Corti 2004 1/8

NCTS Advanced Course in Alg Geom

"flips and higher dim alg-geom"

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What is a flip?

a flip is a special type of "codim ≥ 2" birational map (i.e. small)

Examples of flips:

consider $B = \mathbb{C}^4 \circlearrowleft \mathbb{C}^\times$ with weights $(-k, -1, 1, 1)$

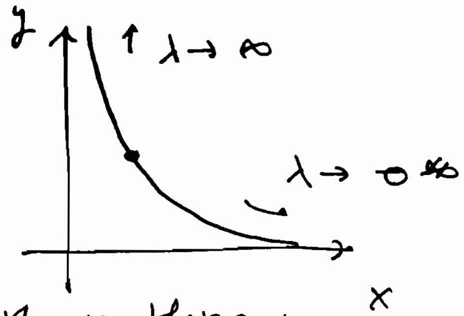
$$(x_0, x_1, y_0, y_1) \longmapsto (\lambda^{-k} x_0, \lambda^{-1} x_1, \lambda y_0, \lambda y_1)$$

Quotient $B/\mathbb{C}^\times$?

There are 2 "reasonable" ways to do the quotient, they are related by a flip.

the naive quotient $B/\mathbb{C}^\times$ is not a Hausdorff space: some orbit in x-axis and in y-axis become close.

(look at $\lim_{\lambda \to \infty}$, $\lim_{\lambda \to 0}$)



Solution: Mumford: either throw away x-axis or y-axis!

GIT: There are 2 $\mathbb{C}^\times$-linearized line bundles on B, called $L_\pm$

$L_\pm$ is characterized by:

$\Gamma(B, L_+^{\otimes n})^{\mathbb{C}^\times} = \{ f \mid f(\lambda a) = \lambda^n f(a) \}$

$\Gamma(B, L_-^{\otimes n})^{\mathbb{C}^\times} = \{ f \mid f(\lambda a) = \lambda^{-n} f(a) \}$

i.e. $L_- = L_+^*$.

$B_+^{ss} = \{ b \mid \exists n > 0,\ s \in \Gamma(B, L_+^{\otimes n})^{\mathbb{C}^\times},\ s(b) \neq 0 \}$

$\quad = B \setminus \{ y_0 = y_1 = 0 \}$

there is a reasonable Hausdorff quotient

$B_+^{ss}/\mathbb{C}^\times =: X^+$.

Similarly $B_- = B \setminus \{ x_0 = x_1 = 0 \}$, $B_-/\mathbb{C}^\times = X_-$.

for example, X_- is covered by 2 affine charts:

$U_-^0 = \{ x_0 \neq 0 \} \simeq \mathbb{C}^3/\mu_k$

"coordinates" $1, x_1, y_0, y_1$

μ_k action $\mapsto \zeta^{-1} x_1, \zeta y_0, \zeta y_1$

note: it is singular.

$U_-^1 = \{ x_1 \neq 0 \} \simeq \mathbb{C}^3$, smooth.

X_-, X_+ have morphisms to

$Z = \mathrm{Proj}\ \mathbb{C}[B]^{\mathbb{C}^\times}$

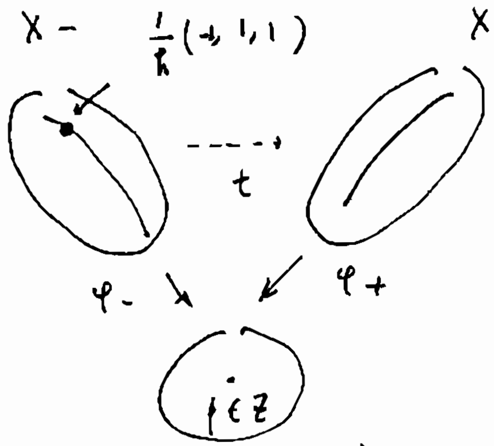
$$\begin{tikzcd} X_- \arrow[rr, dashed, "t"] \arrow[dr] & & X_+ \arrow[dl] \\ & Z & \end{tikzcd}$$

t is an example of flip, the simplest known flip.

exceptional set in X_- is $\{ y_0 = y_1 = 0 \}$

$\quad = \mathbb{P}^1(-k, -1) \simeq \mathbb{P}^1$.

exc set in X_+ is also $\mathbb{P}^1$.



$\varphi_-(\mathbb{P}^1) = p$

$\varphi_- : X_- \setminus \mathbb{P}^1 \simeq Z \setminus p.$

Warning: this is a <u>very</u> easy example.

more complicated example:

$B = (f=0) \subset \mathbb{C}^5 \circlearrowleft \mathbb{C}^*$, $(a_0, a_1, b_0, b_1, b_2)$

(or complete intersection in $\mathbb{C}^6$, codim 3 ...)

see work of Brown, Reid, <u>Mori</u>.

<u>definition</u> (K-opposite)

see Kollar "flips, flops, minimal models etc"

start with a small$^\vee$ proper birational morphism

$\varphi : X_- \to Z$

and $k_- \in H^2 X_-$ (rat'l coefficient) negative

i.e. $\int_C k_- < 0$ for all alg curve $C \subset X_-$ st $\varphi(C) = pt$

the k-opposite is a diagram

$$\begin{tikzcd} X_- \arrow[rr, dashed, "t"] \arrow[dr, "\varphi_-"'] & & X_+ \arrow[dl, "\varphi_+"] \\ & Z & \end{tikzcd}$$

and $k_+ \in H^2 X_+$ st.

(1) φ_+ is a small proper birat'l morphism

(2) k_+ is positive

(3) $k_- = k_+$ on $X_- \setminus Exc(\varphi_-) = X_+ \setminus Exc(\varphi_+)$ //

1.4 Note : Another way to say (3) is

$$\begin{array}{ccc} H^2 X_- & \xrightarrow{\rho} & H_{2d-2}\, X_- \quad (\text{Poincaré duality}) \\ & & \| \\ H^2 X_+ & \xrightarrow{\rho} & H_{2d-2}\, X_+ \quad ; \quad \rho k_- = \rho k_+ . \end{array}$$

It is not hard to see that X_+ is unique if it exists. but will not be done here.

Conjecture on existence of flips :

If X_- has terminal singularities, $k_- = c_1(K) \in H^2$, then the opposite exists, and is called a flip classically.

If X is non-singular, $K_X = \Lambda^{top} T_X^*$ is a line bundle. In the example, it is easy to make sense of K as an "orbifold" line bundle and

for $k > 1$: $\begin{cases} c_1(K) \in H^2(X_-, \mathbb{Q}), \\ c_1(K_-) \cap [\mathbb{P}^1] = -\dfrac{k-1}{k} . \\ c_1(K_+) \cap [\mathbb{P}^1] = k-1 . \end{cases}$

X normal alg. var. / Weil divisors

There is a 1-1 correspondence

$$\mathrm{We}\,X = \left\{ \begin{array}{c}\text{Weil divisors} \\ \text{on } X\end{array} \right\} \Big/ \underset{\text{linear equiv.}}{\equiv} \;=\; \left\{ \begin{array}{c}\text{divisorial} \\ \text{sheaves on } X\end{array} \right\} \Big/ \text{iso} .$$

def : a divisorial sheaf on X is a coh. sh. $\mathcal{F}$

(1) generic of rk 1

(2) torsion-free and saturated i.e. $\mathcal{F} = \bigcap_{p} \mathcal{F}_p$

equivalently: (2') $\mathcal{F} \xrightarrow{\sim} \mathcal{F}^{\vee\vee}$.

notice that the sp law is

$$\mathcal{F}_1, \mathcal{F}_2 \mapsto (\mathcal{F}_1 \otimes \mathcal{F}_2)^{\vee\vee} =: \mathcal{F}_1 [\otimes] \mathcal{F}_2$$

since $\mathcal{F}_1 \otimes \mathcal{F}_2$ may violate both (1) & (2).

X normal, ω_X makes sense as a divisorial sheaf.

def: X proper, ω_X has trace map $t : H^n(\omega_X) \longrightarrow k$ s.t. ($n = \dim X$)

$\forall$ coherent sheaf $\mathcal{F}$,

$$H^n(\mathcal{F}) \times \mathrm{Hom}(\mathcal{F}, \omega) \longrightarrow H^n(\omega_X) \xrightarrow{t} k$$

is a perfect pairing.

Notice: a predualizing sheaf of X is a sheaf ω_X st the above holds.

note: in terms of Grothendieck duality

$\omega_X = \mathcal{H}^{-n}(\mathcal{D}_X)$; $\mathcal{D}_X$ the dualizing cpx.

Proposition (Hartshorne's book): If X is normal, then ω_X is a divisorial sheaf.

pf: <u>exercise</u>: ω is torsion free

will show saturated:

$$0 \to \omega \to \omega^{\vee\vee} \to \mathcal{F} \to 0$$

codim Supp $\mathcal{F} \geq 2$

$$\begin{array}{ccccccc} H^{n-1}\mathcal{F} & \to & H^n\omega & \xrightarrow{\sim} & H^n(\omega^{\vee\vee}) & \to & H^n\mathcal{F} \\ \| & & | d & & | d & & \| \\ 0 & & & & & & 0 \\ & 0 \leftarrow & \mathrm{Hom}(\omega,\omega) & \leftarrow & \mathrm{Hom}(\omega^{\vee\vee}, \omega) & \leftarrow & 0 \end{array}$$

so $\exists\, s : \omega^{\vee\vee} \to \omega$ splitting this seq.

this $\Rightarrow \mathcal{F} = (0)$! $\square$

P.6 This $\Rightarrow$ $\mathrm{Tr}: H^n(\omega_X) \xrightarrow{\sim} k$.

this fact is NOT clear to hold without going through the above.

So ω_X is a divisorial sheaf, i.e. a linear equivalence class of Weil div. traditionally called "the" canonical divisor.

def'n: a Weil div sheaf $\mathcal{F}$ is $\mathbb{Q}$-Cartier
$\Leftrightarrow$ $\mathcal{F}^{[\otimes n]}$ is a line bundle for some $n>0$.

i.e. if $\mathcal{F} = \mathcal{O}(D)$, I saying $\mathcal{O}(nD)$ is everywhere locally principal.

def'n: X has terminal (canonical) sing if

(1) K_X is $\mathbb{Q}$-Cartier

(2) for all $f: Y \to X$ resolution of singularities if I write
$K_Y = f^* K_X + \sum\limits_{E_i \text{ exceptional}} a_i E_i$ $\quad a_i \in \mathbb{Q}$
then $a_i > 0$ ($a_i \geq 0$).

Example: $\dim X = 2$,

$\begin{cases} \text{terminal} \Leftrightarrow \text{non-singular} \\ \text{canonical} \Leftrightarrow \text{Du Val (RDP, ADE, simple sing ...)} \end{cases}$

$\dim X = 3$, there is a reasonable explicit description of terminal singularities, a "good" general theory of canonical singularities.

NOT for higher dim.

the "canonical cover":

X affine, $\mathcal{L}$ is a $\mathbb{Q}$-Cartier div. sh.
$\exists\, r > 0$, minimal st $\mathcal{L}^{[r]} \simeq \mathcal{O}_X$, called index $(\mathcal{L})$. $\pi : Y = \underline{\mathrm{Spec}} \bigoplus_{i=0}^{r-1} \mathcal{L}^{[i]} \longrightarrow X$.
$\pi : Y \to X$ is finite and $\pi^*\mathcal{L}$ is Cartier on Y
μ_r cyclic, Galois cover
i.e. If $X^0 \subset X$ is largest st. $\mathcal{L}|_{X^0}$ is a line bundle and $Y^0 = \pi^{-1} X^0$, $\pi_0 : Y^0 \to X^0$ then $\pi^{0*}\mathcal{L}|_{X^0}$ extends to a line bd on Y.
See Reid [YPG].

$\dim X = 3$: Reid proves

- If $p \in X$ is a terminal singularity and K_X is a line bundle, then $p \in X$ is a cDV sing.
i.e. $\simeq \{ f(x,y,t) + z\, g(x,y,z,t) = 0 \} \subset \mathbb{C}^4$
a Du Val sing.
(This is a VERY HARD result.)
- Mori: If $r = \text{index } K_X > 1$, the canonical cover is cDV, and there is an explicit description of all possibilities.

the main case reads $(xy + f(z^r, t) = 0)$
$\subset \frac{1}{k}(a, -a, 4, 0) = \mathbb{C}^4/\mu_k$ — for certain f's.
weight of action. //

In general X terminal (can) then the can. cover also has terminal (can) sing.
(Exercise).

P. 8 Two main research directions in HDCG.

① To do explicit stuff in dim 3

② To do the general theory in dim ≥ 4.

Summary:

(a) def of opposite and flip

(b) terminal & canonical singularities

If X alg var (nonsingular, projective)
canonical ring

$$R = \bigoplus_{n \geq 0} H^0(X, mK_X)$$

the canonical model of X is : $\bar{X} = \operatorname{Proj} R$,
provided R is f.g.

X is general type if $|nK| : X \dashrightarrow \mathbb{P}^{P_n - 1}$
is birat'l for some $n > 0$.

Theorem: R f.g, X general type
$\Rightarrow \bar{X}$ has canonical singularities.
[proof in YPG].

- log terminal & log canonical singularities.

pairs (X, B)

- normal $\quad B = \sum b_i B_i$; $0 < b_i \leq 1$, $b_i \in \mathbb{Q}$.

prime divisors

Why do we care for pairs ?

(1) when all $b_i = 1$, and X proj, we care about $U = X \setminus B$.
It is well-known that, if $B \subset X$ normal crossing div, then
$H^0(X, n(K_X + B))$ depends only on U.
(so $R(U, K_U) = \bigoplus_{n \geq 0} H^0(X, n(K_X + B))$,
this leads to the Iitaka program.)

(2) Kodaira's adjunction formula.

$f : X \to B\,C$ elliptic surface (fiber genus $=1$)

$$K_X = f^*\left(K_C + M + \sum \frac{m_i - 1}{m_i} P_i\right)$$

$M = f_* K_{X/C}$ a (semi-positive) line bundle

m_i = multiplicity of fiber over P_i.

$$R(X, K_X) = R\left(C, K_C + M + \sum \frac{m_i - 1}{m_i} P_i\right)$$

- pairs also appears naturally in constructions

(3) $\pi : X' \to X$ finite

$$K_{X'} = \pi^*\left(K_X + \sum \frac{e_i - 1}{e_i} B_i\right)$$

branched div.

(4) Restriction/adjunction

$S \subset X$: $K_S + \mathrm{Diff}_S = (K_X + S)|_S$.

to be discussed later.

(5) Resolution :

$X = \mathbb{C}^2/\mathbb{Z}/n = \frac{1}{n}(1, q)$ notation : action with weights $1, q$.

$f : Y \to X$ min resol.

$f^* K_X = K_Y + \sum a_i E_i$, $a_i \in \frac{1}{n}\mathbb{Z}$.

key point : ter. can. sing. do not always behave well in various constructions, need log-version.

(6) subadjunction,

X non-normal + reasonable assumptions

$\nu : \tilde{X} \to X$

normalization can often make sense of a formula $\nu^* K_X = K_X + \Delta$.

Rmk: (3) may be better written using div on X': $K' + R = \pi^*(K + B)$

see Kollár's paper "sing of pairs".

def'n (X, B) has klt (plt, lc) sing if for all good resolutions of sing. $f: Y \to X$, I can write

$K_Y = f^*(K_X + B) + \sum a_j E_j$ where

all $a_j > -1$ ($a_j > -1$ if E_j is exceptional, $a_j \geq -1$.)

$a_j =:$ discrepancy of divisor $E_j = a(E_j)$

note $a(B_i) = -b_i$, so for klt $b_i < 1 \ \forall i$.

note: the definition implicitly requires that $K_X + B$ be $\mathbb{Q}$-Cartier.

Example: $\dim X = 2$,

if $B = \emptyset$, X has klt $\Leftrightarrow$ X has plt $\Leftrightarrow$ locally analytically $X = \mathbb{C}^2/G$ for G = finite group.

pf is based on:

- Proposition: $\pi: X' \to X$ finite étale in codim 1, $B \subset X$, $B' = \pi^* B$, then $K_X + B$ is klr (plt, lc) $\Leftrightarrow$ $K_{X'} + B'$ is so.

This is the reason that pairs is more suitable for constructions. pf is not hard.

p.12 pf of example:

$\Leftarrow$ follows immediately from prop.

$\Rightarrow$: let $X' \to X$ be the canonical cover, then X' has plt sing and $K_{X'}$ is Cartier (by construction).

all $a_j > -1$, they are integers $\Rightarrow a_j \geq 0$

X' has canonical sing.

$\Rightarrow X' = \mathbb{C}^2/G$ where $G \subset SL(2,\mathbb{C})$.

$\Rightarrow X = X'/\mu_k$ also has quotient sing.

Inversion of adjunction:

adjunction formula, in good generality, we can make sense of the following

prime divisor $S \hookrightarrow X$, "$K_S = K_X + S|_S$"

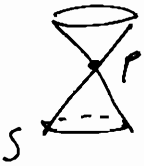
e.g. S, X nonsingular, (only need X)

$$0 \to N_S^{\vee}X \to T_X^{\vee}|_S \to T_S^{\vee} \to 0$$

$$\Rightarrow K_S = (K_X \otimes N_S X)|_S = (K+S)|_S.$$

However, in general we may need
when X is singular

$$K_S + \mathrm{Diff} = (K_X + S)|_S.$$

Example, $X =$ 

$\subset \mathbb{P}^3$, quadric cone.

$S =$ ruling $\subset X$

$K_S \neq K_X + S|_S$! $\longrightarrow 2S \sim \mathcal{O}(1)$, so $S \sim \mathcal{O}(\frac{1}{2})$!

$\mathcal{O}_{\mathbb{P}^1}(-2)$ $\quad$ $\mathcal{O}_{\mathbb{P}^3}(-4+2)$

Thus $K_S + \frac{1}{2}P = K_X + S|_S$.

Theorem: provided that everything make sense,

$K + S + B$ plt $\Leftrightarrow$ $K_S + \underbrace{\mathrm{Diff} + B|_S}_{B_S}$ is klt.
in a nbd of S

pf see [FA, § 17.7]

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problem ⇑ is hard (need connectedness)

Minimal Model Program:

X normal projective

(X, B) either $\begin{cases} B = \phi & \text{ter. can} \quad \text{or} \\ B \neq \phi & \text{klt, plt} \end{cases}$

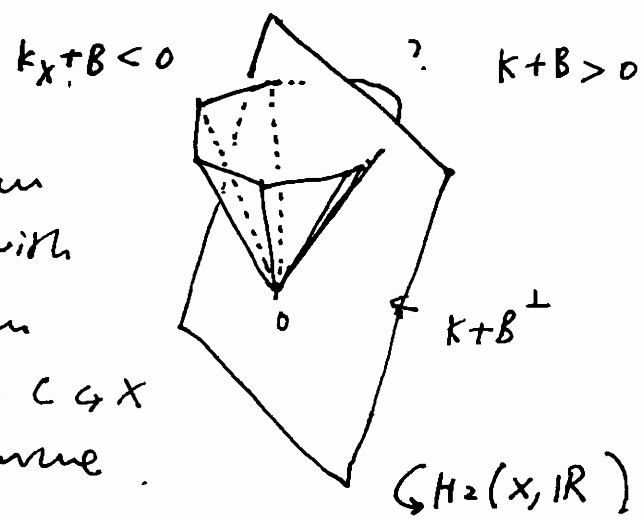
Mori cone:

$$\overline{NE}(X) = \overline{\sum_{C \subset X \text{ curve}} \mathbb{R}_+[C]} \subset H_2(X, \mathbb{R}).$$

Theorem of the cone:

$\overline{NE}(X)$ is locally finitely generated in the half space $\{K + B < 0\}$.

(if $a \in H_2(X, \mathbb{R})$, $(K+B).a := c_1(K+B) \cap a$.)



let $R \subset \overline{NE}(X)$ be an extremal ray with $(K+B).R < 0$, then $R = \mathbb{R}_+[C]$ where $C \subseteq X$ is a rational curve.

Contraction Theorem

Let $R \subset \overline{NE}(X)$ be an extremal ray with $(K+B).R < 0$. There exists a morphism:

$\varphi = \mathrm{Cont}_R : X \to Y$ st for $C \subset Y$ any curve, $\varphi(C) = pt \Leftrightarrow [C] \in R$.

def'n: X is $\mathbb{Q}$-factorial if any Weil div on X is $\mathbb{Q}$-Cartier.

Types of cont_R:

$$\varphi_R \text{ birational } \begin{cases} \varphi \text{ contracts a divisor } E & ① \\ \varphi \text{ small} & ② \end{cases}$$

φ fibering : $\dim Y < \dim X$. ③

Rmk: ① If E is $\mathbb{Q}$-Cartier then $E = \mathrm{Exc}\,\varphi$ and $(Y, \varphi(B))$ have same singularities as (X, B). E is not auto $\mathbb{Q}$-Cartier!! ($E = \mathrm{Exc}\,\varphi$ is auto).

Note: $\mathbb{Q}$-factorial is a local property in Zariski top. but not in the analytic top.

* Example: $X = (xy + zt = 0) \subset \mathbb{C}^4$ is NOT $\mathbb{Q}$-factorial. eg. $(x = 0 = z) \subset X$ is not $\mathbb{Q}$-Cartier.

However, let $X_4^3 \subset \mathbb{P}^4$ be non-singular outside a ordinary double pt, then X is $\mathbb{Q}$-fact.

indeed : $H_4(X,\mathbb{Q}) = \mathbb{Q}$

$$\begin{array}{c} Bl_p X = Y \hookrightarrow Bl_p \mathbb{P}^4 = \tilde{\mathbb{P}} \\ \pi \downarrow \\ X \end{array}$$

the key point : $Y \subset \tilde{\mathbb{P}}$ is ample, hence by Lefschetz,

$\mathbb{Q}^2 = H^2 \tilde{\mathbb{P}} \simeq H^2 Y = \mathbb{Q}^2$

$\Rightarrow H_4(X,\mathbb{Q}) = \langle E, \pi^* \mathcal{O}(1) \rangle = \langle \mathcal{O}(1) \rangle$

This explains the subtlety of the global nature of $\mathbb{Q}$-factoriality.

MMP : (X,B) $\mathbb{Q}$-factorial, klt (or plt)

$K+B$ nef ? if not, $\varphi_R = \text{cont}_R : X \to Y$

$$\begin{cases} \varphi_R \text{ divisorial} \Rightarrow \text{conti with } (Y, \varphi(B)) \\ \varphi_R \text{ fibering} \Rightarrow \text{stop} \\ \varphi_R \text{ small} \Rightarrow \text{flip} \end{cases}$$

Notice that in dealing with flips, we usually need to consider analytic topology, hence loose the $\mathbb{Q}$-factorial condition i.e. we do flips always without $\mathbb{Q}$-fact assumption.

Conjecture : flips exists and terminate

why do we believe that flips terminate ?

Theorem : let $X \overset{t}{\dashrightarrow} X'$ be a flip.

$$\begin{array}{ccc} X & \overset{t}{\dashrightarrow} & X' \\ & \varphi_R \searrow \quad \swarrow \varphi' & \\ & Z & \end{array}$$

φ_R small contr. of $R = \mathbb{R}^+[P]$, Then

$a(E, K+B) \le a(E, K'+B')$ and $<$ at least one philosophy: the larger the discrepancies, the better the singularity.

pf of thm:

$$\begin{tikzcd} & W \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[dr, "\varphi"'] & & X' \arrow[dl, "\varphi'"] \\ & Z & \end{tikzcd}$$

$$K_W + \tilde{B} + E$$
$$= f^*(K+B) + \sum a_i E_i \quad (\text{all } a_i > 0)$$
$$= g^*(\underbrace{K'+B'}_{\varphi'\text{- positive}}) + \sum a_i' E_i$$

$g^*(K'+B') \equiv_f \sum (a_i - a_i') E_i$ is f-nef

General lemma:

$f: W \to X$ birational proper morphism, exceptional divisor E_i.

$A := \sum \alpha_i E_i$ f-nef $\Rightarrow$ $\alpha_i \leq 0$. And if moreover $A \not\equiv 0$, then some $\alpha_i < 0$.

pf of lemma: If W is a surface, this is well-known and elementary. in fact $(E_i \cdot E_j) < 0$. The general case is reduced to the surface case by slicing with hyperplane. $\square$

§1. Reduction to pl-flips

def'n: $(X, S+B)$, $B = \sum b_i B_i$, $0 < b_i < 1$
(S prime div. w coeff $= 1$)

a flipping contr. for $K+S+B$ is pl if:
(i.e. f small & $K+S+B$ is f-neg)

$$\begin{array}{c} X \\ f \downarrow \\ Z \end{array}$$

(1) X is $\mathbb{Q}$-factorial
(2) S is f-negative

(i) Note that $S \supset \operatorname{Exc} f$. indeed if $C \subset X$ is contracted, then $S.C < 0 \Rightarrow S \supset C$.

(ii) Philosophy: pl-flips are easier than general flips: since by (i), "everything happens in S".

Notice this is already in Shokurov's 1991 paper, but it takes 10 yrs to justify that pl-flips is easier.

Theorem: pl flips exists $\Rightarrow$ klt flips exist.
Pf [FA, 18.11, 18.12]:

Step 1: fix resolution $Z' \to Z$ and div. $F_j \subset Z'$ generate $N^1(Z'/Z)$.
choose $\bar{H} \subset Z$ reduced Cartier st.

p. 18

$$\begin{cases} f^* \bar{H} = H \supset \mathrm{Exc}\, f \\ \bar{H} \supset \mathrm{Sing}\,(Z, f(B)) \\ \bar{H} \supset \pi(F_j) \end{cases}$$

$\Rightarrow$ (*) If $h : Y \to Z$ is any proper bi-nat'l morphism with exc div E_i, then $N'(Y/Z)$ is gen by E_i and components of $\bar{H}'$.
$(\cdot)'$ always means proper transf.

Let $f : X \to Z$ be a ker flipping contr. for $K_X + B$ ($B = \sum b_i B_i$),

Step 2. choose a resolution

$$\begin{tikzcd} Y \arrow[r, "g"'] \arrow[rr, bend right, "t"'] & X \arrow[r, "f"'] & Z \end{tikzcd}$$

which is an iso over $Z \setminus \bar{H}$.

Run a MMP for $K_Y + B' + H' + \sum_i E_i$, notice in this case all flips needed are all pl-flips:

if $C \subset Y$ flipping curve, then

$0 = (t^* \bar{H} \cdot C) = (\sum H_j' + \sum \beta_i E_i, C)$
all $\beta_i > 0$

and at least one of

this $H_j' \cdot C$; $E_i \cdot C \neq 0$ $\Rightarrow$ at least one of this is < 0.

At the end, I may assume

$K_Y + B' + \sum H_j' + \sum E_i$ is nef.

Note: a $\pm$ flip is a flip for
$K + S_1 + S_2 + B$ with S_1 negative,
S_2 positive. step 2 uses only $\pm$ flips.
philosophy: $\pm$ flips are "even easier"
than general pl-flips.

step 3. I have a partial resolution

$$Y \xrightarrow[g]{} X \xrightarrow[f]{} Z \quad \text{which is iso over } Z \setminus \bar{H}$$
$$\text{and } K_Y + B' + \Sigma E_i + H' \text{ nef}$$

If $K_Y + B' + \Sigma E_i$ is already nef, then run a MMP for $K_Y + B' + \Sigma E_i$. The lc model of the final product is the flip I want to construct.
[cf. Reid "surface of small degree"].
substract little bits of H', look at

$$K_Y + B' + \Sigma E_i + (1-\varepsilon) H'$$

with ε largest st this is nef. (if $\varepsilon = 1$ then I'm finished.)

$\Rightarrow$ There exists $R \subset \overline{NE}(Y/Z)$
with $H' \cdot R > 0$, $(K_Y + B' + \Sigma E_i) \cdot R < 0$.
operate on this R:
If R is a flipping ray, then it is pl:
$0 = (f^* \bar{H} . R) = (H' + \Sigma \beta_i E_i . R) \Rightarrow$ one of
$E . R < 0$, "almost" end of pf. $\square$

- Notice in Step 3, since H' do not have coeff 1, it is not a $\pm$ flip, it is a pl-flip.
- There is also an issue with termination of pl-flips, which is not discussed yet. Usually this needs to assume a lower-dim'l MMP to get it. Thus "almost".

§2. Set up for construction of pl-flips.

B-divisors (after Shokurov)

def'n: X normal var. a model of X is a proj birat'l morphism $f: Y \to X$.

a b-divisor is an element $\mathbb{D} \in \varprojlim_{\substack{Y \to X \\ \text{models}}} \text{W-Div}\, Y$.

basically, $\mathbb{D} = \sum d_i D_i$

where D_i is a valuation of $k(X)$ with center on X.

if $Y \to X$ is a model, $\mathbb{D}_Y := \sum_{\substack{D_i \subset Y \\ \text{is a divisor}}} d_i D_i$

$=$ trace of $\mathbb{D}$ on $Y \in \text{W-Div}\, Y$.

example. Cartier closure.

D : $\mathbb{Q}$-Cartier div on X,

$\bar{D}$ = Cartier closure of D is a b-divisor st.

$\bar{D}_Y = f^* D$ for $f: Y \to X$.

* def'n: $\mathbb{M}$ on X is mobile, if $\exists\, Y \to X$ st.

(1) $\mathbb{M} = \overline{\mathbb{M}_Y}$ (as an integral div.)

(2) $|\mathbb{M}_Y|$ is free from base points.

Example: D integral div on X.
$M = \mathrm{Mob}\, D$ (the mobile part of D)
is defined by: for all $f: Y \to X$

$$\begin{cases} M_Y \in \mathrm{Mob}\, f_*^{-1}|D| \\ f(M_Y) \subset D \end{cases}$$

notice that this def applies to also non-Cartier D,
for D Cartier, this is the same as $\mathrm{Mob}\,|f^*D|$.

Now fix a pl-contraction:

$$(X, S+B) \xrightarrow[f]{} Z$$

choose a general element $D \in |r(K+S+B)|$
form $M_i = \mathrm{Mob}(iD)$. $\quad$ ($r \in \mathbb{Z}$ to get Cartier.)
note $M_{i+j} \geq M_i + M_j$.

def'n: $D \rightsquigarrow \mathcal{O}_X(D)$

$$H^0(X, \mathcal{O}_X(D)) = \{ f \in k(X) \mid \mathrm{div}_X f + D \geq 0 \}$$

can form algebra:

$$R = \bigoplus_{i \geq 0} H^0(X, M_i) = \bigoplus H^0(X, iD)$$

Notice that these 2 algebras are the same!

Fact: flip exists $\Leftrightarrow$ R is finitely generated.

Mobile restriction:

M mobile on X, $S \subset X$ codim $=1$, $S \not\subset \mathrm{Supp}\, M_X$.
$\Rightarrow$ it makes sense to restrict $\mathrm{res}_S M = M^0$.
Take $f: Y \to X$ high enough to make $\star$.
then $M^0 := \overline{M_Y|_{S'}}$.

Consider $\mathbb{M}_i^\circ = \mathrm{res}_S \mathbb{M}_i = \mathrm{res}_S \mathrm{Mob}_X iD$

and $R^\circ = \bigoplus H^0(S, \mathbb{M}_i^\circ)$.

($R \to R^\circ$ is integral)

Lemma: R° f.g $\Rightarrow R$ f.g.

Consider a sequence of mobile divisors on X, $\mathbb{M}_i$, s.t. ① $\mathbb{M}_i \neq 0$

② $\mathbb{M}_{i+j} \geq \mathbb{M}_i + \mathbb{M}_j$

denote $\mathbb{D}_i = \frac{1}{i}\mathbb{M}_i$.

A pbd algebra is an alg of the form

$$R(X, \mathbb{D}_\bullet) = \bigoplus_{i \geq 0} H^0(X, \underset{iD_i}{\overset{\shortparallel}{\mathbb{M}_i}}).$$

Rmk: We "may" always assume Z affine, hence any div on X is mobile, since $f: X \to Z$ small and Z affine.

Limiting criterion:

$R(X, \mathbb{D}_\bullet)$ is f.g. $\Leftrightarrow \exists\, i_0 > 0$ st. $\mathbb{D}_{i i_0} = \mathbb{D}_{i_0}, \forall i \geq 1$.

i.e. $\mathbb{D}_\bullet$ "stablizes" (pf: Exercise.)

Key properties:

1. $\mathbb{D}_\bullet$ is bounded (ie. $\exists\, \mathbb{D}$ st. $\mathbb{D}_i \leq \mathbb{D}\ \forall i$).
2. Saturation.

def^n: D, C: $\mathbb{Q}$-divisors on X. D is C-saturated if $\mathrm{Mob}\lceil D + C\rceil \leq D$.

(we require though C is not nec. ef. but $\lceil D+C\rceil \geq 0$)

Key example: $f : Y \to X$ proj. birat'l.

D : $\mathbb{Q}$-Cartier, integral Weil div on X and $D \geq 0$. $C = \sum \beta_i E_i \geq 0$ on Y, E_i are f-exceptional. Then

$\Rightarrow$ f^*D is C-Saturated !

This is a complicated way to say

$$f_* \mathcal{O}_Y(\lceil f^*D + \sum \beta_i E_i \rceil) = \mathcal{O}_X(D).$$

in general only get $\subseteq$. If $\beta_i \geq 0$ then $\Rightarrow$ "=".

(to say "=" is the same as saying $f_*\mathcal{O}_Y$ is saturated.)

$\mathbb{D}$ is $\mathbb{C}$-saturated if

$$\mathrm{Mob} \lceil \mathbb{D}_Y + \mathbb{C}_Y \rceil \leq \mathbb{D}_Y$$

on all high-enough models $Y \to X$

i.e. All $Y \to Y_0 \to X$ where $Y_0 \to X$ is the same fixed model.

Example: (X, B) klt pair

$\mathbb{A} = \mathbb{A}(X, B)$ = discrepancy st.

$K_Y = f^*(K_X + B) + \mathbb{A}_Y$

when $\mathbb{C} = \mathbb{A}$, I simply say $\mathbb{D}$ is saturated.

def'n: $\mathbb{D}_\bullet$ is asymptotically saturated if for all i, j,

$$\mathrm{Mob} \lceil j \mathbb{D}_{i\,Y} + \mathbb{A}_Y \rceil \leq j \mathbb{D}_{j\,Y}$$

on high (depending on i, j) models.

def'n: $R(X, \mathbb{D}_\bullet)$ (for a pair (X, B)) is a Shokurov algebra if $\mathbb{D}_\bullet$ is bounded and asymptotically saturated.

Conjecture: (X, B) klt pair, $f: X \to Z$ proj, birat'l, $-(K+B)$ nef$/Z$: affine
$\Rightarrow$ All Shokurov algebras on X are f.g.

(in applications, we use $X \leftarrow S$, $Z \leftarrow f(S)$.)

Example: $X = \mathbb{A}^1$, $B = b\,0$ (origin) $(0 < b < 1)$

$A = -b\,0$, $D_i = d_i\,0$

condition $(i+j)\,d_{i+j} \geq i d_i + j d_j$

$\Rightarrow$ $d = \lim d_i$ exists.

asymptotical saturated means

$\lceil j d_i - b \rceil \leq j d_j$

$i \to \infty$ get $\lceil jd - b \rceil \leq jd$ $\forall j$,

this $\Rightarrow$ $d \in \mathbb{Q}$ and $d_j = d$ $\forall j$ sufficiently divisible.

Next Time:

$$\begin{array}{ll} X & \text{pl flipping contraction}, \quad S+B \\ \downarrow & \\ Z & \quad D \sim r(K+S+B) \end{array}$$

$R^0 := R(S, \mathbb{D}_i^0)$ is a Shokurov algebra

$\mathbb{D}_i^0 = \frac{1}{i}\,\mathrm{res}_S\,\mathrm{Mob}_X(iD)$. i.e. asymp. sat. holds.

Flips : lecture 4 (VII, VIII)

Recap : mobile b-div $\mathbb{M}$ on X

i.e. $\exists\ f : Y \to X$ prop birat'l st

(1) $\mathbb{M} = \overline{\mathbb{M}_Y}$

(2) $|\mathbb{M}_Y|$ is free

(I say that $\mathbb{M}$ "descends to Y")

Example : $X = \mathbb{P}^2$, $\mathbb{M}_X$ = line on $\mathbb{P}^2$

① $\mathbb{M} := \overline{\mathbb{M}_X}$ (linear sys $|\mathcal{O}_{\mathbb{P}}(1)| = |\mathcal{O}_X(\mathbb{M})|$)

② $Y = Bl_p\, \mathbb{P}^2$, $p \in \mathbb{M}_X$, $\mathbb{M}_Y$ = proper transf

$\mathbb{M} := \overline{\mathbb{M}_Y}$ (linear sys $|m_p(1)| = |\mathcal{O}_X(\mathbb{M})|$)

mobile part :

let $\mathbb{D}$ on X a b-div,

let $V \subset H^0(X, \mathcal{O}(\mathbb{D})) \subseteq \{\varphi \in k(X) \mid \operatorname{div}_X \varphi + \mathbb{D} \geqslant 0\}$

$\mathbb{M} = \mathbb{M}\mathrm{ob}_V\, \mathbb{D}$ is defined as

$$\operatorname{mult}_E \mathbb{M} := -\min_{\varphi \in V} \operatorname{mult}_E \varphi .$$

(on a curve $\mathrm{Mob}\, D = D - \mathrm{Fix}\, V$)

In the example, $p \in L \subset \mathbb{P}^2$ be a line,

$V = H^0(\mathbb{P}^2, \mathcal{O}(1))$ $\quad \mathbb{M}\mathrm{ob}_V\, L = ①$

$V = H^0(\mathbb{P}^2, m_p(1))$ $\quad \mathbb{M}\mathrm{ob}_V\, L = ②$

Key point : $\quad \mathbb{M}\mathrm{ob}_V\, D = \mathbb{M}\mathrm{ob}_V\, \hat{D}$.

proper transf .

Start of Today:

$$\begin{array}{ccc} X & \supset & S \\ {\scriptstyle f}\downarrow & & \downarrow \\ Z & \supset & T \end{array}$$

pl-flipping contraction for $K+S+B$; FACT: flip exists $\Longleftrightarrow$

$R = R(X, M_\cdot) := \bigoplus_{i \geq 0} H^\cdot(X, \mathcal{O}(M_i))$ is f.g.

where $M_i = \mathrm{Mob}\, iD$

$D \sim r(K+S+B)$

If D general, $S \not\subset \mathrm{Supp}\, D$. It makes sense to form restriction $M_i^\circ = \mathrm{res}_S M_i$.

(Notice this requires to be in a high model)

$R^\circ = R(S, M_\cdot^\circ)$.

FACT: R is f.g $\Longleftrightarrow$ R° is f.g.

A word on this: R° = quotient of R by a principal ideal (ef for S), since we in fact work with $\rho(X/Z) = 1$, all div are linearly-equiv.

(or R = int closure of R° ...) Not too hard.

def'n: $D_i = \frac{1}{i} M_i$ asymptotically saturated

if $\mathrm{Mob} \lceil j D_{iY} + A_Y \rceil \leq j D_{jY}$

on $Y \to X$ "high enough". (depends on i, j)

(Crucial pt: If can get Y indep of i, j then can prove flip conj.)

Thm 1. R° is asymptotically sat.
(i.e. is a Shokurov algebra)

Conjecture: If (X, B) is a klt pair, $f: X \to Z$ a weak log-Fano morphism (i.e. $-(K_X + B)$ is f-nef, f-big), then every Shokurov alg on X is f.g.
(Even the case $Z = pt$ is very interesting)

This conj is OK if dim $= 2$, and it implies existence of 3-fold flips, details see Corti's paper.

This is today's Thm 2.

---

proof of Thm 1:

Lemma: $(X, S+B)$ weak log-Fano plt pair, M is except. sat. on $X \Rightarrow M^\circ = \mathrm{res}_S M$ is $A(S, B_S)$-sat.
$(\mathrm{Mob}\lceil M^\circ_T + A_T \rceil \le M^\circ_T$ on high $T \to S$)

This is essentially the case $i = 1$. The general case is the same, only notationally more involved.

pf (X-method):

Take $f: Y \to X$ that satisfies M.

Let $T = f_*^{-1} S \subset Y$. Then

$M^\circ_T = M_Y|_T$ (by def'n of M°).

I want to understand

$H^0(T, \lceil M_T^\circ + A_T \rceil)$ in terms of restriction from Y.

$K_T = f^*(K_S + B_S) + A_T^{(S, B_S)}$ (def^{n} of A_T)

$K_Y + T|_T$ " $K + S + B|_S$ by def^{n} of B_S (adjunction formula)

$K_Y = f^*(K + S + B) + A(X, S+B)_Y$

$\Rightarrow A(S, B_S)_T = (T + A(X, S+B)_Y)|_T$.

So, $(M_Y + T + A_Y)|_T = M_T^\circ + A_T$

we are ok if

① $H^0(Y, M_Y + T + \lceil A_Y \rceil) \twoheadrightarrow H^0(T, (\cdot\cdot)|_T)$,

② $\mathrm{Mob}(M_Y + \underbrace{T + \lceil A_Y \rceil}) \leq M_Y$.

this together is exceptional so ② holds.

for ①, $H^1(Y, M_Y + \lceil A_Y \rceil) = 0$: since it

$= H^1(Y, K_Y + M_Y + \lceil -f^* K + S + B \rceil)$

$= 0$ by Kawamata-Viehweg van thm.

done. $\square$

The following outline in the 4-fold case is not finalized yet. it may contains error. See notes to be handed later.

4 fold flips :

$$\begin{array}{ccc} X & \subset & \tilde{X} \\ f \downarrow & & \downarrow \tilde{f} \\ Z & \subset & \tilde{Z} \end{array} \qquad \left\{ \begin{array}{l} \tilde{M}_{\bullet} = \mathrm{Mob}\, i\tilde{D} \\ D \sim r(\tilde{K} + X + \tilde{B}) \end{array} \right.$$

$$\Rightarrow M_{\bullet} = \mathrm{res}_X \tilde{M}.$$

4-fold pl-flipping contr. for $\tilde{K} + X + B$

$R = R(X, M_{\bullet})$

$D_{\bullet} = \frac{1}{i} M_{\bullet}$

We want to show that $\exists\, n$ st $D_{ni} = D_i\ \forall i$.

* Method to show the f.g. conj for $f : X \to Z$ needs the addition data that it is inside the flipping conj. (so this does not really prove Shokurov's conj)

Lemma : (X, B) klt pair, $f : X \to Z$ bi-rat'l w. log Fano $\Rightarrow \exists\, B < \bar{B}$ st. $K + \bar{B} \sim 0$ and $(X, \bar{B})$ is log canonical and $\exists !\ E$ with $a(E, K + \bar{B}) = 1$.

Pf: Idea: choose some D, take

$\bar{B} = B + t_0 D$

$t_0 := \sup \{ t \mid K + B + tD \text{ log can.} \}$ $\square$

Theorem 1: If E is as in the lemma, (there exists $M_i' \sim M$ st.) $\mathrm{mult}_E M_i' = 0\ \forall i$.

(ie. it makes sense to restrict to E.

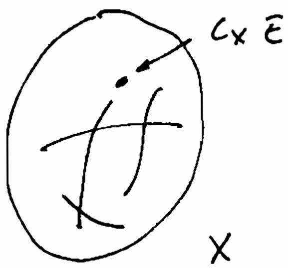
the (...) part is auto since $M_{\bullet}$ is general.

P.30 what is good about this?

$\mathrm{res}_E\, M_\cdot = M^\circ_\cdot$ is asymp sat.

(if E is created in the lemma, then good Kodaira vanishing works.)

$\dim E = 2$, hence I know that $D^\circ_\cdot$ stabilizes, i.e. $D_i^\circ = D_j^\circ = D^\circ$ $\forall i, j$ "on E"

** 

I would like:

$\exists$ Zariski nbd U of $C \times E$ st. $D_\cdot|_U$ stabilizes.

This is the "whole idea", similar to Grothendieck's proof of the Lefschetz thm.

However, this can't work directly, since for each i, need certain model Y and the U can be made. for i varies, then U is not fixed.

Theorem 2. There exists models $Y_m \xrightarrow{g_m} Z$ resolving D_m ($D_m = \overline{D_{m}}_{Y_m}$), st. and Zariski open subset $U_m \subset Y_m$ such that if I write: (← here is the reason we want $K + \bar{B} \sim 0$)

$g_m^*(K_Z + \bar{B}_Z) = K_{Y_m} + \bar{B}_m^+ - \bar{B}_m^-$, with both being > 0,

the the following holds:

① $c_E Y_m = E_m$ is a divisor $\subset U_m$ and $U_m \cap \bar{B}_m^- = \phi$.

② $D_m|_{U_m}$ descends to U_m, i.e. $D_m|_{U_m} = (D_m|_{U_m})_{U_m}$.

③ $D_{im}|_{U_m} = D_m|_{U_m}$ $\forall i \geq 1$.

Rmk: Already U_1 makes ** works:

i.e. $g: Y \to Z$ resolves D_1 & $U \subset Y$ st

$g^*(K_Z + \bar{B}_Z) = K_Y + \bar{B}^+ - \bar{B}^-$, then

① $c_Y E = E'$ divisor $\subset U$ and $U \cap B^- = \phi$

② $D_1|_U$ descends to U.

③ $D_i|_U = D_1|_U$ all $i \geq 1$.

i.e. works for D_1, and almost works for D_i, only over U. The U_m is try to enlarge U.

Theorem 3. There exists a plt pair $(Z', \bar{B}')$ and $h: Z' \to Z$ proper bi-rat'l, $(K_Z + \bar{B}') = h^*(K_Z + \bar{B}_Z)$. and a Zariski open $U' \subset Z'$ st. ①, ②, ③ holds.

Key point: $\exists$ only finite such pairs $(Z', \bar{B}')$.

Theorem 4: There exists such a pair $Z', \bar{B}'$ where $U = Z'$.

Next time will explain thm 2. Thm 3 is not a big deal. Will say something about thm 4.

Corti 2004, 2/9

"Last lecture on flips".

$$\begin{tikzcd} X \arrow[d, "f"'] & \subset & \tilde{X} \arrow[d, "\tilde{f}"] \\ Z & \subset & \tilde{Z} \end{tikzcd}$$

pl contr. for $\tilde{K} + X + B$

$K + B = \tilde{K} + X + B|_X$; $\tilde{R}$

$R = \mathrm{res}_X \tilde{R}$.

Quasistable models (Shokurov calls this "distab" model, but this is strange and I (Corti) can't use it!)

$$\begin{tikzcd} X \arrow[rr, dashed, "\varphi"] & & \mathbb{P} \\ & Y \arrow[ul] \arrow[ur] & \end{tikzcd}$$

φ rat'l, resol. of base locus of a linear system by Y.

rmk: for non-sing surfaces, simply blowing-up the pt with largest mult. then repeat. Can do this in a min way.

Take Y to be "minimal" resol. of base locus. (How?)

X, mobile b-divisor M on X, the stable model is

$$\bar{Y} = \mathrm{Proj}_{\mathcal{O}_X}\Big(\underbrace{\bigoplus_{i \geq 0} H^0(X, \mathcal{O}_X(iM))}_{R}\Big)$$

(if $Y \to X$ is st. M_Y is free, $M = \bar{M}_Y$ then $R = R(Y, M_Y)$.)

 basically the stable model is the normalized closure of the graph of $\varphi = \varphi_{|M|}$: (due to Zariski)

$$\begin{array}{ccc} & \Gamma_\varphi \leftarrow \Gamma_\varphi^\nu = \bar{Y} & \\ \swarrow & & \searrow \\ X & \overset{\varphi}{\dashrightarrow} & P \\ \varphi_{|M|} & & \end{array}$$

Any min resol. must go through $\bar{Y}$.

Def'n. Let (X, B) be a klt pair and M a mobile b-divisor on X. A lt (lc) quasi-stable model of M is a model $g : Y \to X$ st. one of the following equivalent condi holds : (where $F_i \subset Y$ the g-exc div's)

① $K_Y + B' + \Sigma F_i$; $(Y, \Sigma F_i + B')$ is lt (lc)
is nef over $\bar{Y}$.
(ample)

We do not treat lt before. but we only require the lc case.

② $(Y, \Sigma F_i + \tilde{B})$ is lt (lc) and
$K_Y + \Sigma F_i + B' + c_0 M_Y$ is nef (ample) over X.
(where $c_0 = 2 \dim X + 1$).

③ $(Y, \Sigma F_i + \tilde{B})$ is lt (lc) and
$K_Y + \Sigma F_i + B' + c M_Y$ is nef (ample) over X
for all $c \geq k_0$. (k_0 some constant)

---

① is most natural.

explanations :

$1 \Rightarrow 2$: $Y \to \bar{Y} \to X$ so $|\mathbb{M}_Y|$ is free & $\mathbb{M} = \bar{\mathbb{M}}_Y$

I want to show that

$K_Y + \Sigma F_i + B' + c_0 \mathbb{M}_Y$ is nef $/X$. If not,

$\exists$ extr. ray $R \subset \overline{NE}(Y/X)$ with

$$R \cdot (K_Y + \Sigma F_i + B') < 0 \quad (\text{since } \mathbb{M}_Y \text{ free})$$

Indeed ; $R = \mathbb{R}_+[\Gamma]$ where "$-\Gamma \cdot (K_Y + \Sigma F_i + B') < c_0$"

then $\Gamma \cdot \mathbb{M}_Y = 0$ (can't be > 0). $\uparrow$ by Kawamata (?)

then Γ maps to ... in X $\times$

construction of $Y \to X$:

(i) Take $Y \to \bar{Y}$ a resolution. Run $K_Y + \Sigma F_i + B'$ MMP over $\bar{Y}$. Or

(ii) Take $Y \to X$ a resolution. Run $K_Y + \Sigma F_i + B' + c_0 \mathbb{M}_Y$ MMP over X.

Rmk : lt quasi-stable model exists in dim $= n$ if MMP exists in dim $= n$.

In some cases, existence of $\pm$-flips and termination is sufficient.

(see " reduction to pl flips ", there is one step which uses only $\pm$ flips.)

$\pm$ flips are NOT discussed here. We will simply use it when needed.

 $X, \mathbb{M}$

Rmk: $g: Y \to X$ lc quasi-stable model

$$K_Y + \Sigma F_i + B' = g^*(K_X + B) + \underbrace{\Sigma a_i F_i}_{\text{all } a_i > 0}$$
ker

write $F = \frac{1}{c_0} \Sigma a_i F_i$, then

$\frac{1}{c_0}(K_Y + \Sigma F_i + B' + c_0 \mathbb{M}_Y) = F + \mathbb{M}_Y$ is ample$/X$.

Shokurov calls the pair (Y, F) the "distab" model of $(X, B, \mathbb{M})$.

Indeed, F in the pf of next statement.

BACK TO 4-FOLD FLIPS:

Lemma. Let $\mathcal{R}$ be exceptionally asymptotically saturated and $\tilde{E}$ a prime divisor st.

1. $\mathrm{mult}_{\tilde{E}} \hat{\mathbb{D}}_i = 0$ all i,
2. $\hat{\mathbb{D}}_i^\circ = \mathrm{res}_{\tilde{E}} \mathbb{D}_i$ is constant in i.

$\Rightarrow$ If $g: \tilde{Y} \to \tilde{Z}$ is the lc quasi stable model & $\tilde{F} \subset \tilde{Y}$ is as before, then $C_{\tilde{Y}} \tilde{E} \subset \tilde{Y} \setminus \tilde{F}$.

pf: There exists $p > 0$ integer st.

$j \tilde{M}_1 + p\tilde{F}$ is mobile for $j > 0$ and divisible.
free

saturation says

$\mathrm{Mob}(j\tilde{M}_1 + p\tilde{F}) \le \tilde{M}_j$.

So $\tilde{\mathbb{D}}_1 + \frac{p}{j}\tilde{F} \le \tilde{\mathbb{D}}_j$.

$\tilde{E}$ does not appear in $\tilde{\mathbb{D}}_j \Rightarrow \hat{E}$ does not appear in $\tilde{F}$.

restricting to $\hat{\tilde{E}}$ get $\mathrm{res}_{\tilde{E}} \overline{\tilde{F}} = 0$

$\Rightarrow \quad c_{\tilde{F}} \hat{E} \cap \tilde{F} = \phi$.

Theorem 1 (= thm 2 last time): slightly diff

$$\begin{array}{ccc} X & \subset & \tilde{X} \\ f \downarrow & & \downarrow \tilde{f} \\ Z & \subset & \tilde{Z} \end{array}$$

4 fold

pl-contr; $\tilde{R}$ etc.

- choose $\tilde{B}_1 > \tilde{B}$ st.

$K_{\tilde{X}} + X + \tilde{B}_1 \sim 0$ & $B_1 = \tilde{B}_1 | X$. not klt, only plt

Abuse notation: $B_1 = f(B_1) \subset Z$.

There exists a model $g: Y \to Z$, $\phi \neq U \subset Y$ open, $m > 0$ integer st. the following conditions. Where we define $B_1^* = B_1^+ - B_1^-$ by

$$g^*(K + B_1) = K_Y + B_1^*$$
$$= K_Y + B_1^+ - B_1^-.$$

(1) $U \cap B_1^- = \phi$,

(2) If E is a valuation st. $\mathrm{res}_E \mathbb{D}_{m_i} = \mathrm{res}_E \mathbb{D}_m$ for all i, then $c_Y E \subset U$.

(3) $\mathbb{D}_m |_U$ descends to U.

(4) $\mathbb{D}_{m_i} |_U = \mathbb{D}_m |_U$ for all $i \geq 1$.

proof: recall $\tilde{\mathbb{D}}_\bullet$, $\mathbb{D}_\bullet$. we find E, $\mathbb{D}_\bullet^\circ = \mathrm{res}_E \mathbb{D}_\bullet$ which we want to prove stabilization.

 by thm 1 of last time (no pf),
there exists E, $m > 0$ s.t.
$\mathrm{res}_E \mathbb{D}_{m_i}$ is independent of i.
let $\hat{g} : \hat{Y} \to \tilde{X}$ be the log-canonical
quasi-stable model of $\tilde{M}_m$
(this uses only $\pm$ flips, details left
to readers to check.).
Let $\tilde{F} = \frac{1}{c_0}\left(K_{\hat{Y}} + \sum \tilde{F}_i + \hat{B}_1' - \hat{g}^*(K_{\tilde{X}} + \tilde{B}_1)\right)$
as before.
So $\tilde{F} > 0$ supported on the whole exc div's
of $\hat{Y} \to \tilde{X}$, and $\tilde{F} + M_{\hat{Y}}$ ample $/\tilde{X}$.
I claim that Y = normalization of
$\hat{g}_X^{-1}(X) \to \hat{X}$ and $U = Y \setminus \tilde{F}$
satisfy the conditions.
Lemma $\Rightarrow$ (2). $B_1^* = \tilde{B}_1^*|_X$ and $\hat{B}_1^-$ is exc.
hence $\subset \tilde{F}$, hence $\tilde{B}_1^-|_U = 0$, so (1).
(3) is ok by construction.
(4) : ~~by (2)~~ $\mathrm{tr}_{\tilde{U}} \mathbb{D}_{m_i}|_{\tilde{U}} = \mathrm{tr}_{\tilde{U}} \mathbb{D}_m|_{\tilde{U}}$.
$\tilde{U} := \hat{Y} \setminus \tilde{F} \to \tilde{X}$ is small, therefore
and restricting to U :
$\mathrm{tr}_U \mathbb{D}_{m_i}|_U = \mathrm{tr}_U \mathbb{D}_m|_U$.
Not too hard then to show
$\mathbb{D}_{m_i}|_U = \mathbb{D}_m|_U$. $\square$

Key point is in the lemma: Exceptionally asymp sat is a very strong condition. $\tilde{R}$ being exc. asymp. sat $\not\Rightarrow$ R is lc asymp sat. this is NOT enough to get $C_{\tilde{Y}}\hat{\tilde{E}} \subset \tilde{Y} \setminus \tilde{E}$ in the Y, X level. This is also a reason why we get stuck in dim > 4.

Theorem 2. As in th 1, except that also $B_1^- = \phi$. i.e. (Y, B_1^*) is klt.

pf (sketch): Run a MMP for $K_Y + B_1^+$, this eventually eliminates B_1^- & does not change U. $\square$

Theorem 3. As in th 2, but $U = Y$.

Lemma: let $Y \supsetneq U, m$ as in th 2. There exists $m | m'$, $m < m'$ and E, $C_Y E \not\subset U$, st. $res_E \mathbb{D}_{m_i} = res_E \mathbb{D}_m$, $\forall i$.

idea: to inflat C more:

$K + \bar{B} = K + B + t_0 D$

$t_0 = \max \{ t | K + B + tD \text{ lc} \}$

now work with $B < \bar{B} < \bar{\bar{B}}$

(messy bit: this creates non lc sing.; need to make sense they happen "far away") $\square$

$Y_1 \to X$, $U_1 \subset Y_1$, m_1. if $U_1 = Y_1$ ok.

otherwise using thm 1 to construct

$Y_2 \to X$, $U_2 \subset Y_2$, $m_1 | m_2$. If $U_2 = Y_2$ ok,

otherwise $Y_3 \to X$, $U_3 \subset Y_3$, $m_2 | m_3$ ....

At some point, $U_m = Y_m$:

If $U_n \neq Y_n$ $\forall n$, then get infinite sequence

But there are only* finitely many klt pairs

Y^*; B_1^* with $K_{Y^*} + B_1^*$ = pull back $(K_Y + B_1')$,

so I may assume that $Y_i = Y$.

then $U_1 \subsetneq U_2 \subsetneq U_3 \subset \cdots$

is a contradiction $\square$

Exercise: (for *): (X, B) is a klt pair

$\{E \mid a(E, K+B) \leq 0\}$ is finite.

End