrak{g}$ via adjoint rep'n

$$p \longmapsto \omega_n(p)$$

is an isomorphism.

Atiyah class of a vector bundle.
torsor, exact obstruction to the existence of a connection on a V.b. E $\in H^1(X, \Omega^1 \otimes \operatorname{End} E)$

α) locally $\exists$ connection by declaring a given local basis as a flat basis.

$u_i \quad (e_i)$; $u_j \quad (e_j)$

$$\nabla_i - \nabla_j : E \to \Omega^1 \otimes E \qquad \in H^1(X, \Omega^1 \otimes \mathrm{End}\, E)$$

(β) 2nd way to think of it:

$X \times X \supset \Delta^{(1)}$

1st inf'l nbd of Δ

obtain $\mathcal{P}(E)$ = sheaf of principal part of E

get $0 \to \Omega^1 \otimes E \longrightarrow \mathcal{P}(E) \to E \longrightarrow 0$

$$\in \mathrm{Ext}^1(E, \Omega^1 \otimes E) = H^1(X, \Omega^1 \otimes \mathrm{End}\, E).$$

(γ) let $P \xrightarrow{\pi} X$ be principal G-bundle asso. to E

$GL(r)$

has a G-action

$\Rightarrow \pi_*(\Omega^1_P)$ have G-action

$\Rightarrow$ take inv. $\pi_*(\Omega^1_P)^G$ hence get

Atiyah extension:

$$0 \to \Omega^1_X \to \pi_*(\Omega^1_P)^G \to \mathrm{End}\, E \longrightarrow 0$$

we will use this to define chern classes.

to be <u>continued</u>.

Weil homomorphism à la B-K:

Atiyah extension of a v.b. E/X

$p: P \to X$ principal G-bundle, $G = GL(r)$

$\mathfrak{g} = \mathrm{Lie}\, G = M_{r\times r}(\mathbb{C})$

$$0 \to \Omega^1_X \to \Omega^1_{X,E} \to \mathrm{End}\, E \to 0$$

$$\Omega^1_{X,E} = (p_* \Omega^1_P)^G \quad : \text{ G-inv form in principal bundle.}$$

BK $\to$ differential graded algebra (DGA) $\supset \Omega^\bullet_X$

$$d: \mathcal{O}_X \cong \Omega^0_{X,E} \longrightarrow \Omega^1_{X,E}$$

$$\Omega^n_{X/E} = \bigoplus_{a+b=n} \Omega^{a,b}_{X,E} \supset \Omega^{n,0}_{X,E} \supset \Omega^n_X$$

$$\Omega^{a,b}_{X,E} := \Lambda^{a-b} \Omega^1_{X,E} \otimes S^b \,\mathrm{End}\, E$$

$$\Omega^{a,b}_{X,E} \xrightarrow{d'} \Omega^{a+1,b}_{X,E} \quad \leftarrow \text{comes from d on P}$$

$$\Omega^{a,b}_{X,E} \xrightarrow{d''} \Omega^{a,b+1}_{X,E}$$

$$\omega_1 \wedge \cdots \wedge \omega_{a-b} \otimes \varphi_1 \cdots \varphi_b \mapsto \sum (-1)^j \omega_1 \wedge \cdots \hat{\omega}_j \wedge \cdots \omega_{a-b} \cdot (\mathrm{Im}\, \omega_j)\, \varphi_1 \cdots \varphi_b$$

$$(d')^2 = (d'')^2 = 0, \quad d'd'' = d''d', \quad d = d' + d''$$

$$(\Omega^\bullet_X, d) \longrightarrow (\Omega^\bullet_{X,E}, d) \text{ sub complex}$$

is a filtered quasi-isomorphism (purely alg'ly):

$$(\Omega^{\geq p}_X, d) \longrightarrow (F^p \Omega^\bullet_{X,E}) = \bigoplus_{a \geq p} \Omega^{a,b}_{X,E} \text{ as an algebra}$$

Hodge filtration

$$\begin{array}{lll} \Omega^{\geq n}: & \Omega_X^n \longrightarrow & \Omega_X^{n+1} \\ & \downarrow & \downarrow \\ F^n: & \Omega^{n,0} & \Omega^{n+1,0} \oplus \Omega^{n,1} \\ & \downarrow & \downarrow \\ & \Omega^{n,1} & \Omega^{n+1,1} \oplus \Omega^{n,2} \\ & \downarrow & \\ & \Omega^{n,2} & \end{array}$$

$$\Omega_X^1 = \operatorname{Ker} \Omega^1_{X/E} \longrightarrow \operatorname{End} E$$

$$\Omega^{1,0}_{X,E} \qquad \Omega^{1,1}_{X,E} = \Lambda^0 \Omega^1_{X/E} \otimes S^1 \operatorname{End} E \cong \operatorname{End} E .$$

Weil homomorphism:
Apply for the def. of $\Omega^{a,b}$ to the classifying simplicial scheme (space) of principal G-bundles together with its universal G-bundle. $EG \longrightarrow BG$.

top: top. space M is a BG up to some dim N
if ① $\exists$ G prin. bundle $EG \to BG$, EG contractible

$$\begin{array}{ccc} P = X \times_{BG} EG & \longrightarrow & EG \\ \exists \quad \downarrow & & \downarrow \\ X & \longrightarrow & BG \end{array}$$

② $\forall$ $P \to X$, X mfd, $\dim X \leq N$, the above diagram exists.

To do this algebraically, we need to replace alg. V. by simplicial schemes.

Algebraically, BG exists (with ① and ② without bounds on dim X) in the category of simplicial (schemes) varieties.

ie. X. simplicial variety is a contravariant functor (Hodge III)

$$(\Delta^{\cdot}) \longrightarrow \{\text{schemes}\} \; ; \; n \mapsto X_n$$

satisfies obvious relations in ∂_i maps: $X_{n+1} \longrightarrow X_n$. (n+2 such maps)

sheaf $\mathcal{F}$: a complex of sheaves:

$n \longmapsto \mathcal{F}_n$: cpx of sheaves on X_n st.

$$\mathcal{F}_{n+1} \longleftarrow \partial_i^* \mathcal{F}_n$$

topological realization of X. : (mfd)

$$\coprod X_n \times \Delta^n \quad : \quad \begin{tikzcd} X_{n+1} \times \Delta^{n+1} \arrow[r, bend left, "\partial_i"] & X_n \times \Delta^n \arrow[l, bend left, "\delta_i"] \end{tikzcd}$$

eg. $\mathbb{C}, \pi$ simplicial sheaf ($X/\mathbb{C}$ for simplicity)

$$H(X., \text{simplicial sheaf } \mathbb{C}, \pi) \cong H(\text{top reali}, \mathbb{C}, \pi)$$

BG:

$$\begin{tikzcd} G^{l+1} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & {} \\ G^{l+1}/G \cong G^l & \end{tikzcd}$$

forgetful maps

↑ right transl.

$$\begin{tikzcd} G \times G \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] \arrow[d] & G \arrow[d] \\ G \arrow[r, shift left] \arrow[r, shift right] & \cdot = \mathrm{Spec}\, k \end{tikzcd}$$

$$G \times G/G \overset{g}{\underset{h}{\rightrightarrows}} G/G$$

$(g, h)\delta = (g\delta, h\delta)$

right transl. (Hodge III)

BK: Atiyah class can be extended to simplical cat.

$$(*) \quad 0 \to \Omega^1_{BG} \to \Omega^1_{BG, E_{uu}} \to \operatorname{End} E_{uu} \to 0$$

$$\qquad\qquad X, E \qquad \mathfrak{g}^*$$

Koszul complex:

(*) - resolution of $\Lambda^n \Omega^1_X = \Omega^n_X$ follows

$$0 \to \Omega^n_X \to \Big[\Lambda^n \Omega^1_{X,E} \to \Lambda^{n-1} \Omega^1_{X,E} \otimes \mathfrak{g}^* \to \Lambda^{n-2}\Omega^1_{X,E} \otimes S^2 \mathfrak{g}^* \to \cdots \to S^n \mathfrak{g}^* \to 0 \Big]$$

in general; in particular on BG

$\Rightarrow$ connecting morphism. $X = BG$

$$\begin{array}{ccc} H^0(X, S^n \mathfrak{g}^*) & \longrightarrow & H^n(X, \Omega^n_X) \\ X = BG \ \Vert & & \\ (S^n \mathfrak{g}^*_{speck})^G & & \\ \text{SI} & & \\ \text{inv. poly.} / k & & \end{array}$$

$$X_1 \overset{\delta_2}{\underset{\delta_1}{\rightrightarrows}} X_0 \ ; \ G \times G \overset{g}{\underset{h}{\rightrightarrows}} G$$

$$\mathcal{F}_1 \longleftarrow \delta_1^* \mathcal{F}_0 \quad \text{what's } H^0 \text{ ?}$$

Prop: BG $(G = GL(-))$

$$\begin{array}{ccc} H^{2n}_{DR}(X = BG) & & H^n(X, \Omega^n_X) \\ \uparrow \wr & \nearrow & \\ \mathbb{H}^n(X, \Omega^{\geq n}_X) & & \end{array}$$

$H^i(BG, \Omega^j_{BG}) = 0$ for $i \neq j$.

cor: conn. hom. $(S^n \mathfrak{g}^*_k)^G \xrightarrow[w_n]{\sim} H^n(BG, \Omega^n_{BG}) \simeq H^{2n}_{DR}(BG)$

In fact, for $k = \mathbb{C}$, $G(\mathbb{C})$:

Def'n : $(S^n \mathfrak{g}^*)^G_{\mathbb{Z}} = \mathrm{Ker}\left((S^n \mathfrak{g}^\times_{\mathbb{C}})^G \to H^{2n}(BG(\mathbb{C}), \mathbb{C}) \to H^{2n}(BG(\mathbb{C}), \mathbb{C}/\mathbb{Z}(n)) \right)$

$$H^{2n}_{DR}(BG) \simeq H^{2n}(BG(\mathbb{C}), \mathbb{C})$$

Rmk: Up to now, mostly are true for alg. gp G up to 2-torsion (Stiefel-Whitney class)

$n \rightsquigarrow \mathrm{Tr}\, M^n$ (–, Newton class). done. $\square$

Chern-Simons (classical Theory):

firstly, chern character via Chern-Weil theory.

question (CS): $P \xrightarrow[p]{} X$ principal G-bundle

find ? functorial + good property for products in the following:

$p^* E \simeq \oplus^r \mathcal{O}_\infty$ (canonical trivialization)

(E, ∇), P_n -inv polynomial

$d? = P_n(\nabla, \nabla, \cdots, \nabla) \in H^0(P, \Omega^{2n}_\infty, cl)$ closed

$\exists ? \in H^0(P, \Omega^{2n-1}_\infty)$.

Answer, let $F(A) = dA - A^2 = \nabla^2$

$$\underset{\text{transgression}}{TP_n} = n \int_0^1 P_n(A, \underbrace{F(tA), \cdots, F(tA)}_{n-1})\, dt \quad !$$

ex. $n=1$, $P_1(M) = \mathrm{Tr}\, M$

$$TP_1 = \int_0^1 P_1(A)\,dt = \int_0^1 (\mathrm{Tr}\, A)\,dt = \mathrm{Tr}\, A$$

$$\mathrm{Tr}(dA) = d\,\mathrm{Tr}\,A, \quad P_1(\nabla^2) = d\,TP_1, \quad \mathrm{Tr}\,(dA - A^2).$$

$P_2 = \mathrm{Tr}\, M^2$:

$$TP_2 = 2\int_0^1 \mathrm{Tr}\,(A, F(tA))\,dt$$

$$= 2\int_0^1 \left[t\,\mathrm{Tr}\,AdA - t^2(\mathrm{Tr}\,A^3)\right]dt$$

$$= 2\left(\tfrac{1}{2}\mathrm{Tr}\,AdA\right) - \tfrac{2}{3}(\mathrm{Tr}A^3) = \mathrm{Tr}\,AdA - \tfrac{2}{3}\,\mathrm{Tr}\,A^3.$$

Easy to check this is the answer.

Summary: P_n – inv. poly.

$(E, \nabla) \rightsquigarrow H^0(P, \Omega_\infty^{2n-1})$

$d(TP_n) = P_n(\nabla^2)$.

If P_n has zero periods $(\in (S^n g^*_{\mathbb{C}})^n_{\mathbb{Z}})$

$\to$ class of $P_n(\nabla^2) \in H^{2n}(X, \mathbb{C}/\mathbb{Z}(n))$

$\to$ $\exists$ chain $u \in C^{2n-1}(X, \mathbb{C}/\mathbb{Z}(n))$ st.

$TP_n - p^* u =$ coboundary.

Answer: $\nabla^2 = 0 \Rightarrow P_n \in H^0(P, \Omega^{2n-1}_{\infty, cl})$

$$\downarrow$$

$$[P_n] \in H^{2n-1}_{DR}(P)$$

$\to [P_n] \in H^{2n-1}(P, \mathbb{C}/\mathbb{Z}(n))$ well-defined on X.

Def: class $H^{2n-1}(X, \mathbb{C}/\mathbb{Z}(n)) \ni c_n(E, \nabla) = c_n(\nabla)$.

does not live algebraically.

Cheeger - Simons :
(really the beginning of new notions)
differential characters, cohomology theory.

X : C^∞ mfd :

$$\begin{tikzcd}
\mathbb{H}^{2n}(X, \mathbb{Z}(n) \to \Omega^0_\infty \to \Omega^1_\infty \to \cdots \to \Omega^{2n-1}_\infty) \arrow[dr] & & & \\
 & \mathrm{Ker}(H^0(X, \Omega^{2n}_{\infty, cl})) \arrow[r] & \dfrac{H^{2n}_{DR}(X)}{\mathrm{Im}(H^{2n}(X,\mathbb{Z}(n)))} \arrow[r] & 0 \\
H^{2n-1}(X, \mathbb{C}/\mathbb{Z}(n)) \arrow[uu] & \mathrm{CW.}\ Ch(\nabla^2) \arrow[u] & & \\
0 \arrow[u] & CS(E,\nabla;\ \nabla^2=0) \arrow[ul] & &
\end{tikzcd}$$

once $\nabla^2=0$. then comes from CS.

Ancestor of Deligne - Beilinson Coh. ~78.

<u>To be continued</u>.

coh. of Cheeger - Simons (diff. characters). C^∞

$$\begin{tikzcd}
& H^{2n}(X, \mathbb{Z}(n) \to \mathcal{O}_\infty \xrightarrow{d} \Omega^1_\infty \to \cdots \to \Omega^{2n-1}_\infty) \arrow[rrd] & & \\
H^{2n-1}(X, \mathbb{C}/\mathbb{Z}(n)) \arrow[ru] & & & \mathrm{Ker}\,(H^0(X, \Omega^{2n}_\infty, d) \to H^{2n}(X, \mathbb{C}/\mathbb{Z}(n))) \to 0 \\
0 \arrow[u] & & & \\
C.S. \arrow[uu] & c_n((E,\nabla)) \arrow[uuu] \arrow[r] & c_n(\nabla^2, \cdots, \nabla^2) & \\
& & C.W. &
\end{tikzcd}$$

in case $c_n(\nabla^2, \cdots, \nabla^2) = 0$.

Argument roughly:

$\exists$ classifying space of (E, ∇) ($\dim X \leq \cdots$, $\mathrm{rk}\, E \leq \cdots$)

$\to$ univ. case, also classifying space for bundles M

$\Rightarrow$ $H^{2p-1}(M, \mathbb{Z}) = 0$.

$\to$ diff character = diff form

$\to$ Chern-Weil form

$\to$ pull back. done. $\square$

Weil Homomorphism:

$$S^n(\mathfrak{g}^*)^G \longrightarrow H^{2n}_{DR}(BG) \xrightarrow{[E]} H^{2n}_{DR}(X)$$

but
$$\begin{tikzcd}
S^n(\mathfrak{g}^*)^G \arrow[r] & S^n(\mathfrak{g}^*_{E_{nn}}) = \Omega^{n,n}_{BG, E_{n,n}} \arrow[d, "d = d' + d''"] \\
\oplus \mathbb{Z} \arrow[u, phantom, "\wr\wr"] & \Omega^{n+1,n} = \Omega^1 \otimes S^n(\mathfrak{g}^*_E).
\end{tikzcd}$$

$c_n, c_1 c_{n-1}, \cdots$

Here Beilinson: $\Omega^{a,b} := \Lambda^{a-b}\Omega^1_{X,E} \otimes S^b \mathfrak{g}^*$.

obtains
$$\begin{tikzcd} (S^n\mathfrak{g}^*)^G \arrow[r] \arrow[rr, bend right, "w"'] & \Omega^{n,n}_{(X,E),\mathrm{cl}} \arrow[r] & c(F^n\Omega^\bullet_{X,E})[2n] \end{tikzcd}$$

w induces: $$(S^n\mathfrak{g}^*)^G \xrightarrow[{[E]}]{} H^{2n}_{DR}(X)$$

Beilinson: Weil coh. (dep. on E) analytic theory

$$U_E(n) := \mathrm{cone}\left(\mathbb{Z}(n) \oplus (S^n\mathfrak{g}^*)^G[-2n] \xrightarrow{2 \oplus w} \Omega_{X,E}\right)[-1]$$

$$\begin{array}{lll} & \rightsquigarrow \mathbb{H}^{2n}(U_E(n)) \longrightarrow & \ker\left\{ H^0(X,(S^n\mathfrak{g}^*)^G) \oplus H^{2n}(\mathbb{Z}(n)) \right. \\ H^{2n-1}(\mathbb{C}/\mathbb{Z}(n)) & \qquad\quad \psi & \qquad \left. \longrightarrow H^{2n}(\Omega^\bullet_{X,E}) \right\} \\ \quad\uparrow & \qquad c_n(E) & \\ \quad 0 & \qquad c_{n-1}(E)\,c_1(E) \ldots & \end{array}$$

in particular, on (BG) ..

given ∇ on E, $\longleftrightarrow$ split $\Omega^\bullet_X \rightleftarrows \Omega^\bullet_{X,E}$

$\longleftrightarrow$ $\Omega^1_X \rightleftarrows \Omega^1_{X,E}$

$$\begin{tikzcd} \mathcal{O}_X \arrow[r] \arrow[d, phantom, "\wr"] & \Omega^1_{X,E} \arrow[r] \arrow[d, shift left, "\nabla"] & \Lambda^2\Omega^1_{X,E} \oplus \mathfrak{g}^*_E \arrow[r] \arrow[d, shift left, "\nabla"] \arrow[d, "\nabla^2", bend left] & \cdots \\ \mathcal{O}_X \arrow[r] & \Omega^1_X \arrow[r] \arrow[u, shift left] & \Omega^2_X \arrow[r] \arrow[u, shift left] & \cdots \end{tikzcd}$$

$\nabla^2 = 0 \iff$ filtered q. isom.

$\exists \nabla \iff$ split q. is. $\Omega^\bullet_X \longrightarrow \Omega^\bullet_{X,E}$

$\exists \nabla, \nabla^2 = 0 \iff$ split filtered q. is.

$$\begin{array}{ccc} \exists \nabla,\ G(E) & & UE(n) \\ \downarrow \nabla & & \nabla \downarrow \end{array}$$

$$\mathbb{H}^{2n}(\text{cone}\,(\mathbb{Z}(n) \oplus F^{2n} \longrightarrow \Omega^{\bullet}_{X,E})[-1])$$

= Cheeger - Simons diff. char.

Chern - Simons diff forms:

(2n−1) - diff. form on principal bundle, ∇

$\nabla^2 = 0 \longrightarrow H^{2n-1}(\mathbb{C}/\mathbb{Z}(n))$.

Algebraization — alg. diff. char:

$X/\operatorname{spec} k$, char $k = 0$:

$$AD^n(X) := \mathbb{H}^{2n}_{Zar}(X, \mathcal{K}_n \to \Omega^n_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{2n-1}_X)$$

alg. diff forms

Eventually: $\mathcal{K}_n$ Zariski sheaf of Milnor K-theory

F: field / or local ring: $\quad K^M_n(F) = \dfrac{F^* \otimes_{\mathbb{Z}} \otimes \cdots \otimes F^*}{(\ldots x \otimes \cdots \otimes (1-x) \otimes \cdots)}$ $\quad x \in K^\times - \{1\}$.

$\mathcal{K}_n = \operatorname{Im} K^M_n(\mathcal{O}) \longrightarrow K^M_n(k(X))$

additive sp.

For X smooth, $CH^n(X) = \mathbb{H}^n(X, \mathcal{K}_n)$

$$K_1 = \mathcal{O}_X^\times \xrightarrow[d\log]{} \Omega^1$$

$$\begin{array}{ccc} K^M(\mathcal{O}) & \longrightarrow & \Omega^n \\ \downarrow & & \uparrow d\log \\ \mathcal{K}_n(\mathcal{O}) & \longrightarrow & K^M_n(k(X)) \end{array}$$

(cf. Gabber's result. all char. are torsion)

$$\begin{tikzcd} 0 \arrow[r] & AD^n_{flat}(X) \arrow[r] \arrow[d] & AD^n(X) \arrow[r] \arrow[d] & \operatorname{Ker}\left\{ H^0(\Omega^{2n}_{cl}) \to \frac{H^n(X, \Omega^{2n})}{\operatorname{Im} CH^n(X)} \right\} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^{2n-1}(\mathbb{C}/\mathbb{Z}(n)) \arrow[r] & CS_{an}\operatorname{Diff}^n(X) \arrow[r] & * & \end{tikzcd}$$

The strongest wh. theory:

$$\begin{tikzcd} AD^n_{flat}(X) \subset AD^n(X) \arrow[r, "c_n(E,\nabla) \mapsto c_n(E)"] \arrow[d, "/\mathbb{C}"'] & CH^n(X) \arrow[dl] \arrow[d] \arrow[dr, "/\mathbb{C}"] & \\ H^{2n-1}(\mathbb{C}/\mathbb{Z}(n)) & \left(H_{ef} \quad H_{\mathcal{D}} \right. & \left. H(\mathbb{Z}) \quad \ldots \right) \end{tikzcd}$$

construction: (B-K, Weil homomorphism)

generalized splitting principal.

Example: $\operatorname{rk} E = 2$:

$$\begin{tikzcd} P = \mathbb{P}(E) \arrow[d, "\pi"'] & 0 \to \Omega^1_{P/X}(1) \to \pi^*E \to \mathcal{O}(1) \to * \\ X & \nabla \rightsquigarrow \quad \Omega^1_{\mathbb{P}} \xrightarrow[\uparrow \ \text{alg.}]{\tau} \pi^*\Omega^1_X \end{tikzcd}$$

define $\qquad \tau \circ \nabla$ on π^*E — respects $*$.

① $(\mathcal{O}(1), \tau\cdot\nabla) \in H^1(\mathbb{P}, \mathcal{O}^\times \xrightarrow[\tau\, d\log]{} \pi^*\Omega^1_X)$

② Product: $H^2(\mathbb{P}, K_2 \xrightarrow[\tau\, d\log]{} \frac{\Omega^2_{\mathbb{P}}}{\Omega^2_{\mathbb{P}/X}}$

$$\xrightarrow[\tau\, d\log]{} \pi^*\Omega^3_X = \frac{\Omega^3_{\mathbb{P}}}{\langle \Omega^1_{\mathbb{P}/X} \rangle}).$$

in general case may have horizental direction using this kind of ideas:

For $X/\mathbb{C}$, flat bundle (E, ∇), $\nabla^2 = 0$

$$H^{2n-1}(X, \mathbb{C}/\mathbb{Z}(1)) \qquad H^{2n}(X, \mathbb{Z})$$

$$\mathbb{C}^\times \qquad\qquad \uparrow$$

$$\| \qquad\qquad K_0(\mathbb{C})$$

$$K_1(\mathbb{C}) \qquad\qquad \|$$

$$H^{2n}(X, \mathbb{Z}) \otimes K_0(\mathbb{C}).$$

$$H^{2n-1}(X, \mathbb{Z}) \otimes K_1(\mathbb{C}).$$

$\longrightarrow$ $H^{2n-2}(X, \mathbb{Z}) \otimes K_2(\mathbb{C})$ mod torsion

$H^{2n-p}(X, \mathbb{Z}) \otimes K_p^M(\mathbb{C})$ $\quad p \leq n$

$H^n(X, \mathbb{Z}) \otimes K_n^M(\mathbb{C})$. $\longrightarrow$ conjectures.

while AD complicated.

Simpler part:

$$AD^n(X) \longrightarrow AD^n(\operatorname{Spec} \overset{\eta}{\overset{\|}{k(X)}}) = \begin{cases} \dfrac{\Omega_\eta^{2n-1}}{d\,\Omega_\eta^{2n-2}} & n \geq 2 \\[2ex] \dfrac{\Omega_\eta^1}{d\log k(X)^\times} & n = 1 \end{cases}$$

S. Bloch: Close at generic pt $\longrightarrow$

given by TP eq'n à la Chern-Simons

(E, ∇) alg, trivialized by Zariski covering of X

$\nabla \longleftrightarrow$ matrix of 1-forms A, compute that

$$TP_n(A) - TP_n(gAg^{-1} + dg\,g^{-1}) \quad \text{closed in } U$$

g-inv in U

locally exact. $n \geq 2$.

locally $d\log$ exact $n = 1$.

$\Rightarrow \quad Tp_n(A) \in H^0(X, \Omega^{2n-1}/d\,\Omega^{2n-2}) \qquad n \geq 2$

$\in H^0(X, \Omega^1/d \log \mathcal{O}^\times) \qquad n = 1$

Bloch Ogus: $\quad C \begin{cases} \Omega_\eta / d\,\Omega_\eta^{2n-2} & n \geq 2 \\ \Omega_\eta / d \log k(X) & n = 1. \end{cases}$

---

class at generic pt $\rightarrow$ just like

$$CH^n(X) \longrightarrow \text{Griffiths}^n(X)$$

relation $\sim$ by $\mathbb{P}^1$ deformation ; $\sim$ by 1 parameter deformation

THEOREM : (Bloch, —) :

$f : X \to S$ projective morphism / spec k, char $k = 0$

X, S smooth, $d = \dim f = \dim X - \dim S$,

smooth / Spec $k(S)$, Y = rel. NCD $= f^{-1}(\Sigma) + Z$

with Σ NCD on S.

Given $D : E \to \Omega^1_X(\log Y) \otimes E$ with conditions :

$$\begin{array}{ccccc} \text{here} & & \Omega^2_{k(X)} & \longrightarrow & \Omega^2_{k(X)/S} \\ \downarrow & & \uparrow & & \\ \Omega^2_S \otimes k(X) & \longrightarrow & * & \longrightarrow & \Omega^1_S \otimes \Omega^1_{k(S)/S} \end{array}$$

assume $\nabla^2 : E \to f^*\Omega^2_S(\log \Sigma) \otimes E$

$\cap\ \Omega^2_X(\log Y)$

(condition for the existence of a Gauß-Manin conn.)

classes W_n = class at generic pt associated to $P_n(M) = \mathrm{Tr}\, M^n$. Then

I. $w_n\Big(\Sigma (-1)^i R f_* \big(\Omega^{\bullet}_{X/S}(\log Y) \otimes E, \nabla_{X/S} \big), GM \text{ conn.} \Big)$

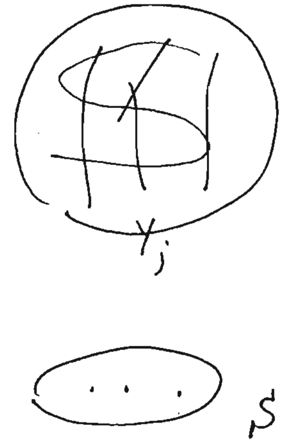
$= (-1)^d f_* \big[c_d \big(\Omega^1_{X/S}(\log Y), \mathrm{res}_Z \big) \cdot w_n(E, \nabla) \big], \; n \geq 2.$

II. $n = 1$: true if $\otimes \mathbb{Q}$, but also true $/\mathbb{Z}$ if replace (E, ∇) by $\big(E - (\mathrm{rk}\, E)\mathcal{O}, \nabla_E (\mathrm{rk}\, E)\, d \big)$.

comments:

① LHS: coh. sheaf $R^i f_* \Omega_{X/S} \log Y \otimes E$ not loc. free. (TP th) extends to coh. sheaves with connections.

② RHS: the product "$\cdot$":



$(E, \nabla)|_Z$

cycles like multi-sections with coeff.

$\cdot \quad \begin{array}{c} w_n \\ \cap \end{array} \quad \dfrac{H^0(X, \Omega^{2n-1}(\log Y))}{d\, H^0(X, \Omega^{2n-2}(\log Y))} \quad n \geq 2$

$\cdot \quad \dfrac{H^0(X, \Omega^1(\log Y))}{d \log H^0(X, \mathcal{O}^{\times})}$

take repr of $c_d(\Omega^1_{X/S}(\log Y))$ does not contain Y_j.

then $(E, \nabla)|_S$ + trace not defined.

Instead of this "section S",

$CH^d(X) = H^d(X, K_d) \leftarrow \mathbb{H}^d\big(X, K_d \to \oplus K_d|_{Y_i} \to \oplus_{i<j} K_d|_{Y_i \cap Y_j} \to \cdots\big)$

res $\Omega^1_X(\log D) \longrightarrow \oplus \mathcal{O}_{Z_i}$

if write $CH^d(X, Z) = \mathbb{H}^d(X, K_d \to \oplus K_d|_{Y_i} \to \cdots)$

then get pairing

$$f_*\left(CH^d(X, Z) \cdot \{\text{chern-Simons coh.}\}\right)$$

(3) In Previous lect. R.R.:

$$\begin{array}{lll} f: X \to S & & ch(Rf_*E),\ ob(S) \\ \quad \uparrow & & \quad \parallel \\ \quad E & ch(E),\ ob(X) & F(f, ch\, E) \end{array}$$

the thm is of this shape.

- LHS depends on $X, U \xrightarrow[f]{} S, (E, \nabla)|_U$
- RHS is OK: CS classes are recongnized on $\operatorname{Spec} k(X)$, in particular on U.

information of $c_d(\Omega^1_{X/S}(\log Y), res_Z) \in CH^d(X, Z)$ depends only on (X, U)

$\rightsquigarrow$ RHS eq'n is of Grothendieck type.

ON THE PROOF:

(1) Reduction to case $X = \mathbb{P}^1 \times S \xrightarrow{\gamma} S$ and S = fcn field $\underbrace{(S \sim \operatorname{Spec} k(S))}$ $=: K$

$X \supset Y$ horizontal divisor, modulo torsion

$Y \in \mathbb{P}(K)$, ΣY_i, y_i rat'l point of the base.

(2) Reduction to the case: to allow more general coherent sheaves with this type of connections.

need deal with torsion $\rightarrow$ no torsion in E.

ie. $E \simeq \oplus \mathcal{O}(n_i)$. hence may assume

$E \simeq \oplus \mathcal{O}$ trivial bundle (alg. trivial bundle)

(Bolibud: $S = \mathbb{C}$, $\mathbb{P}^1$ − pts (ICM '94)

$\rho : \pi_1(U(\mathbb{C})) \longrightarrow GL(r, \mathbb{C})$,

if ρ is not irreducible, then $\forall (E, \nabla)$

$$\nabla : E \longrightarrow \Omega^1_{\mathbb{P}^1}(\log \Sigma) \otimes E$$

st. $E^{\nabla}|_U \cong \rho$.

$E \neq \mathcal{O}(m) \otimes$ trivial bundle.)

③ Boils down to the following kind of eq'ns

$\Leftarrow$ higher trace formula in dlog − Milnor's K-theory:

$N \geq \delta \geq r \geq 1$: let $a_i \neq 0$

consider $\omega = \sum_i d\log(z - a_i) - (\delta + 1)\dfrac{dz}{z} = \dfrac{F(z)\,dz}{z\prod(z - a_i)}$

$\mathrm{res}_{a_i}\omega = \mathrm{res}_{\infty}\omega = 1$, F has roots β_i.

key lemma: $\sum d\log(\beta_1 - a_1) \wedge \cdots \wedge d\log(\beta_i - a_i)$

$$= \sum_1^r (-1)^{j-1} d\log F(a_j) \wedge d\log(a_j - a_1) \wedge \cdots \wedge \widehat{d\log(a_j - a_j)} \wedge \cdots \wedge d\log(a_j - a_r)$$

this eq'n $\Leftrightarrow$ R.R.

To be continued.

RR Thm for rk 1 irregular connections on curves

Deligne ($\sim$73)

$U \hookrightarrow X$ smooth alg v. $/\mathbb{C}$
open
local system on U:

$$V_\rho \longleftrightarrow \rho : \pi_1(U) \longrightarrow GL(N, \mathbb{C})$$

How to extend to all X ?

Analytic way: (Riemann-Hilbert correspondences)

$$V_\rho \xleftrightarrow{1-1} (E, \nabla)_{ana.} \quad \nabla \text{ flat connection}$$

$$V_\rho \longrightarrow (E = \mathcal{O}_{an} \otimes_{\mathbb{C}} V_\rho , \ d \otimes \wedge)$$

solution of system of differential equations.

$E^{\nabla} \longleftarrow (E, \nabla)$. Require strong topology.

$(E, \nabla)_{ana}$ underlies an algebraic structure:

$$((E_{alg}, \nabla_{alg})) \otimes_{\mathcal{O}_{X_{Zar}}} \mathcal{O}_{X_{an}} = (E, \nabla)_{ana}.$$

but (E_{alg}, ∇_{alg}) not unique! many choices.

Yes, if one requires ∇_{alg} at ∞ $(= X - U)$ has only logarithmic poles, then (E_{alg}, ∇_{alg}) is unique!

Related to connection with regular singularities at ∞.

i.e. $\exists$ extension of (E_{alg}, ∇_{alg}):

$$\overline{E}_{alg}, \overline{\nabla}_{alg} : \overline{E}_{alg} \longrightarrow \Omega^1(\log \infty) \otimes \overline{E}_{alg}$$

alg. vector bundle on X.

"top"$_{/U}$ $\longleftrightarrow$ "reg. sing." connections are controlled by topology.

For irregular singularities, invariants of connections are only partly controlled by topology.

Rank 1 case:

$$X \longrightarrow \operatorname{Spec} K = S$$

(L, ∇) connection, $U_{/K} \subset X \supset D = X - U$

K = function field$_{/k}$ char $k = 0$.

take $\overline{L} \xrightarrow{\overline{\nabla}} \Omega^1_X(*D) \otimes \overline{L}$ an extension.

$D = \sum D_i$ defined over K.

$$\nabla_{X/S} : \overline{L} \longrightarrow \Omega^1_{X/S}(*D) \otimes \overline{L}$$

rank 1 sheaf

n_i = smallest $n_i \in \mathbb{N}$ st.

$$\nabla_{X/S} : \overline{L} \longrightarrow \Omega^1_{X/S}\left(\sum n_i D_i\right) \otimes \overline{L}, \text{ but not}$$

$$\mathcal{D} = \Omega^1_{X/S}\left((n_i - 1) D_i + \sum_{j \neq i} n_j D_j\right) \otimes \overline{L}$$

call $\omega = \Omega^1_{X/S}$. have

$$\nabla_{X/S} : \overline{L} \longrightarrow \omega\left(\sum n_i D_i\right) \otimes \overline{L}$$

Recall:

$$(L,\nabla)|_{\mathcal{U}} \in \mathbb{H}^1(U, \mathcal{O}^\times \xrightarrow{d\log} \Omega^1_U)$$
$$(\bar{L},\bar{\nabla}) \in \mathbb{H}^1(X, \mathcal{O}^\times \longrightarrow \Omega^1_X(*D))$$
$$(\bar{L},\bar{\nabla})_{X/S} \in \mathbb{H}^1(X, \mathcal{O}^\times \longrightarrow \underset{\omega(\mathcal{D})}{\overset{\shortparallel}{\Omega^1_{X/S}(\textstyle\sum n_i D_i)}})$$

- want to make $*D$ more precise, call $f^{-1}(\omega(\mathcal{D}))$.

$$0 \longrightarrow f^*\Omega^1_S(*D) \to \Omega^1_X(*D) \twoheadrightarrow \Omega^1_{X/S}(*D)$$

relative connection.

Assume: $\nabla^2 : \bar{L} \to f^*\Omega^2_S(*D) \otimes \bar{L}$ (vertically)

eg. ∇ flat.

$\Rightarrow$ In fact, $\bar{\nabla} : \bar{L} \to \Omega^1_X(\log D)(\mathcal{D}-D) \otimes \bar{L}$

computations $\Rightarrow$

$$(\bar{L},\bar{\nabla}) \in \mathbb{H}^1(X, \mathcal{O}^\times \to \Omega^1_X(\log D)(\mathcal{D}-D))$$

R.R: $\mathcal{U} \hookrightarrow X \longrightarrow \operatorname{Spec} K = S$

(L,∇) on $\mathcal{U}$, ∇^2 ...

Katz: take $(\bar{L},\bar{\nabla})$ as before with extra condition that

$$\left(\bar{L} \xrightarrow{\bar{\nabla}_{X/S}} \omega(\mathcal{D} \otimes \bar{L})\right) \underset{\text{q. isom.}}{\longrightarrow} \left(L|_U \xrightarrow{\nabla_{X/S}} \omega_{U/S} \otimes L|_U\right)$$

$\uparrow$
a ... if ∇ with pole everywhere along D.

Gauß-Manin connections = ?

$$
\begin{tikzcd}
 & \bar{L} \arrow[r, "\sim"] \arrow[d, "\nabla"] & \bar{L} \arrow[d, "\nabla_{X/S}"] \\
f^*\Omega^1_S \otimes \bar{L}(\mathcal{D}-D) \arrow[r] \arrow[d] & \Omega^1_X(\log D)(\mathcal{D}-D) \otimes \bar{L} \arrow[r] \arrow[d, "\nabla"] & \omega(\mathcal{D}) \otimes \bar{L} \\
f^*\Omega^1_S \otimes \omega(\mathcal{D}-D) \otimes \bar{L} \arrow[r, "\sim"] & \dfrac{\Omega^2_X(\log D)(\overset{2\downarrow}{2}\mathcal{D}-D) \otimes \bar{L}}{f^*\Omega^2_S(*D) \otimes \bar{L}} &
\end{tikzcd}
$$

think Leibnitz rule

$$GM^i : R^i f_*(\bar{L} \to \omega(\mathcal{D}) \otimes \bar{L}) \longrightarrow \Omega^1_S \otimes R^i f(\bar{L}(\mathcal{D}-D) \to \omega(\mathcal{D})\bar{L}(\mathcal{D}-D))$$

$$H^i(\nabla_{X/S}) = R^i f(L|_U \to \omega_{U/S} \otimes L) \qquad \uparrow \qquad (L|_U \longrightarrow \omega_{U/S} \otimes L|_U)$$

GM connection

want to compute:

$$\det\left(\textstyle\sum (-1)^i H^i(\nabla_{X/S})\right)$$
$$= \left(\Lambda^{\mathrm{rk}H^0} H^0, \Lambda^{\mathrm{rk}H^0} GM^0\right) \otimes \left(\Lambda^{\mathrm{rk}H^1} H^1, \Lambda^{\mathrm{rk}H^1} GM^1\right) \cdots$$

rk 1 connection on $\operatorname{Spec} K = S$

$$\in H^1(\operatorname{Spec} K, \mathcal{O}^\times \longrightarrow \Omega^1_S) = \Omega^1_K / d\log K^\times .$$

Know already:

thm: ① Regular case (no sing.) $f : X \rightleftarrows S$ (L, ∇)

$$(\det)^{-1} = f_*\left(c_1(\omega) \cdot c_1(L, \nabla)\right)$$

in $\operatorname{Pic}(K)$ $\qquad$ $H^1(X, \mathcal{O}^\times \to \Omega^1)$

product into $H^2(X, \mathcal{K}_2 \to \Omega^2_X)$.

then to $\mathbb{H}^2(X, K_2 \to f^*\Omega^1_S \otimes \omega)$

$$\begin{array}{c} \downarrow Tr \\ \Omega^1_S / d\log \mathcal{O}^\times_S \end{array}$$

trace

② regular (i.e. logarithmic) singularities :

$(\det)^{-1} = f_*\left(c_1(\omega(D)) \cdot c_1(\bar{L}, \bar{\nabla})\right)$

$\mathrm{Pic}(X, D) \times \mathbb{H}^1(X, \mathcal{O}^\times \to \Omega^1_X(\log D))$

$\uparrow$

// generalized jacobians of Serre.

$\{(M, s) \mid s : \mathcal{O}'_D \xrightarrow{\sim} M_D\}$

$\mathbb{H}^1(X, \mathcal{O}^\times \twoheadrightarrow \mathcal{O}^\times_D)$

$c_1(\omega(D))$; res: $\mathcal{O}_D \xrightarrow{\sim} \omega_D|_S \cong \mathcal{O}_S$

the product we have

$$\begin{tikzcd} \mathcal{O}^\times_X \times (\mathcal{O}^\times_X \to \Omega^1_X) \arrow[r] \arrow[d] & (K_2 \to \Omega^2_X) \arrow[d, "Tr"] \\ \mathcal{O}^\times_X \to \Omega^1_X(D) \arrow[ur] & K_2 \to f\Omega^1_S \otimes \omega_{X/S} \end{tikzcd}$$

$\mathcal{O}^\times_X \to \mathcal{O}^\times_D$

③ Irregular case :

$\mathrm{Pic}(X, \mathcal{D}) \times \mathbb{H}^1(X, \mathcal{O}^\times \to \Omega^1(\log D)(\mathcal{D} - D))$

not D. in order to pair.

? need char. class $c_1(\bar{L}, \bar{\nabla})$ here.

like $c_1(\omega)$, $(c_1(D), \mathrm{res})$, but now ...

Need new idea.

$$\begin{tikzcd}
\bar{L} \arrow[r, "\nabla_{X/S}"] \arrow[dr, "\text{Leibnitz formula}"'] & \omega(\mathcal{D}) \otimes \bar{L} \arrow[d] \\
& \frac{\omega(\mathcal{D})}{\omega} \otimes \bar{L} = \omega_{\mathcal{D}} \otimes \bar{L}|_{\mathcal{D}}
\end{tikzcd}$$

$\to$ $\mathcal{O}_{\mathcal{D}}$ linear map.

pp $\nabla_{X/S} : \bar{L}|_{\mathcal{D}} \xrightarrow{\sim} \omega_{\mathcal{D}} \otimes \bar{L}_{\mathcal{D}}$; $\mathcal{O}_{\mathcal{D}} \to \omega_{\mathcal{D}}$.

the class is :

$c(\omega(\mathcal{D}), pp\nabla_{X/S})$! Now can write down

THEOREM : (S. Bloch, —) :

$$(\det)^{-1} = f_*\left(c(\omega(\mathcal{D}), pp\nabla_{X/S}) \cdot c(\bar{L}, \bar{\nabla})\right).$$

More precisely, $\exists$ cup product + trace $j : U \hookrightarrow X$.

$$\begin{tikzcd}
Pic(C, \mathcal{D}) \times H^1(X, j_*\mathcal{O}^\times \to \Omega^1(\log D)(\mathcal{D} - D)) \arrow[d] \\
\Omega^1_K / d
\end{tikzcd}$$

Some Comments :

① Reg. Sing. Case : then $D = \mathcal{D}$.

$c(\omega(D), pp\nabla_{X/S}) \in Pic(X, D)$ – today

$c(\omega(D), res_D) \in Pic(X, D)$ – tuesday

global geometry (Atiyah class)

$\Rightarrow$ 2 RHS are the same.

② Another formulation :

$N = 2g - 2 + \sum m_i = \deg \omega(\mathcal{D})$

on $Pic^N(X, \mathcal{D})$ $\exists$ special K point :

$G(\omega(\mathcal{D}), pp\,\bar{\nabla}_{X/S})$ some choice of $\bar{\nabla}$

Formulation:

$$X - |D| \xrightarrow{\alpha} Pic^1(X, \mathcal{D})$$

$$\begin{array}{c} \psi \\ x \longmapsto \mathcal{O}_X(x), \ \theta \hookrightarrow \mathcal{O}_X(x) \end{array}$$

Similarly for $Pic^N(X, \mathcal{D})$ ...

$$\begin{tikzcd} Pic^N(X,\mathcal{D}) \arrow[d] \\ S \end{tikzcd}$$

Gives rk 1 connections

choice of x_0

$Pic^0(X, \mathcal{D}) \simeq Pic^N(X, \mathcal{D})$

comm. alg. gp

$(M, s) \longrightarrow (M(Nx_0), s)$

Def'n: ① invariant relative rank 1 connection, i.e.

$$\begin{tikzcd} & G \times G \arrow[r, "\mu"] \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & G \\ G & & G \end{tikzcd}$$

$$\mu^*(L, \nabla) = p_1^*(L, \nabla) \otimes p_2^*(L, \nabla)$$

② absolute inv. rk 1 connection

$\rightarrow$ Same def + triviality along 0-section.

Prop: Via α^*:

(1,1) correspondence between

inv. rk 1 conn

rel: $\longleftrightarrow$ $L \to \omega(\mathcal{D}) \otimes L$

abs: $L \to \Omega^1(\log D)(\mathcal{D} - D) \otimes L$

(rk 1)

Vect $\longleftrightarrow$ Vect.

In particular, via α^*, (L, ∇) thought of as a rk 1 vertical conn. on $Pic^N(C, \mathcal{D})$. Then RHS

$(\bar{L}, \bar{\nabla})\,|$ special point.

One word about the proof:

Invariant connection

$\to$ New R.R. thm:

$$\begin{array}{ccc} Pic^N(X,\mathcal{D}) & & (M,\Delta) \\ \downarrow g & & \downarrow \\ Pic^N(X) & & M \end{array}$$

$B = g^{-1}(\omega(\mathcal{D})) \simeq$ torsor under $\mathcal{O}_{\mathcal{D}}^{\times}$ ($\simeq \prod G_a \times \prod G_m$)

(L,∇) on B.

$d+\beta$, β = "exact form".

Kontsevich: Comparing in some cases:

DR coh $\longleftrightarrow$ $d + df\beta$. f exact β

Higgs coh $\longleftrightarrow$ df $\Vert\!\!\to$ dim are the same.

"K's thm" come out to the diff. form.

though the pf has nothing to do with Kontsevich.

(3) Further Comment: (Higher rank case.)

Assume (E,∇) irregular on $U \subset X \to S$, $r = \mathrm{rk}\,E > 1$

curve.

$(\bar{E},\bar{\nabla})$: $pp\,\nabla_{X/S}: \bar{E}_{\mathcal{D}} \to \omega_{\mathcal{D}} \otimes \bar{E}_{\mathcal{D}}$

(polar part)

$\det pp\nabla_{X/S} : (\det E)|_{\mathcal{D}} \xrightarrow{\sim} \omega_{\mathcal{D}}^{r} \otimes (\det \bar{E})|_{\mathcal{D}}$

S'l see, $\exists$ ex. for which

$(\det)^{-1} \neq f_*(u\,\omega(\mathcal{D}), pp \det V) \cdot u \det(\bar{E},\bar{\nabla})$!

(this is what we have for $\bar{\nabla}$: log sing.) END

Need again new ideas. (in progress).