A. Corti 2/13 2004 at
NCTS at Taida Room 308.

Mori Fiber Spaces.

Def^{n}: A Mori fiber space (Mfs) is a variety X with terminal $\mathbb{Q}$-fact. singularities, and an extr. contr. $f: X \to S$ of fibering type. (i.e. surj, $f_* \mathcal{O}_X = \mathcal{O}_S$) that means:

$-K_X$ is f-ample, $\rho(X) - \rho(S) = 1$

(rk $N^1 X$ − rk $N^1 S = 1$) $\dim S < \dim X$

f is not necessarily flat or even equidimensional.

I mostly do $\dim X = 3$:

$\dim S = 2$ "conic bundle" (f not always flat)

$\dim S = 1$ "del Pezzo fibration"

(X_η is a dP_k of degree $1 \le k = K_{X_\eta}^2 \le 9$)

$S = pt$: X is a Fano 3-fold, $\rho = 1$ (poss. sing.)

Example: typical 3-fold ter. sing is :

$$\{ xy + f(z^r, t) = 0 \} \subset \frac{1}{r}(a, -a, 1, 0)$$

require to be isolated sing. $\qquad \mathbb{C}^4/\mu_r$

$$(x, y, z, t) \mapsto (\varepsilon^a x, \varepsilon^{-a} y, \varepsilon z, t)$$

I mostly work with $\frac{1}{r}(a, -a, 1)$.

At leastly locally the example is deformed into bunches of $\frac{1}{r}(a, -a, 1)$'s.

We already commented on $\mathbb{Q}$-factoriality.

if $X_4^3 \subset \mathbb{P}^4$, $\mathbb{Q}$-factorial $\iff$ $H^2 X \to H_4 X$ is an isom. $/\mathbb{Q}$

(a global topological property.)

Q: classification of terminal quartic 3-fold not nec. $\mathbb{Q}$-factorial?
(Tunable, but not yet been done)

Examples of 3-fold Mfs.

Fano 3-folds.

- 17 deformation families of non-singular Fano 3-folds with $\rho = 1$.
(A modern reference is still lacking, even Iskovskikh / Iskovskikh's Fano 3-folds I, II require existence of lines which is proven by not so standard method!!)
- Many 100s of families of singular Fano 3-folds.

$$\begin{cases} -K^3 = 2g-2+\sum \dfrac{a_i(r_i-a_i)}{r_i} \\ h^0(X,-K) = g+2 \end{cases}$$

discrete invariants:

$g \leftarrow$ in the sm case, the genus of $\underbrace{\text{the curve as}}$ double hyp. sect. Sing. case also relate to it.

$\{(r_i, a_i)\}$

about 7000 of such.

let Hilbert function $P_X(t) = \sum h^0(X, -nK)\, t^n$.

eg. $g=2$, $\frac{1}{2}(1,1,1)$

the Hilbert fcn identifies X as a hypersurface $X_5 \subset \mathbb{P}(1^4, 2)$ i.e. $\mathbb{P}(\underbrace{1,1,1,1}_{4},2)$

$x_1, \cdots, x_4, y$

$= (x, y + a_2(x) y + b_5(x) = 0)$

$\mathbb{P}(a_0, \cdots, a_n) := \mathbb{C}^{n+1} \setminus \{0\} / \mathbb{C}^\times$

$\lambda \curvearrowright (\lambda^{a_0} x_0, \cdots, \lambda^{a_n} x_n)$.

eg. $(y=1) = \frac{1}{2}(1,1,1,1)$;

$X_5 \cap (y=1) \simeq \frac{1}{2}(1,1,1)$ locr x_2, x_3, x_4

is the unique sing.

There are 95 families of Fano 3-folds hypersurfaces in WP.

eg. $g=3$, $\frac{1}{2}(1,1,1)$

$\Rightarrow X_{3,3} \subset \mathbb{P}(1^5, 2)$.

merely examples.

eg. $g=4$, $\frac{1}{2}(1,1,1)$

$$\begin{tikzcd}
X \text{ codim } 3 \text{ in } \mathbb{P}(1^6, 2) \arrow[d, dashed] \arrow[dr, dashed] & \\
\bar{Y}_{2,3} \arrow[r, hook, "\text{``guess''}"'] & \mathbb{P}5
\end{tikzcd}$$

$\bar{Y}_{2,3} \supset \Pi^2$

a 2-plane

$= \{x_1 = x_2 = x_5 = 0\}$

eq'n of $\bar{Y}$:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$p \in X \simeq \frac{1}{2}(1,1,1)$

$E \subset Y = Bl_p X$

non-sing space

$-K_Y^3 = -K_X^3 - \frac{1}{2} = 6$

I "guess" $-K_Y$ nef, big

$\bar{Y} = \operatorname{proj} \bigoplus_{n \geq 0} H^0(Y, \mathcal{O}_Y(-nK_Y))$

is a Fano of degree

$-K_{\bar{Y}}^3 = -K_Y^3 = 6$

$\deg a_i = 1$, $\deg b_i = 2$ in $x_1, \cdots, x_5$ in $\mathbb{P}5$.

complete intersection.

p. 44 $\quad y$ new variable of wt 2

$$y\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ -a_1b_3 + a_3b_1 \\ a_1b_2 - a_2b_1 \end{pmatrix} \quad 3 \text{ eq'n}$$

get total 5 eq'n in $(x_1, \dots, x_6, y)$, get

$$X_{2,3,3,3,3} \underset{\text{cod } 3}{\hookrightarrow} \mathbb{P}(1^6, 2).$$

The equations are the 4×4 Pfaffine of a 5×5 anti sym matrix of homog forms on $\mathbb{P}(1^6, 2)$:

$$\left(\begin{array}{cc:ccc} 0 & y & a_1 & a_2 & a_3 \\ & 0 & b_1 & b_2 & b_3 \\ \hdashline & & 0 & x_1 & x_2 \\ & -* & & 0 & x_3 \\ & & & & 0 \end{array}\right)$$

Theorem of Eisenbud of Bushbaum that codim 3 Gorenstein must be Pfaffin of the form. Here we do get this directly.

Rmk: $\quad \operatorname{pf}\begin{pmatrix} 0 & a & b & c \\ & 0 & d & e \\ -* & & 0 & f \\ & & & 0 \end{pmatrix} = af - dc + be$

$$\det = \operatorname{pf}^2.$$

in the 5×5 case there are $\underbrace{4}_{\text{more}}$ of such ?

This is a prototype for very many calculations of "unprojection".

For Fano 3-folds of codim > 3, say 4. then do not have Bushbaum-Eisenbud thm, but still can do "unprojection" from cod 3 to cod 4. Are worked out by thesis of Papadakis (M. Reid).

For bigger cases, the comm. algebra has not been worked out yet (many research problems in this case).

Typical examples of conic bundles:

$S = \mathbb{P}^2$, $E = E^3$ a rk 3 v.b.

$X = (f_2 = 0) \subset \mathbb{P}(E)$

$f_2 : \mathrm{Sym}^2 E \to L$ this gives a conic bundle in classical sense.

For singular case $\frac{1}{r}(a, -a, 1)$, $r > 1$ with higher index $r > 1$:

Thm: Still $\exists$ a birat'l map

$$\begin{tikzcd} X \arrow[r, dashed, "\varphi"] \arrow[d, "\mathbb{P}^1"'] & X' \arrow[d, "\mathbb{P}^1"] \\ S & S \end{tikzcd}$$

any curve

$\varphi : X_\eta \simeq X'_\eta$

s.t. X' is Gorenstein

$\Rightarrow X' \hookrightarrow \mathbb{P}(E)$
$(f_3 = 0)$.

Theorem (Corti): φ is a composite of links of 4 types. (long time ago). For dim = particular case: $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$.

(i)
$$\begin{tikzcd} & Z \arrow[r, dashed, "\text{flips}"] \arrow[dl, "\text{div. cont.}"'] & X' \arrow[d, "\text{mfs}"] \\ X \arrow[d] & & \\ S & & S' \arrow[ll] \end{tikzcd}$$

$\leftarrow$ look at this closely!

Example:

$$\begin{tikzcd} \mathbb{P}^2 \arrow[d] & F_1 \arrow[l] \arrow[d] \\ \{pt\} & \mathbb{P}^1 \arrow[l] \end{tikzcd}$$

this is the only example in dim 2.

(ii) flips

$$\begin{tikzcd}[column sep=small] & Z \arrow[dl, "\text{div cont.}"'] \arrow[rr, dashed] & & Z' \arrow[dr, "\text{div. cont.}"] & \\ X \arrow[d] & & & & X' \arrow[d] \\ S \arrow[rrrr, equal] & & & & S \end{tikzcd}$$

Example:

$$\begin{tikzcd}[column sep=small] & Z \arrow[dl] \arrow[dr] & \\ \mathbb{F}_k \arrow[d] & & \mathbb{F}_{k\pm1} \arrow[d] \\ \mathbb{P}^1 \arrow[rr, equal] & & \mathbb{P}^1 \end{tikzcd}$$

(iii)

$$\begin{tikzcd} X \arrow[r, dashed, "\text{flips}"] \arrow[d] & X' \arrow[d] \\ S \arrow[dr] & S' \arrow[d] \\ & T \end{tikzcd}$$

Example:

$$\begin{tikzcd} \mathbb{P}^1 \times \mathbb{P}^1 \arrow[r, equal] \arrow[d, "p_1"] & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[d, "p_2"] \\ \mathbb{P}^1 \arrow[dr] & \mathbb{P}^1 \arrow[d] \\ & \operatorname{Spec} k \end{tikzcd}$$

(iv)

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d] & Z \arrow[dr, "\text{div. cont.}"] & \\ S \arrow[rr] & & X' \arrow[d] \\ & & S' \end{tikzcd}$$

The proof is in 3 steps:

(1) define ~~in~~ a "discrete" invariant $\deg \varphi$.

(2) show there exists a link

$$\begin{tikzcd} \varphi_1 : X \arrow[r, dashed] \arrow[d] & X_1 \arrow[d] \\ S & S_1 \end{tikzcd} \quad \text{st. } \deg(\varphi \circ \varphi_1^{-1}) < \deg \varphi$$

(3) show that deg can't go down infinitely times.

$\deg \varphi = (\mu, \lambda, e)$:

choose a v.a linear system $\mathcal{H}'$ on X'.

let $\mathcal{H} = \varphi_*^{-1} \mathcal{H}'$ proper transform.

X/S is a Mfs $\Rightarrow$ $\mathcal{H} \subset |-\mu K + A|$ with A a pull back from base S.
this defines μ.

eg. for $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ a deg d linear system
then $\mu = d/3$.

Notice that μ is uniquely det. by $\mathcal{H}'$.

$\lambda = \frac{1}{c_0}$, $c_0 = \max\{c \mid K_X + c\mathcal{H}$ has canonical singularities$\}$

this means: choose $f: Y \to X$ resolution of $(X, \mathcal{H})$.

$$\begin{cases} K_Y = f^*K_X + \sum a_i E_i \\ \mathcal{H}_Y = f^*\mathcal{H} - \sum b_i E_i \end{cases} \qquad (\mathcal{H}_Y = \text{proper transf})$$

$$K_Y + c\mathcal{H}_Y = f^*(K_X + c\mathcal{H}) + \sum (a_i - cb_i) E_i$$

call $(X, c\mathcal{H})$ canonical $\iff a_i - cb_i \geq 0$

so $c_0 = \min\{a_i/b_i\}$ and $\lambda = \max\{b_i/a_i\}$.

$\lambda \downarrow$ means "less singular".

Exercise: $\dim X = 2 \Rightarrow \lambda = \max\{\mathrm{mult}_p H \mid p \in X,\ H$ general member of $\mathcal{H}\} \in \mathbb{Z}$

if $\dim X = 3$, $\lambda \in \mathbb{Q}$, and it is difficult to form an intuitive picture of what λ means.

$e := \#\{E_i \mid a_i - c_0 b_i = 0\}$.

(μ, λ, e) is ordered lexicographically.

Let $\varphi: \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, $\mathcal{H}$ 2-dim'l linear system $\subset \mathcal{O}_{\mathbb{P}^2}(n)$.

Theorem: $\exists$ a point p of

$$\mathrm{mult}_p \mathcal{H} = m > n/3$$

(This is weaker than classical estimate $\exists$ 3 pts $p_1\ p_2\ p_3$ st $\sum m_i > n$.)

Pf ~~Assume~~ this thm.

$$\begin{tikzcd} & E_i \quad Y \arrow[dl] \arrow[dr] & \\ \mathbb{P}^2 & & \mathbb{P}^2 \end{tikzcd}$$

$$(H - \sum \nu_i E_i^*)^2 = 1$$
$$n^2 - \sum \nu_i^2$$

$$(H - \sum \nu_i E_i^*)\cdot(-K - \sum E_i) = 3$$
$$3n - \sum \nu_i$$

P. 48

$$\begin{cases} \sum v_i^2 = n^2 - 1 \\ \sum v_i = 3n - 3 \\ n^2 - 1 = \sum v_i^2 \leq v_1 \sum v_i = 3n-3 \ (v_1) \end{cases}$$

$$v_1 \geq \frac{1}{3} \cdot \frac{n^2-1}{n-1} = \frac{1}{3}(n+1) > \frac{n}{3}.$$

why do this? Remember we want to do step 2.

$$\begin{tikzcd} & Bl_p = \mathbb{F}_1 \arrow[d, "p_1"] \arrow[dl] \arrow[ddr, dashed, "\varphi \circ \psi_1^{-1}"] & \\ \mathbb{P}^2 \arrow[ur, dashed, "\psi_1"] \arrow[d] \arrow[rr, dashed, "\varphi"] & {} \arrow[dl] & \mathbb{P}^2 \\ \{pt\} & & \end{tikzcd}$$

$\deg \varphi = (\frac{n}{3}, \lambda, e)$

$\mathcal{O}(n) \sim \mathcal{O}(-\frac{n}{3} K)$

? $\deg(\varphi \circ \psi_1^{-1})$

$= (\frac{n-m}{2}, \lambda_1, e_1)$

$$\frac{n-m}{2} < \frac{n}{3} \iff m > \frac{n}{3}.$$

So we get now:

$\mathbb{F}_1 \dashrightarrow \mathbb{P}^2$

$\begin{cases} m > \mu \text{ or} \\ \text{other cases} \end{cases}$

$m > \mu$ use

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & & \\ \mathbb{F}_1 \arrow[rr, dashed] & & \mathbb{F}_{0/2} \arrow[r, dashed] & \mathbb{P}^2 \end{tikzcd}$$

or

though μ does not $\searrow$ but better sing and $e \searrow$.

<u>To be continued.</u>

A. Corti. 2004 1/16 at 東大

Mori fiber space.

$$\begin{array}{ccc} X & \overset{\varphi}{\dashrightarrow} & X' \\ \downarrow & & \downarrow \pi' \\ S & & S' \end{array}$$

given 2 Mfs' and a birational map

choose complete linear system $\mathcal{H}' \sim |-\mu' K' + \pi'^* A'|$; v.a.

$\mathcal{H} = \varphi_*^{-1} \mathcal{H} \subset |-\mu K + \pi^* A|$

A' ample.

no control, can be < 0

def'n: φ is square if it sends a general fiber of π isomorphically to a general fiber of π'.

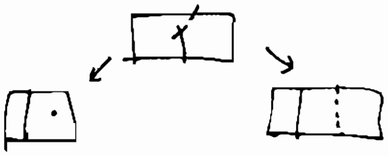
i.e. $\exists$ comm diagram

$$\begin{array}{ccc} X & \overset{\varphi}{\dashrightarrow} & X' \\ \pi \downarrow & \circlearrowright & \downarrow \pi' \\ S & \dashrightarrow & S' \end{array}$$

birational

& $\varphi : X_\eta \simeq X'_{\eta'}$.

Example: $\mathbb{F}_k \dashrightarrow \mathbb{F}_{k\pm 1}$ is square



this is elementary.

Theorem

(1) $\mu \geq \mu'$ and $\mu = \mu' \iff \varphi$ is square.

(2) If $(X, \frac{1}{\mu}\mathcal{H})$ has canonical singularities (all $b_i \leq \mu a_i$ as in last time) and $K + \frac{1}{\mu}\mathcal{H} = \frac{1}{\mu}\pi^* A$ is nef, then φ is a square isomorphism.

This can be used in the factorization theorem.

p.50 example : $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$,

$\mathcal{H} \subset |\mathcal{O}(n)| = |-\frac{n}{3} K_{\mathbb{P}^2}|$; thm says that

(1) $n \geq 1$ unless φ isomorphism

(2) either $(X, \frac{1}{\mu}\mathcal{H})$ not canonical or φ is an isomorphism.

"not can" means : $c_0 = \max\{c \mid (X, c\mathcal{H})\} < \frac{1}{\mu}$,

i.e. $\max(\operatorname{mult}_p \mathcal{H}) = \lambda = \frac{1}{c_0} > \mu = n/3$

so either $\exists p$ with $\mu_p > \frac{n}{3}$ or φ is an isom.

(This is originally to Max Noether / Castelnuovo)

Pf : Pick common resolution of singularities (can be quasi-stable model if needed)

$$\begin{tikzcd} & Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "\varphi"] \arrow[d] & & X' \arrow[d] \\ S & & S' \end{tikzcd}$$

(*) $K_Y + \frac{1}{\mu'}\mathcal{H}_Y = q^*(K_{X'} + \frac{1}{\mu'}\mathcal{H}') + E_q$

$E_q > 0$, q-exceptional

$=$ discrepancy of $a(-, K_{X'})$.

$= q^*(\frac{1}{\mu'} A') + E_q$.

consider a curve $C \subset X$ st $\pi(C) = pt \in S$,

C is general that it "avoids exceptional set"

i.e. p is an isomorphism in a nbd of C.

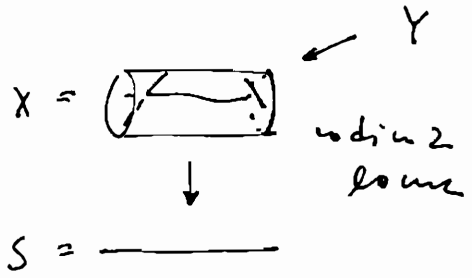
$C.(K_X + \frac{1}{\mu'}\mathcal{H})$

$= C.p_*(K_Y + \frac{1}{\mu'}\mathcal{H}_Y)$

$= p^*C.(K_Y + \frac{1}{\mu'}\mathcal{H}_Y)$

$= p^{-1}(C).(q^*(\frac{1}{\mu'}A') + E_q)$

≥ 0. (**)



So $K_X + \frac{1}{\mu'} \mathcal{H} . C \geq 0$

i.e. $\frac{1}{\mu'} \geq \frac{1}{\mu}$ $(K + \frac{1}{\mu}\mathcal{H} . C = 0)$

so $\mu \geq \mu'$.

Also > 0 unless $q^{-1}C . q^*(\frac{1}{\mu} A') = 0$.

i.e. $p^{-1}C \subset q^*$(fiber of π')

i.e. φ (fiber of π) $\subset$ fiber of π',

with more work, get φ square. Get (1).

For (2), use $(*)$ plus $A' \geq 0$ in $(**)$.

Assume $X, \frac{1}{\mu}\mathcal{H}$ canonical, A nef

Run the same argument but reverse,

$$K_Y + \frac{1}{\mu}\mathcal{H}_Y = p^*(K_X + \frac{1}{\mu}\mathcal{H}) + E_p$$

& A is nef, $E_p \geq 0$ & p-exc.

$\Rightarrow \mu' \geq \mu \Rightarrow \varphi$ square.

with more work can show that φ = isom. $\square$

Corollary: If φ is not a square isom,

there are 2 cases:

(1) $(X, \frac{1}{\mu}\mathcal{H})$ does not have can. sing.

(2) $(X, \frac{1}{\mu}\mathcal{H})$ has can. sing. But

$K_{X'} + \frac{1}{\mu}\mathcal{H}$ is not nef.

example: $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$; have always in (1),

$\downarrow$

$S = \{pt\}$ i.e. $\exists\, p$ with

$m = \text{mult}_p \mathcal{H} > 4/3$.

We blow-up

$$\begin{array}{ccc} \mathbb{P}^2 \leftarrow & F_1 & \dashrightarrow \mathbb{P}^2 \\ & \downarrow & \\ & \mathbb{P}^1 & \end{array}$$

What can we do in 3-fold case?

P. 52

Case (1). let $c_0 = \max\{c \mid K + c\mathcal{H} \text{ can}\} < \frac{1}{\mu}$

let $f : Y \to X$ be a "maximal blow-up"
\ extr. divisorial contr.

$K_Y + c_0\mathcal{H} = f^*(K + c_0\mathcal{H})$.

Construction of f: resolve $Z \to X$ and
run $K_Z + c_0\mathcal{H}_Z$ MMP.

$$\begin{tikzcd} & Y \arrow[dl, "f"'] \\ X \arrow[d] & \\ S & \end{tikzcd}$$

$K_Y + c_0\mathcal{H}_Y$ is negative over S,
run a MMP for $K_Y + c_0\mathcal{H}_Y$ over S
get: (after a bunch of flips ...)

(1.1)
$$\begin{tikzcd} & Y \arrow[dl, "\text{d.c.}"'] \arrow[r, dashed] & Y_1^* \arrow[dd, "\text{Mfs}"] \\ X \arrow[d] & & \\ S & & S_1^* \arrow[ll] \end{tikzcd}$$

or

(1.2)
$$\begin{tikzcd} & Y \arrow[dl, "\text{d.c.}"'] \arrow[r, dashed] & Y_1 \arrow[dr, "\text{d.c.}"] & \\ X \arrow[d] & & & X_1 \arrow[d, "\text{Mfs}"] \\ S \arrow[rrr, equal] & & & S \end{tikzcd}$$

in both cases
$\mu_1 \le \mu$ &
$(\mu, \lambda, e) > (\mu_1, \lambda_1, e_1)$

needs some work.

Case (2).
$$\begin{tikzcd} X \arrow[d] \arrow[r, "\text{contR}"] & W \arrow[d] \\ S \arrow[r] & T \end{tikzcd}$$

$K + \frac{1}{\mu}\mathcal{H}$ has extr. ray R, $(K + \frac{1}{\mu}\mathcal{H}).R < 0$.

then $\exists$ morphism $S \to T$ making diagram comm.

Now run a MMP for $K_X + \frac{1}{\mu}\mathcal{H}$ over T.

$\rho(X) - \rho(T) = 2$.

(2.1)
$$\begin{tikzcd} X \arrow[d] \arrow[rr, dashed] & & X_1 \arrow[d, "\text{Mfs}"] \\ S \arrow[dr] & & S_1 \arrow[dl] \\ & T & \end{tikzcd}$$

(2.2)
$$\begin{tikzcd} X \arrow[d] \arrow[r, dashed] & Y_1 \arrow[dr, "\text{d.c.}"] & \\ S \arrow[r] & T & X_1 \arrow[l, "\text{Mfs}"] \end{tikzcd}$$

This finishes the outline.

The above is the so called "Sarkisov Program".

Open Questions :

(1) describe relations between links in terms of "2-links" (elementary relations).
e.g. Quartic 3-folds with one node.
no relations.
This is a very realistic project.

(2) there is a proof of the Sarkisov program
((μ, λ, e) can't descrease so many times,
This is subtle because $\lambda \in \mathbb{Q}$.)
The pf known is non-effective.
Find an effective proof.
(This should be harder than (1).)

def'n: If $X \to S$ is a Mfs, the <u>pliability</u> of X/S

$$\mathcal{P}(X/S) = \left\{ \begin{matrix} \text{Mfs} \\ Y \to T \end{matrix} \;\middle|\; \begin{matrix} \exists \text{ birational map} \\ X \dashrightarrow Y \end{matrix} \right\} \Big/ \begin{matrix} \text{square} \\ \text{birat/} \\ \text{equiv.} \end{matrix}$$

Question : is $\mathcal{P}(X/S)$ an alg. var.
(It takes sometime to understand what the question asks !)

effectivity statement in (2) suggests that $\mathcal{P}(X/S)$ is an alg. var by boundedness argument.

- X/S is birationally rigid if $\# \mathcal{P}(X/S) = 1$.

p.54 There are good criteria for birational rigidity, but not yet optimal.

- Examples: (1). $X = X_4^3 \subset \mathbb{P}^4$

  X is $\mathbb{Q}$-factorial, has ordinary nodes $(xy + zw = 0)$ $\Rightarrow$ X is birationally rigid. (by M. Mella)

  (2) $\begin{array}{c} X \\ \downarrow \\ S \end{array}$ conic bundle, X non-singular

  $\Delta \subset S$ discriminant

  $4K_S + \Delta \geq 0 \Rightarrow$ birat'l rigid.

However, very little is known about $\mathcal{B}(X/S)$ when X is not rigid.

- Example: Start with $X = X_4^3 \subset \mathbb{P}^4$ a quartic 3-fold with a singular point $p: xy + z^3 + w^3 = 0$.

$$X = (\underbrace{x_0^2 + x_1 x_2 + x_0 a_3(x_1, \cdots, x_4) + b_4(x_1, \cdots, x_4)}_{F} = 0) \subset \mathbb{P}^4$$

here p is at $(1, 0, 0, 0, 0)$.

$$x_1 F = (x_0 x_1)^2 x_2 + (x_0 x_1) a_3 + x_1 b_4$$

substitute $y = x_0 x_1$:

get strange looking thing:

$$\bar{Y}_5 = (x_2 y^2 + a_3 y + x_1 b_4 = 0) \subset \mathbb{P}(1^4, 2)$$

is a nice Fano 3-folds $-K_{\bar{Y}_5} = \mathcal{O}_{\bar{Y}_5}(1)$:

$$\begin{tikzcd}
& Y \arrow[dl, "\text{weighted blow-up}"'] \arrow[dr] \arrow[rr, dashed, "\text{flop}"] & & Y' \arrow[dl, "\text{flopping contr.}"'] \arrow[dr] & \\
X \arrow[rr, dashed] & & \bar{Y}_5 & & X'
\end{tikzcd}$$

$\bar{Y}_5 = \left(y(x_2 y + a_3) + x_1 b_4 = 0 \right)$

$y_2 = \dfrac{x_2 y + a_3}{x_1} = -\dfrac{b_4}{y}$

$X'_{3,4} = \begin{cases} x_1 y_2 = x_2 y + a_3 \\ y_2 y = -b_4 \end{cases} \subset \mathbb{P}(1^4, 2^2)$.

Theorem: $\mathcal{P}(X) = \{X, X'\}$. Only 2 things!

Question: What is $\mathcal{P}(\mathbb{P}3)$?

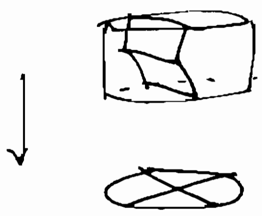
Conjecture: (by Chokovsky). First, it is not hard to get:

$X \downarrow S$ conic bundle, non-singular st. $2K_S + \Delta$ not nef

$\Rightarrow$ either X is rational or X is birat'l to cubic 3-fold.

Example: $S = \mathbb{P}^2$. $2K_S + \Delta$ not nef

$\Leftrightarrow -6 + \deg \Delta < 0 \Leftrightarrow \deg \Delta \leq 5$.



over Δ I have a 2–1 cover "étale", $\pi : N \to \Delta$

indeed admissible only i.e. étale over non-sing pt.

$\pi_* \mathcal{O}_N = \mathcal{O}_\Delta \oplus L$, $\deg L = 0$ & $L^2 = \mathcal{O}_\Delta$.

say $S = \mathbb{P}^2$ & $\deg \Delta = 5$.

$K_\Delta = \mathcal{O}(5-3) = \mathcal{O}(2)$.

so $L(1)$ is a $\underline{\theta\text{-characteristic}}$ of Δ

i.e. a line bundle st square = can. class.

$h^0(\Delta, L(1)) = 0$: X is rat'l.

$h^0(\Delta, L(1)) = 1$: X is bi-rat'l to a cubic 3-fold.

p.56 when $X = \mathbb{P}^3$ or $X = X_3^3 \subset \mathbb{P}^4$:
$\mathcal{P}(X)$ contains the moduli space of pairs $(\Delta, \mathcal{L})$ where $\Delta_5 \subset \mathbb{P}^2$ is a quintic and $\mathcal{L}_5$ is θ characteristic.

$\binom{5+2}{2} - 1 = 21 - 1 = 20$.

Guess : $\dim \mathcal{P} \geqslant 20 \Rightarrow \dim \mathcal{P} = 20$ and X is birat'l to $\mathbb{P}^3$ or $X_3^3 \subset \mathbb{P}^4$.

Mori Fiber Space — examples & conjecture

* Thm: X non-sing, $f: X \to S$ a conic bundle Mfs, dim X arbitrary. If $4K_S + \Delta \geq 0$
$\Rightarrow X/S$ is birationally rigid.
($\Delta :=$ disc(f).)

Example: When $S = \mathbb{P}^2$, deg $\Delta \geq 12 \Rightarrow$ bi-rat'l rigid.

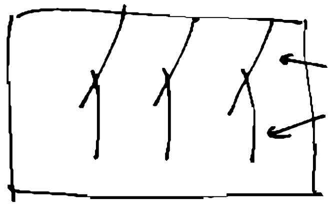
* this is very old result (Sarkisov 1982) and never been improved. In $\mathbb{P}^2$ case, we have

Conjecture: When $S = \mathbb{P}^2$, deg $\Delta \geq 8$
$\Rightarrow$ birat'l rigid.

Conjecture (Iskovskikh):
dim $X = 3$, $2K_S + \Delta \geq 0 \Rightarrow X$ is not rat'l.

Pf: I assume that X is a surface, over a (non-closed) field k (This is the only Mfs conic bundle in dim 2).

Apply the Sarkisov prog:



assume a non-square birational map:

$$\begin{array}{ccccc} \mathcal{H} & X & \overset{\varphi}{\dashrightarrow} & Y \\ & f\downarrow & & \downarrow \\ & S & & S \end{array}$$

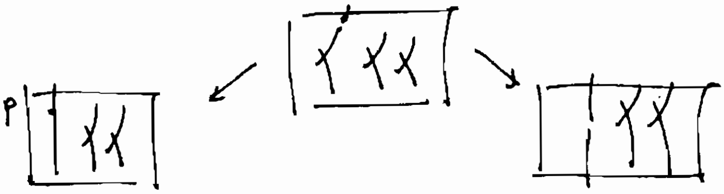
φ is given by a linear system $\mathcal{H} \subset |-\mu K + f^* A|$

and there are 2 cases

(1) $(X, \frac{1}{\mu}\mathcal{H})$ not canonical

(2) $(X, \frac{1}{\mu}\mathcal{H})$ canonical, but A not nef.

In case (1), apply an elementary transf for conic bundle $X \dashrightarrow X_1$



to a conic bundle, $f_1 : X_1 \to S$, $\Delta_1 = \Delta$.

a point of mult $> \mu$ can not occur in sing fiber

($\mathcal{H}.f = 2\mu$ + small argument $\Rightarrow$

p is not on a sing fiber) so Δ not changed.

After finitely many applications of case (1), (square maps), may get case (2)

i.e. $K_X + \frac{1}{\mu}\mathcal{H} = f^* A$ and $\deg A < 0$.

let's change notation $A \leftarrow -A$ and $\deg A > 0$.

$$1 < \frac{1}{\mu^2}\mathcal{H}^2 = (-K - f^*A)^2 = K^2 + \underbrace{2Kf^*A}_{<0} < K^2$$

here we may assume $S \otimes \bar{K} = \mathbb{P}^1$,

so $K^2 = 8 - \deg\Delta \Rightarrow \deg\Delta < 8$.

i.e. we have proved that if there is a non-square map

$$\begin{array}{ccc} X & \dashrightarrow & Y \\ \downarrow & & \downarrow \\ S & & T \end{array}$$

then $\deg\Delta < 8$ $\square$

There is a similar result about dP fibrations $f : X \to S$ of degree $d \le 3$. But in general square birat'l maps of dP fibrations are not well-understood (this is a good research problem.).

The pf proceeds slightly diff lines (Pukhlikov).

Intro / Tutorial to scrolls:

a scroll is a $\mathbb{P}^n$ bundle over $\mathbb{P}^k$.

$\mathbb{F} = \mathbb{C}^{k+n+2} /\!/ \mathbb{C}^\times \times \mathbb{C}^\times$ a geometric quot.

$(t_0, \dots, t_k)(x_0, \dots, x_n)$ one action is

$\to (\lambda t_0, \dots, \lambda t_k, \lambda^{-a_0} x_0, \dots, \lambda^{-a_n} x_n)$

another action to $(t_0, \dots, t_k, \mu x_0, \dots, \mu x_n)$

where $(\lambda, \mu) \in \mathbb{C}^\times \times \mathbb{C}^\times$, and $a_i > 0$ integers.

usually normalized st. $0 = a_0 \le a_1 \le \dots \le a_n$.

notation $\mathbb{F}(0, a_1, \dots, a_n) = \mathbb{F}_{a_1, \dots, a_n}$.

There is an obvious morphism

$\pi : \mathbb{F} \to \mathbb{P}^k$ and all fibers are $\mathbb{P}^n$.

($\mathbb{F} = (\mathbb{C}^{k+1} \setminus \{0\}) \times (\mathbb{C}^{n+1} \setminus \{0\}) / \mathbb{C}^\times \times \mathbb{C}^\times$ indeed.)

Line bundles on $\mathbb{F}$:

$\mathrm{Pic}\, \mathbb{F} = \mathbb{Z}^2$, basis by L, M

$L = \pi^* \mathcal{O}_{\mathbb{P}^k}(1)$, M = relative $\mathcal{O}(1)$.

$H^0(\mathbb{F}, eL + dM) = \{ f : \mathbb{C}^{k+n+2} \longrightarrow \mathbb{C}$ polynomials,

st. $f((\lambda, \mu) \cdot p) = \lambda^e \mu^d f(p) \}$

eg. $H^0(\mathbb{F}, L) = \{ f \mid f((\lambda, \mu) p) = \lambda f(p) \}$.

$= \langle t_0, \dots, t_k \rangle$.

$H^0(\mathbb{F}, M) = \{ f \mid f((\lambda, \mu) \cdot p) = \mu f(p) \}$

with basis $\langle m_{ij}(t_0, \dots, t_k) x_i \mid m_{ij}$ homog monomials of degree $a_i \rangle$

a best introduction to scrolls is M. Reid's "Chapters on Surfaces".

Notice: scrolls are all toric varieties.

p.60 many useful examples of Mfs X/S and links X'/S' arise as linear sections of scrolls $\mathbb{F}$ and diff GIT quotients.

Proposition: let X be non-singular, $f: X \to \mathbb{P}^2$ a conic bundle Mfs, with disc $\Delta \subset \mathbb{P}^2$ of degree $d = 7$. Then X can be described as follows:

1. $X_{2M+L,\, M+L,\, M+L} \subset \mathbb{P}^2 \times \mathbb{P}^4 \longrightarrow \mathbb{P}^2$,
   complete int of 3 hyp surface
   (there is also a proj to $\mathbb{P}^4$) $h^0(\Delta, \mathcal{L}(2)) = 0$.
2. $X_{2M+L,\, M+L} \subset \mathbb{F}_{0,0,1}$, $h^0(\Delta, \mathcal{L}(2)) = 1$.
3. $X_{2M+L} \subset \mathbb{F}_{0,1}$, $h^0(\Delta, \mathcal{L}(2)) = 2$.
4. $X_{2M+L} \subset \mathbb{F}_{0,2}$ $h^0(\Delta, \mathcal{L}(2)) = 3$.

Let $N \to \Delta$ be the associated admissible cover, $\pi_* \mathcal{O}_N = \mathcal{O}_\Delta \oplus \mathcal{L}$.

(pf is an exercise in toric varieties.) $\square$ see BCZ

eg. in case 4. $\mathbb{F}_{0,2} = \mathbb{F}(0,0,2)$,
$X = (f = 0)$, $f \in H^0(2M + L)$.

$$f(t_0, t_1, t_2, x_0, x_1, x_2) = (x_0, x_1, x_2) \begin{pmatrix} \text{"1"} & \text{"1"} & \text{"3"} \\ & \text{"1"} & \text{"3"} \\ & & \text{"5"} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$$

Symmetric matrix of homog forms of the indicated degrees in the t variables.

for 1, 2, 3, these varieties are non-rigid (examples and calculations in BCZ)

Q: Maybe the one in 4 is birat'lly rigid?

Example :

$X^3 = X_{3M-2L} \subset \mathbb{F}_{1,3,3}$ (over $\mathbb{P}^1$)

in BCZ it is shown that general X^3 is non-sing.

We conj that X^3 is birat'lly rigid. This does not follows from known results.

Theorem : $X := X^{2c}_{4c+2} \subset \mathbb{P}(1^{2c}, 2, 2c+1)$, $c > 1$.

X non-singular $\Rightarrow$ X birat'l rigid.

first example : $c = 2$, X a 4-fold.

$X_{10} \subset \mathbb{P}(1^4, 2, 5)$ (since $X_6 \subset \mathbb{P}(1^2, 2, 3)$ is a dP$_1$, not a Mfs.)

pf [KSC (Kollar - Smith - Corti) : rat'l and nearly rat'l varieties CUP]

$K_X = \mathcal{O}_P(-2c-2-(2c+1)+4c+2) = \mathcal{O}_X(-1)$

so $-K_X = \mathcal{O}(1)$.

$$(-K_X)^{2c} = \frac{4c+2}{2(2c+1)} = 1.$$

This weighted proj sp has 2 sing.

but X does not pass thr them : $\mathbb{P}(1^{2c}, \underset{\uparrow\, P_0}{2}, \underset{\uparrow\, P_1}{2c+1})$

Equation for X : $x_1, \dots, x_{2c}, y, z$

$$z^2 + y^{2c+1} + a_1(x) y^{2c} + \cdots + a_{4c+2}(x)$$

$P_0, P_1 \notin X$ and general X is non-singular

Idea of proof :

If X is non-rigid, there exists a linear system $\mathcal{H} \subset |\mathcal{O}_X(n)|$ st.

$(X, \frac{1}{n}\mathcal{H})$ not canonical, in particular :

(exercise) $\exists\, p \in X$ st $\operatorname{mult}_p \mathcal{H} > n$.

p.62 consider $C_p = D_1 \cdots D_{2c-1}$

where $D_i \in |\mathcal{O}_X(1)|$ ($p t$) are general elements passing through p.

Pick $H \in \mathcal{H}$ general. then

" $\mathrm{mult}_p H < C_p . H = n$ " then $*$?

problem: there is a unique such C_p and it can be $C_p \subset B_s \mathcal{H}$ then the estimate fails.

(Notice this $\mathcal{O}(1)$ has base locus.)

$X \cap \{x_0 = \dots = x_{2c} = 0\} = X \cap \overline{P_0 P_1} = \{P_\infty\}$

and $P_\infty = Bs\, \mathcal{O}_X(1)$. The argument does work for P_∞.

Lemma: If $P \neq P_\infty$, let $X' = Bl_p X \xrightarrow{\varphi} X$.

There are 2 cases:

1. $\mathrm{mult}_p C_p = 1$.

$\overline{NE}^1(X') = \mathbb{R}_+[E] + \mathbb{R}_+[A - E]$, $A := \varphi^* \mathcal{O}_X(1)$.

2. $\mathrm{mult}_p C_p = 2$.

$\overline{NE}^1(X') = \mathbb{R}_+[E] + \mathbb{R}_+[A - 2E]$, and

$\exists!\, D \in |A - 2E|$ st. if $D \neq H \geq 0$ is irreducible $\Rightarrow H \in \overline{NE}^1(X')$.

(D is irreducible.)

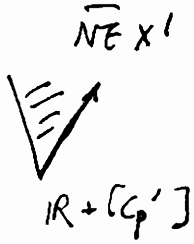
Even case 1 needs 10 more min to explain.

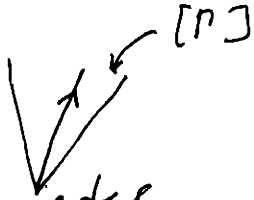
idea of pf: assume $\mathrm{mult}_p C_p = 1$,

then $C_p^* = D_1 \cdots D_{2c-1}$

$C_p' = D_1' \cdots D_{2c-1}'$

Claim: $\mathbb{R}_+[C_p'] \subset \overline{NE}X'$ is an extr ray: through $K_{X'} . C_p' > 0$.





If not, then note
$C_p'.(A-E) = 0$, so $\exists$
$\Gamma \subset X'$ st. "closer" to the edge,
necessarily $\Gamma.(A-E) < 0$

$\Rightarrow \Gamma \subset D_i'$ but then $\Gamma = C_p$ because
$C_p = Bs\,|A-E|$.

Now because $C_p' = Bs\,|A-E|$ and $C_p'(A-E) = 0$, then C_p' is nef. i.e. $C_p'.D \geq 0$ if $D \geq 0$.

If $D \sim nA - \nu E \geq 0$ then

$$0 \leq D.C_p = n - \nu$$

$\Rightarrow D \in \mathbb{R}_+[E] + \mathbb{R}_+[A-E]$. case 1 is done. $\square$

The "thinking in Mori cone" is really essential in the pf.

Corti 3/8 2004 at Taida

One angle to Mirror Symmetry —
to understand unpublished work of
Vaisly GOLYSHEV.

X Fano mfd, mirror sym relates

$\left\{ \begin{array}{l} \text{enumerative geom} \\ \text{of rat'l curves in } X \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{VHS of } H_n(Y_t) \\ f : Y \to \mathbb{C}^\times \end{array} \right\}$

(GW - inv.) (the mirror obj to X)

X : a Fano mfd.
"small" quantum coh:
denote by $\langle , \rangle$ Poincaré duality on $H^{ev}X$,
$\rho = \dim H^2 X = 1$.

$$\langle A * B, C \rangle = \langle A \cup B, C \rangle + \sum_k \langle A, B, C \rangle_k q^k$$

where $\langle A, B, C \rangle_k = \# \left\{ f : \mathbb{P}^1 \to X \mid \deg f_* \mathbb{P}^1 = k, \; f(0) \in A, \; f(1) \in B, \; f(\infty) \in C \right\}$

($\deg f_* \mathbb{P}^1 = f_*[\mathbb{P}^1] \cdot D$
where $D \in H^2 X$ a generator.)

This formula defines a "quantum int. prod" on $H^{ev}X$. (Various approaches to make precise this, each one requires a few months!)

eg. on $\mathbb{P}^2$: $QH^*(\mathbb{P}^2, \mathbb{C}) = \mathbb{C}[h, q] / (h^3 = q)$

meaning: 2 curves may not meet in the classical sense, but can be connected thr some $\mathbb{P}^1$'s!

Ref. [Kontsevich - Manin]
* [Dubrovin : Painlevé transcendents etc.. AG/9803107]

 Still need to talk about "big quantum coh"

Gromov-Witten Invariants:

$I^X_{0,n,\beta} : H^{ev}X^{\otimes n} \longrightarrow H^*\overline{M}_{0,n}$

$M_{0,n} = \{ n \text{ pts } x_1, \cdots, x_n \in \mathbb{P}^1 \} / \text{isom}.$

$\overline{M}_{0,n}$ a natural compactification of $M_{0,n}$:
a pt of $\overline{M}_{0,n}$ is a curve C which is a "tree of $\mathbb{P}^1$'s", with n marked pts or labelled $x_1, \cdots, x_n \in C$ st.

$$\omega_C(\Sigma x_i) \text{ is ample on } C$$

[$\Leftrightarrow$ if C' a comp & $C'' = C \setminus C'$, then
$C'.C'' + \#\{x_i \in C'\} \geq 3$ " .)]

$I^X_{0,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_n) = cl\left(\{(\mathbb{P}^1, x_i) \mid \exists f : (\mathbb{P}^1, x_i) \to X, f(x_i) \in \Gamma_i\}\right)$

$\gamma_i = cl\, \Gamma_i \in H^{ev}X$ $\qquad \in H^*\overline{M}_{0,n}$. $f_*\mathbb{P}^1 \sim \beta$

$\beta \in H_2X$.

These satisfy some axioms:

Axioms: (P not $\mathrm{rc} = 1$)

- $I = 0$ if $\beta \notin NE(X)$.
- compatible with S_n-action.
- $I^X_{0,n,\beta}$ is homogeneous of degree $2K.\beta - 2\dim_{\mathbb{C}} X$. $K = K_X$.

(notation: $|\gamma| = m$ if $\gamma \in H^m$)

$|I^X_{n,\beta}(\gamma_1, \cdots, \gamma_n)| = \Sigma|\gamma_i| + 2K.\beta - 2\dim_{\mathbb{C}} X$.

$\{f : \mathbb{P}^1 \to X, f_*\mathbb{P}^1 \sim \beta\}$ has "expected complex dim" $-K.\beta + \dim X$.

- fundamental class.

$$I^X_{n,\beta}(\gamma_1, \cdots, \gamma_{n-1}, 1^0_X) = \pi_n^* I_{n-1,\beta}(\gamma_1, \cdots, \gamma_{n-1})$$

$\uparrow$ fund class of X

where $\pi_n : \overline{M}_{0,n} \to \overline{M}_{0,n-1}$ forgets the n-th pt.

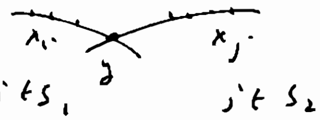
- divisor. If $|\gamma_n| = 2$, then

$$I_{n,\beta}(\gamma_1, \cdots, \gamma_n) = (\beta \cdot \gamma_n) I_{n-1,\beta}(\gamma_1, \cdots, \gamma_{n-1})$$

- Splitting. (Very interesting Axiom!)

$S = \{1, \cdots, n\}$, $S = S_1 \sqcup S_2$, $\# S_i = n_i$

$$\varphi = \varphi_{S_1, S_2} : \overline{M}_{0,n_1+1} \times \overline{M}_{0,n_2+1} \to \overline{M}_{0,n}$$



let $\delta = [\Delta_X] \in H^{2n}(X \times X)$, Künneth:

$\delta = \sum_{i+j=2n} \delta_i \times \delta_j$; $\delta_i \in H^i X$.

$$\varphi^* I_{n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_n)$$

$$= \sum_{\beta = \beta_1 + \beta_2} \sum_{a+b=2n} I_{n_1+1,\beta_1}\Big(\underline{\delta_a} \otimes \bigotimes_{i \in S_1} \gamma_i\Big)$$

$$\times I_{n_2+1,\beta_2}\Big(\underline{\delta_b} \otimes \bigotimes_{j \in S_2} \gamma_j\Big).$$

This causes problem! If we just define GW on H^{ev} then here δ_i can be of odd degree (!!) If we define GW for all H^* (which is needed) then we have to introduce various ± 1 signs.

p. 68 (Big) quantum coh of X.

GW potential

$$\phi_\omega(\gamma) = \sum_{n \geq 3} \sum_\beta e^{-\int_\beta \omega} \frac{1}{n!} I_{n,\beta}(\gamma^{\otimes n})$$

ω a Kahler class, This is a generating series for the I's with parameter in H^*X.

If $\partial_a \in H^*X$ is a basis,

$\gamma = \sum t_a \partial_a \in H^*(X, \mathbb{C})$,

$$\frac{1}{n!} I_{n,\beta}(\gamma^{\otimes n}) := \sum_{\substack{n_0 + \cdots + n_D \\ = n}} \frac{1}{n_0! \cdots n_D!} I(\delta_0^{n_0} \cdots \delta_D^{n_D}) \, t_0^{n_0} \cdots t_D^{n_D}.$$

(ϕ_ω is not known to converge for small t, but for some important examples it is)

$$A_{abc} = \frac{\partial^3}{\partial t_a \partial t_b \partial t_c} \phi$$

quantum mr. prod. $\langle \partial_a * \partial_b, \partial_c \rangle = A_{abc}$

coefficients $\mathcal{O} = \mathbb{C}[[t_0, \cdots, t_D]] = \mathcal{O}^{\wedge}_{0, H^*(X,\mathbb{C})}$

- small quantum coh is obtained by restricting to $\gamma \in H^2 X$ & $q_i = e^{t_i}$.

Example.

for $X = \mathbb{P}^2$, $QH^*(\mathbb{P}^2, \mathbb{C}) = \mathbb{C}[h, q]/h^3 - q$

$\partial_0, \overset{h}{\partial_1}, \partial_2$ standard basis for $H^*\mathbb{P}^2$.

$$\phi(t_0, t_1, t_2) = \frac{1}{2}(t_0^2 t_2 + t_0 t_1^2) + f(t_0, t_1, t_2)$$

where $f(t_0, t_1, t_2) = \sum_{k=1}^{\infty} \frac{N_k \, T_2^{3k-1}}{(3k-1)!} e^{k t_1}$

$N_k = \#\{$rat'l curves of deg k in $\mathbb{P}^2$ thr $3k-1$ general pts$\}$

Set $t_0 = t_2 = 0$, $e^{t_1} = q \Rightarrow QH^*$.

We expect, on a Fano var.
small quantum coh determines the
full (big) quantum coh.

2 approaches to this:

① Kontsevich - Manin ("reconstruction" thm)

✓ ② Dubrovin (ODE's) and
"isomonodromic deformations".

The $\{A_{abc}\}$ give a small nbd $0 \in U \subset H^*(X, \mathbb{C})$ the structure of a Frobenius mfd.
In particular,

$$\nabla^z_{\partial_b} = \partial_b + z \sum_a dx^a A^c_{ab} \otimes \partial_c$$

is a flat connection on U for all $z \in \mathbb{C}^\times$.

(flatness $\Leftrightarrow$ associativity of quantum product).

The Euler vector field

$$E = \sum \left((1 - q_a) t_a + r_a \right) \partial a \quad \in H^*$$

where $\partial_a \in H^{2q_a} X$, $r_a = 0$ unless $q_a = 1$
& $-K_X = \sum r_a \partial_a$

homogeneity of I's can be stated by saying:

$$E\phi = \phi + \text{quadratic terms in } t\text{'s}$$

Dubrovin defines a flat connection on $U \times \mathbb{C}^\times$ |

$$\begin{cases} \tilde{\nabla}_{\partial_a} \frac{d}{dz} = 0 \\ \tilde{\nabla}_{\frac{d}{dz}} \frac{d}{dz} = 0 \\ \tilde{\nabla}_{\frac{d}{dz}} \partial_a = E * \partial_a - \frac{1}{z} \sum \mu_a \partial_a \quad (\mu_a = q_a - 1) \\ \tilde{\nabla}_{\partial_a} \partial_b = \nabla^z \partial_b \end{cases}$$

For a vector $\xi = \sum \xi^a \partial_a$

with $\xi^a = \xi^a(t, z)$ to have $\tilde{\nabla}\xi = 0$:

$$(*) \begin{cases} \dfrac{\partial \xi^b}{\partial t_a} + z A_{ac}^{b} \xi^c = 0 \\ \dfrac{\partial \xi^b}{\partial z} + E^c A_{ca}^{b} \xi^a - \dfrac{1}{z} M_a^b \xi^a = 0 \end{cases}$$

where $E = E^a \partial_a$, $M_a^b = \delta_a^b M_a$.

In good cases (semi-simple qc. expected to be OK on a Fano var).

think of $(*)$ as a ODE on $\mathbb{C}^\times$ with parameter $t \in U$. $(*)_t$ is a isomonodromic deformation of $(*)_0$. As it turns out,

$(*)_0$ is regular at $z = 0$,
irregular at $z = \infty$.

for $\mathbb{P}^2$ at $t = 0$ (small q-coh), $(*)_0$ is

$$(*)_0 \quad \left\{ \frac{d}{dz}\begin{pmatrix} \xi^0 \\ \xi^1 \\ \xi^2 \end{pmatrix} = \left[\begin{pmatrix} 0 & 0 & 3 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} - \frac{1}{z}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\begin{pmatrix} \xi^0 \\ \xi^1 \\ \xi^2 \end{pmatrix} \right.$$

$\Rightarrow \varphi = \dfrac{\xi_2}{z}$ satisfies $\left(z \dfrac{d}{dz}\right)^3 \varphi = 27 z^3 \varphi$.

$D = z \dfrac{d}{dz}$, inv. diff. op. on $\mathbb{C}^\times$

mirror sym for $X = \mathbb{P}^2$. out of q-coh of $\mathbb{P}^2$ get $D^3 - z^3$ on $\mathbb{C}^\times$.

This is turns out motivic! ie. the Picard Fuchs eq'n for some family of var. This can't be "completely true" since P-F eq'ns has only regular singular pts.

Consider the Pencil

$$Y = \begin{cases} x_1 x_2 x_3 = w \\ x_1 + x_2 + x_3 = 1 \end{cases} \circlearrowleft (\mathbb{C}^\times)^4$$

$$Y \xrightarrow{w} \mathbb{C}^\times \qquad \omega = \mathrm{Res}_Y \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \wedge \frac{dx_3}{x_3}$$

$$\psi = \int_{\gamma_t} \omega, \quad H_1(Y_t) \simeq \mathbb{Z}.$$

satisfies a Picard-Fuchs equation:

$$\left[(D - \tfrac{1}{3})(D - \tfrac{2}{3}) - w D^2 \right] \psi = 0.$$

This op is related to $D^3 - z^3$.

fact: poly $z^3 = w$

$\frac{1}{D} FT(?D^3 - w)$ ... something like this.

FT: fourier transf

Theorem (Fano, Iskovskikh, Mori-Mukai ..)
there are 17 deformation families of non-singular Fano 3-folds with $rk = \rho = \dim H^2 X = 1$.

Fano index I, $-K = Ih$, $h \in H^2 X$ primitive.

$I = 4$ $\quad X = \mathbb{P}3$

$I = 3$ $\quad X = Q \subset \mathbb{P}^4$ quadric

$I = 2$ $\quad$ hyp. section of X is a del Pezzo Surface

$X_3 \subset \mathbb{P}^4$ $\quad dP_3$

$X_{2,2} \subset \mathbb{P}^5$ $\quad dP_4$

$X = (1)^3 \cap G(2,5)^6 \subset \mathbb{P}^4$ $\quad dP_5$

$X_4 \subset P(1^4, 2)$ $\quad dP_2$

$X_6 \subset P(1^3, 2, 3)$ $\quad dP_1$

(5 cases)

p. 72 $\quad I = 1$ (Main Cases)

$X^5_{2g-2} \subset \mathbb{P}^{g+1}$, $\quad 2g-2$ = degree of canonical curve.

$-K$ is primitive, general $S \in |-K|$ is a K3 surface

$g=2 \quad X_6 \subset \mathbb{P}(1^4, 3)$

$g=3 \quad X_4 \subset \mathbb{P}^4$

$g=4 \quad X_{2,3} \subset \mathbb{P}^5$

$g=5 \quad X_{2,2,2} \subset \mathbb{P}^5$

$\downarrow$ codim.

$g=6 \quad (\cancel{(1)}^4 \cap C\,G(2,5)) \subset \mathbb{P}^{10}$

$(1)^3 \cap (Q$ $\quad$ cone of ...

$g=7 \quad$ Section of $OG(5,10) \subset \mathbb{P}^{15}$

$g=8 \quad$ " $\quad G(2,6)$

$g=9 \quad$ " $\quad SG(3,6)$

$g=10 \quad$ " $\quad G_2$-variety

$g=12 \quad$ " $\quad$ (wierd).

there is a gap.

Corti. Lecture X (last one)

3/15 2004 at NCTS at Taida

Mirror symmetry for $\mathbb{P}^2$.

from "small" quantum coh we get ODE:

$$\frac{d}{dz}(\xi_0, \xi_1, \xi_2) = (\xi_0, \xi_1, \xi_2)\left[\begin{pmatrix} 0&0&3 \\ 3&0&0 \\ 0&3&0 \end{pmatrix} - \frac{1}{z}\begin{pmatrix} -1 && \\ &0& \\ &&1 \end{pmatrix}\right]$$

for $\varphi = \frac{\xi_0}{z}$ get

(*) $(D^3 - 27z^3)\varphi = 0$; $D = z\frac{d}{dz}$ ($\mathbb{C}^\times$-inv do on $\mathbb{C}^\times$)

Remark: the FT of (*) is isomorphic to:

① $D(D^2 - 27z(Dz)^2)\hat{\varphi} = 0$,

or, to the pull back $w = z^3$ of

② $D\left((D - \frac{2}{3})(D - \frac{1}{3}) - \frac{w}{27}D^2\right)$.

pf: by def'n. $\widehat{\sum f_i(z)\left(\frac{d}{dz}\right)^i} = \sum f_i\left(\frac{d}{dz}\right)(-z)^i$

$\hat{z} = \frac{1}{z}D$, $\hat{D} = -\frac{d}{dz}\cdot z = -D - 1$,

$\widehat{D^3 - 27z^3} = -(D+1)^3 - 27\left(\frac{1}{z}D\right)^3$

($D\cdot z^\gamma = z\frac{d}{dz}z^\gamma = z^\gamma(D+\gamma)$

Here we do things purely algebraically on the ring $\mathcal{D} = \mathcal{D}(\mathbb{G}_m) = \mathbb{C}[z, z^{-1}, D]$ and FT: $\mathcal{D} \to \mathcal{D}$ involution.)

$= -(D+1)^3 - 27\left((D+1)\frac{1}{z}\right)^3$

under $z \mapsto \frac{1}{z}$, $D \mapsto -D$: $\sim (D+1)^3 - 27((D+1)z)^3$

$\sim D^3 - 27(Dz)^3$ iso as DE. $\square$

$$D\left(D^2 - 27 z D z D z\right)$$
$$= D\left(D^2 - 27 D z^2 (D+1)\right)$$
$$= D\left(D^2 - 27 z^3 (D+2)(D+1)\right)$$
$$= D\left(D^2 - 27 w \left(D+\tfrac{2}{3}\right)\left(D+\tfrac{1}{3}\right)\right)$$

$z^3 = w$

$\mathbb{C}^\times \to \mathbb{C}^\times$

$z \mapsto z^3 = w$

$D_z = 3 D_w$

do $w \mapsto \frac{1}{w}$, get

$$= D\left(\left(D+\tfrac{2}{3}\right)\left(D+\tfrac{1}{3}\right) - \tfrac{w}{27} D^2\right).$$

There is a dictionary:

$$\left\{\begin{array}{l}\text{ordinary d.e.}\\ \text{over algebraic}\\ \text{curves } C\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{local system of}\\ \mathbb{C}\text{-vector space}\\ \text{on } C\end{array}\right\}$$

of order m $\qquad\qquad$ of rk m

$\qquad\qquad\qquad\qquad\qquad\qquad$ i.e. repr of $\pi_1(C)$.

$$\updownarrow \qquad\qquad \updownarrow$$

$$\left\{\begin{array}{l}\text{rank } m \text{ v.b. } E \text{ on } C\\ \text{with flat alg. conn:}\\ D : E \to \Omega^1_C \otimes E\end{array}\right\}$$

In nature, local systems arises from the "varying cohomology" of a family of alg. var.

e.g.
$$\begin{array}{l}\mathcal{X}\\ \downarrow f\\ C\end{array} \qquad t \mapsto H^m_c(\mathcal{X}_t, \mathbb{C})$$

this is a local system on C.

The corresponding ODE is the Picard-Fuchs eq'n.

Exercise: $$\int \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} = \varphi(\lambda)$$

show $\varphi(\lambda)$ satisfies D.E of order 2 (hyp geometric type).

Notice that (*) has essential (irregular) singularity at ∞.
That's the reason we take the F.T.

Question: is FT (*) of geometric origin?

i.e. coming from family of alg var

(Singularity from $\begin{array}{c}\mathfrak{X}\\ \downarrow\\ S\end{array}$ must be regular)

Answer: $\mathfrak{X} = \begin{cases} x_1 + x_2 + x_3 = w \\ x_1 x_2 x_3 = 1 \end{cases} \subset (\mathbb{C}^\times)^4_{x_1, \ldots, x_3, w}$
$\downarrow$
$\mathbb{C}^\times_w$

the local system of ② is the local system associate to $H^1_c(\mathfrak{X}_\mu, \mathbb{C})$. So we say that $\mathfrak{X} \to \mathbb{C}^\times$ "is" the mirror of $\mathbb{P}^2$.

Note: $\mathfrak{X} \to \mathbb{C}^\times$ is not proper!

However, under mild assumptions, if $\mathfrak{X} \subset \overline{\mathfrak{X}}$ is a compactification, $\Rightarrow$ $H^1_c(\mathfrak{X}) = H^1(\overline{\mathfrak{X}})$, so it does not matter (for current purpose)

Lefschetz hyp. thm.

The pf is not hard, basically similar to the given exercise.

* cf. N. Katz's Princeton book on local system which explain how can one do FT to transform irregular sing to reg one.

So far, this toy model is all in Givental.
The new contr. is about Fano 3-folds!

X non-singular Fano 3-fold, $\rho = 1$,

deg $d = (-K)^3$. notation $A = -K$.

of Fano index I ($-K = IH$, H primitive).

$$(a_{ij}) = \frac{1}{d}(j+i-1) \, \# \left\{ \begin{array}{l|l} \text{rat'l curves} & \Gamma \text{ meets} \\ \Gamma \subset X, \deg \Gamma & \text{cycles} \\ = j+i-1 & A^{3-i}, A^i \end{array} \right\}$$

plan: do the analogue of the $\mathbb{P}^2$ case.

$$M = \begin{pmatrix} 0 & a_{01}(D_z)^2 & a_{02}(D_z)^3 & a_{03}(D_z)^4 \\ 1 & a_{11} D_z & a_{12}(D_z)^2 & \cdot\cdot \\ 0 & 1 & (a_{22}=a_{11}) D_z & \cdot\cdot \\ 0 & 0 & \cdot\cdot & \cdot\cdot \end{pmatrix} \quad \text{sym along}$$

Def$^{\text{n}}$: Write $\tilde{L} = \det_{\text{right}}(D\,\mathrm{Id} - M) = D \cdot L$

means if a term reads $a\,b\,c$
then we use $c \cdot b \cdot a$

then $L \cdot \varphi = 0$ is the "enumerating DE" of X.

Question: Is $L\varphi = 0$ the pull back under
$w = z^I$ of a d.e. of geometric origin
on $\mathbb{C}^\times$?

Where to look? (for the varieties?)

$X_0(N) = \{ (E, A) \mid E$ is an elliptic curve
$A \subset E[N]$, cyclic order $N \}$

$= \mathfrak{h}^+ / \Gamma_0(N) \cup \{\text{cusps}\}$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \;\middle|\; c \equiv 0 \ (N) \right\} / \pm 1.$$

$$\tau \in \mathfrak{h}^+ \mapsto E_\tau = \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}, \quad A = \left\{ \tfrac{1}{N}, \tfrac{2}{N}, \dots, \tfrac{N-1}{N} \right\}.$$

$w : X_0(N) \circlearrowleft$ Fircke involution
?

$$w(E, A) = (E/A, E[N]/A)$$

$$\tau \mapsto -\frac{1}{N\tau} = \begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix} \cdot \tau$$

over $X_0(N)$, I have a universal family $\mathcal{E} \to X_0(N) \circlearrowleft w$. Let

$$\mathcal{X} = \mathcal{E} \times_{X_0(N)} \mathcal{E}^w / i \leftarrow \text{kummer inversion}$$

$$\downarrow$$

$$X_0(N)^w = X_0(N)/w$$

is a 1-dim'l family of Kummer K3 surfaces. We will look at H^2 of this family.

The transcendental lattice

$$T(\mathcal{X}_t) = NS(\mathcal{X}_t)^\perp \subset H^2(\mathcal{X}_t, \mathbb{C})$$

form a rank 3 local system on $X_0(N)^w$, with monodromy:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} d^2 & 2cd & -c^2/N \\ bd & bc+ad & -ac/N \\ -Nb & -2Nab & a^2 \end{pmatrix}$$

$$w \longmapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Point: X Fano 3 folds as above, <u>we expect</u> that enumerating local system on X is $T(\mathcal{X}_t)|_{\mathbb{C}^\times}$ for some $\mathbb{C}^\times \subset X_0(N)^w$.

P. 78. (for some N, in particular $X_0(N)^w$ would have to be a rat'l curve!)?

Reason: X Fano 3-folds

hyp. sect of X : family of K3 surfaces Y with $\rho = 1$. $\longrightarrow$ mirror to K3 Y' of Picard rk 19 (i.e. moving in a 1-dim'l moduli space)

Mirror sym for K3

D. Morrison: shows that for any such Y', $\exists\, i : Y' \circlearrowleft$ st. $Y'/i = \mathcal{X}_t$.

But this "reason" is only a hint.

def'n: a (I, V)-modular local system is the pull back to $\mathbb{C}^\times$ of $T(\mathcal{X}_t)$ under

$$\begin{array}{l} \mathbb{C}^\times \longrightarrow \mathbb{C}^\times \subset X_0(N)^w \\ w \longmapsto w^I \end{array}$$

(st. $X_0(N)^w$ is $\mathbb{P}^1$.)

def'n. consider a 4×4 matrix

$$M_{k\ell} = \begin{cases} 0 & k > \ell+1 \\ 1 & k = \ell+1 \\ a_{k\ell}(D_z)^{\ell-k+1} & \text{if } k < \ell+1 \end{cases} ; \quad 0 \le k, \ell \le 3$$

where $a_{k\ell} \in \mathbb{C}$ st. $a_{k\ell} = a_{3-k,3-\ell}$.

$\tilde{L} = \det_r(D - M) = D \cdot L$

Then $L \cdot \varphi = 0$ is a d.e. of Type DE^3?

at most $\underline{5}$ singular pts. (or 4?)

Theorem: a (I, N)-modular local system is of type D3 if and only if

$(I, N) = (1,1), (1,2)\ (1,3) \cdots (1,9), (1,11)$

$(2,1), (2,2) \cdots (2,5)$ gap !

$(3,3)$

$(4,2)$

(The pf is a book keeping work, NOT hard.)

later work:

Find modular forms, → D.E. can be found in

Conway Norton: Monsters' Moonshine.

already calculate them.