A Corti NCTS 4/10 2013 p 1
at Taida

MS & Fano mfds

[CCGGK] arXiv 1212 1722 A concise introduction
[ACGK] arXiv 1212 1785 Minkowski poly & mutations
[CCGK] 1303 3288 quantum periods for 3d Fano mfds
` the most technical one

§ 1. Fano mfd & classification
§ 2 Motivation
§ 3 Quantum Periods
§ 4 The (classical) period of a LP (Laurent polynomial)
§ 5 Mirror symmetry
§ 6 Our Main theorems

§1 Examples $n=1$ $\mathbb{P}^1$
$n=2$: del Pezzo surfaces, $\deg d = K_X^2 \geq 1$
X = blow up of $k = 9-d$ general pts on $\mathbb{P}^2$
or $\mathbb{P}^1 \times \mathbb{P}^1$

$n=3$: There are exactly 105 deformation families
17 with $b^2=1$ (Fano 1930's, Iskovskikh 70's, corrected by Mukai, Mori, -)
88 with $b^2 \geq 2$ (Mori-Mukai in 80's)

- A very hard theorem, with a lot of explicit & projective birat'l geom
- Also in any fixed dim n, $\exists$ finitely family (Hard!)

Main motivation of this work is to understand the $n=3$ classification in a diff angle

$b_2=1$ case. the inv is the Fano index f:
$-K_X = fA$ with A primitive in $\operatorname{Pic} X = H^2(X,\mathbb{Z})$)
$1 \leq f \leq 4$.
$f=4 \Rightarrow X = \mathbb{P}^3$
$f=3 \Rightarrow X = Q^3 \subset \mathbb{P}^4$

$f=2 \Rightarrow 5$ families : general $S \in |A|$ is a del Pezzo surf! 2

$K_S = K_X + S|_S = -2A + A|_S = -A|_S$ of degree $1 \leq d \leq 5$

Example : $X = B_5 = (\sigma = 0) \subset Gr(2,5)$

$\sigma \in \mathcal{O}(1)^{\oplus 3}$, $\quad 0 \to S^r \to \mathbb{C}^n \to Q^{n-r} \to 0$

$\Lambda^r S^*$ is ample $:= \mathcal{O}(1)$

HW : Prove B_5 is Fano of index 2

$f=1$: "Main series"

main invariants : $\underset{\text{degree}}{-K_X^3} = 2g-2$ & $h^0(X, -K_X) = g+2$

HW: R.R $\Rightarrow$ this

$2 \leq g \leq 10$ or $g = 12$

eg $g=4$, $\rightsquigarrow V_6 = 2 \cdot 3 \subset \mathbb{P}^5$
degree $2g-2$

$g = 8$, $\rightsquigarrow V_{14} = (\sigma = 0) \subset G(2,6)$, $\sigma \in \mathcal{O}(1)^{\oplus 5}$.

They are all zero loci of section of homog v.b / homog space

~~f~~ $b_2 = 2$ No g

Mori-Mukai construction : (they never published the complete pf only part I)

the blow-up of $\mathbb{P}^3$ with center $\Gamma \subset \mathbb{P}^3$ of $\deg = 7$, $g = 5$ such that I_Γ is generated by cubics

• this is outside the discussion of Hartshorne Ch 4 but really a good exercise :

$$\Gamma = \left\{ rk \begin{pmatrix} l_1 & l_2 & l_3 \\ q_1 & q_2 & q_3 \end{pmatrix} < 2 \right\} \subset \mathbb{P}^3$$

3 eq'ns , not complete int

More than one component in the Hilb scheme

Then
$$X : \left\{ \begin{pmatrix} l_1 & l_2 & l_3 \\ q_1 & q_2 & q_3 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = 0 \right\} \subset \mathbb{P}^3 \times \mathbb{P}^2_{(y_0 : y_1 : y_2)}$$

$\downarrow$

$\mathbb{P}^3$

this does the job of blowing-up

$= (1,1) \cdot (2,1) \subset \mathbb{P}^3 \times \mathbb{P}^2$

complete int

Theorem Let X be a non-singular Fano 3fold, there is a Lie gp $G = \prod_{i=1}^{k} GL_{r_i}(\mathbb{C})$, 2 repr's A, V of G st

$$X = (\sigma = 0) \subset A/\!/G \quad \text{where } \sigma \in H^0(A/\!/G, V)$$

ie section of homog vb

Pf: case by case, post classification

(Hope this works for dim 4, but definitely not true for higher dim' due to moduli is not always unirat'l (same higher genus curve argument) but all Fano from $\sigma = 0$ must have uni-rat'l moduli

· $n \geq 4$? We almost know nothing!)

§2

① What can I learn about Fano mfds by studying their mirrors?

② Perhaps mirrors of Fanos are easier "comb: objects" may I can learn something about classif of Fano 4-folds.

§3. Quantum periods

$$G_X(t) = \sum_{m \geq 0} p_m t^m, \quad p_m := \sum_{-K_X \cdot \beta = m} \langle \psi^{m-2} ev_1^*[pt] \rangle_{X_{0,1,m}}$$

finite sum if X is Fano orbifold $\mathbb{Q}$

Thm $\exists$ poly diff op $Q_X \in \mathbb{Z}\langle t, D\rangle$, regular on $\mathbb{C}^\times$ where $D = t\frac{d}{dt}$, st $Q_X G_X \equiv 0$ ($\Rightarrow G_X$ conv on $\mathbb{C}$)

regularized quantum period

$$\hat{G}_X(t) = \sum (m!) p_m t^m \quad \text{(Fourier-Laplace transf)}$$

$$\hat{Q}_X(t)\hat{G}_X(t) \equiv 0$$

Example $X = \mathbb{P}^2$, $G_X(t) = \sum \frac{t^{3m}}{(m!)^3}$, $\hat{G}_X(t) = \sum \frac{(3m)!}{(m!)^3} t^{3m}$

both are hypergeometric

§4 Classical period of a L.P

Laurent poly : $f : \mathbb{C}^{\times 4}_{x_1, \dots, x_4} \to \mathbb{C}_t$; $f \in [x_1^{\pm 1}, \dots, x_4^{\pm 1}]$

$$\pi_f(t) = \left(\frac{1}{2\pi i}\right)^4 \int_{|x_1| = \dots = |x_4| = 1} \frac{\Omega}{1 - tf} \quad ; \quad \Omega = \bigwedge_{i=1}^4 \frac{dx_i}{x_i}$$

inv form on torus

In the last lecture will discuss

Thm: $\exists$ poly diff op $L_f \in \mathbb{C}\langle t, D\rangle$, with reg sing st $L_f \cdot \pi_f(t) = 0$ (The Picard-Fuchs operator)

in this special case of L.P an explicit pf is possible.

$$\pi_f(t) = \sum t^m \left(\frac{1}{2\pi i}\right)^4 \int_{|x_1| = \dots = |x_4| = 1} f^m \Omega$$

apply residue thm m-times

$$= \sum c_m t^m$$

c_m = const term of f^m

$L = \sum t^k P_k(D)$, $\Phi = \sum a_m t^m$

$L\Phi = 0 \iff$ linear recursive relation $\sum_k P_k(m-k)\, a_{m-k} = 0$, $\forall m$

This is a Huge computation even for Supercomputer

§5 Mirror sym Def^{n}: f is mirror to X if

$\pi_f = \widehat{G}_X$ ($\Rightarrow L_f = \widehat{Q}_X$, $\mathrm{sol}(L_f) \simeq \mathrm{sol}(\widehat{Q}_X)$)

Rmk: This is practical, but it has some disadvantages :

(1) It sometimes gives the Wrong answer,

(2) If X has a mirror, then it has ∞ many,

(we have some understanding, later)

§6 Our thm :

In 3 variables, I will define "Minkowski polynomials" (simple combinatorial def^{n}), there are 3747 MPs but only 165 periods / PF ops

Of these, 88 are of "mfd" type the others are "orbifold" type (More nat'l cat of MS)

Thm the 88 are 1-1 corr. to 88 Fano 3-folds ~~to be~~ with $-K_X$ very ample ... to be continued

Corti, lect 2 4/12 2013

Recap: quantum period

$$\hat{G}_X = \sum m! p_m t^m, \quad p_m = \int_{[X_{0,1,m}]^{vir}} \psi^{m-2} ev^*(pt)$$

classical period of $f \in \mathbb{C}[x_1^{\pm1}, \dots, x_n^{\pm1}]$

$$\pi_f(t) = \sum c_m t^m, \quad c_m = \mathrm{coeff}_1(f^m)$$

MS: $\hat{G}_X = \pi_f(t)$

Today focus on quantum period:

§1 (short) intro to q-coh

§2 computational technique with examples

1.1 Q-prod: X^n Fano mfd

Mori's cone thm: $NEX := \sum_{C \subset X \text{ curve}} \mathbb{R}_{\geq 0}[C] \subset H_2(X;\mathbb{R})$

$NEX = \overline{NEX}$ is a rat'l polyhedral cone

$\exists \ \Lambda = \mathbb{C}[NEX]$ f.g. $\mathbb{C}$ alg (Novikov ring)

new prod str on $H^{ev}(X;\Lambda)$, $*$

$a, b, c, \in H^{ev}(X, \mathbb{C})$,

$$(a*b, c)_{IP} := \sum_{\beta \in NEX} q^\beta \langle a, b, c \rangle_{0,3\beta}$$

a graded ring, $\deg q^\beta = -K_X \cdot \beta > 0 \Rightarrow$ finite sum

1.2 integrable connection on the trivial bundle

fiber $= H^{ev}(X;\mathbb{C})$ over $\mathbb{T} = \mathrm{Spec}\, \mathbb{C}[H_2(X;\mathbb{Z})]$

$= $ a torus $\mathbb{T}_{H^2(X;\mathbb{Z})} = H^2(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^\times$, i.e. $\mathrm{Lie}\,\mathbb{T} = H^2(X;\mathbb{C})$

$\mathcal{E} := \{ s : \mathbb{T} \to H^{ev}(X;\mathbb{C}) \text{ holomorphic} \}$

$$\nabla_\xi s = \xi s - \xi * s, \quad s \in \mathcal{E}, \ \xi \in \mathrm{Lie}\,\mathbb{T}$$

note, ∇ has polynomial coefficients,

associativity of $*$ (WDVV) $\iff \nabla$ is a flat conn

$\Rightarrow \mathcal{E}$ is a $\mathcal{D}$ module, $\mathcal{D} = \{\text{diff op on } \mathbb{T}\}$

Justify the statement that $G_X, \hat{G}_X$ satisfy poly ODE

Local system of solutions: $\mathbb{V} = \mathrm{sol}\, \mathcal{E} = \underline{\mathrm{Hom}}_{\mathcal{D}}(\mathcal{E}, \mathcal{O})$

$\varphi \in \mathbb{V}$, $s \in \mathcal{E} \Rightarrow \varphi(s)$ satisfies $\mathrm{Ann}(s)$

1.3 J-function (first appears in Dijkgraaf, then taken by Givental)

$J_X(q) = 1 + \sum_{\beta \in NEX \setminus \{0\}} q^\beta J_\beta \in H^{ev}(X;\mathbb{C})$, q are variables on the torus.

where $J_\beta = ev_* \frac{1}{1-\psi} = \sum_k \psi^k$ is a finite sum.

Thm (Dijkgraaf): $J_X(q) \in \text{Sol } \mathcal{E}$

in the sense st $\mathcal{E} \longmapsto (J_X(q), s)_{P.D} \in \mathcal{O}$

let $J_X^\circ(q) = (J_X(q), \mathbb{1}) \in \mathcal{O}_{\mathbb{T}}$ $\quad -K_X \in H^2(X,\mathbb{Z})$ $\quad t^{-K_X.\beta} \leftarrow q^\beta$

$\updownarrow$ dual

& $G_X(t) = J_X^\circ(k_X(t))$ $\quad k_X : \mathbb{C}^\times \xrightarrow[\text{cocit}]{} \mathbb{T}$

by all the above, it satisfies a polynomial diff eq^{ns} (for more details, cf the orbifold papers on W.p.s)

Example: $X = \mathbb{P}^2$, $H^{ev}(X,\mathbb{C}) = H^*(X,\mathbb{C}) = \mathbb{C}[P]/P^3$

basis: $\mathbb{1}$, $-K_X = 3P$, $K_X^2 = 9\,[pt]$

Will write everything in this basis

matrix of $-K_X *$ in this basis

$$M = \begin{pmatrix} 0 & 0 & 27t^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \left(\begin{array}{c} \text{i.e.} \\ t \text{ is } e^{2\pi i s} \end{array}\right)$$

the matrix says that $-K_X * K_X^2 = 27\, q\, \mathbb{1}$ $\quad \deg q = -K_X[\text{line}] = 3$

where $27 = (-K_X * K_X^2, pt)$
$= \langle -K_X, K_X^2, pt\rangle_{0,3,[\text{line}]} \underset{\substack{\uparrow \\ \text{div axiom}}}{=} 3\langle K_X^2, pt\rangle_{0,2,[\text{line}]} = 27$

so $\nabla^{\mathbb{P}^2} = \underset{``t\frac{d}{dt}"}{D} - \begin{pmatrix} 0 & 0 & 27t^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

sol of $\nabla \iff (S_0(t), S_1(t), S_2(t))$ s.t $D(S_0,S_1,S_2) = (S_0,S_1,S_2)\begin{pmatrix} 0 & 0 & 27t^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$\Rightarrow S_0$ satisfies an ODE: $\quad DS_0 = S_1$, $\quad DS_1 = S_2$, $\quad DS_2 = 27t^3 S_0$

ie $(D^3 - 27t^3)S_0 = 0$

$J = 1 +$ high order terms also satisfies it

$$\Rightarrow J_{\mathbb{P}^2}(t) = \sum_{m=0}^{\infty} \frac{t^{3m}}{(m!)^3}$$

2.1 Toric varieties

Fix a lattice $\mathbb{L} \simeq \mathbb{Z}^b$, $\mathbb{T}_{\mathbb{L}} = \operatorname{Spec} \mathbb{C}[\mathbb{L}^*]$

$X =$ " $\mathbb{C}^r / \mathbb{T}_{\mathbb{L}}$ " Q: why not try to develop everything from the geom quot. pt of view G.I.T quotient.

where $\mathbb{T}_{\mathbb{L}} \xrightarrow{} \mathbb{C}^{\times r} \curvearrowright \mathbb{C}^r$ diagonal

dual to $\mathbb{L}^* \xleftarrow{D} \mathbb{Z}^r$ given by $D_1, \dots, D_r \in \mathbb{L}^*$, $D_i = D(e_i)$

Always assume the cone $\mathrm{Eff} = \sum \mathbb{R}_{\geq 0} D_i \subset \mathbb{L}^*_{\mathbb{R}}$ is strict (i.e. no subspace), $\Leftrightarrow$ X projective

X depends on the choice of a stability condition

$\chi \in \mathrm{Eff}$

There is a "wall and chamber" decomposition of Eff, called the "secondary fan"

$\chi \in$ maximal chamber $\Leftrightarrow$ X "simplicial"

notation $\langle D_{i_1}, \dots, D_{i_k} \rangle = \sum_{j=1}^{k} \mathbb{R}_{\geq 0} D_{i_j} \subset \mathrm{Eff}$

Walls = union of the all the $\langle D_{i_1}, \dots, D_{i_k} \rangle$ of codim 1 $\subset \mathbb{L}^*_{\mathbb{R}}$

maximal chambers := connected component of Eff of the complement of walls union of all:

Assume $\chi \in$ max chamber

irrelevant ideal:

$$I_\chi = \left(x_{i_1} \cdots x_{i_k} \mid \chi \in \langle D_{i_1}, \dots, D_{i_k} \rangle \right) \subset \mathbb{C}[x_1, \dots, x_r]$$

$$X_\chi = \left(\mathbb{C}^r \setminus V(I_\chi) \right) / \mathbb{T}$$

In terms of G.I.T $\chi \longleftrightarrow \mathbb{T}_{\mathbb{L}}$-linearized (trivial) line bundle on $\mathbb{C}^r$

(χ defines a $\mathbb{T}_{\mathbb{L}}$-action on $\mathbb{C}^r \times \mathbb{C}$
" Spec $\mathbb{C}[\mathbb{L}^*]$

χ character : $\mathbb{T}_{\mathbb{L}} \to \mathbb{C}^\times$,
so $(x, v) \mapsto (\lambda(x), \chi(\lambda)^{\pm 1} v)$).

$V(I_\chi)$ = unstable locus ($\chi \in$ max chamber $\Leftrightarrow$ s.s $\equiv$ s so everything works)

things work out st.

$\mathbb{L}^* = \mathbb{T}_{\mathbb{L}}$-linearized line bundles on $\mathbb{C}^r$ $= \mathrm{Pic}(X_\chi) = H^2(X_\chi, \mathbb{Z})$ & Amp X_χ = chamber contains χ.

However true for "nice" chambers

$-K_X = \sum D_i$ so if we want to make a toric Fano variety, then choose X in the same chamber where $-K$ is

2.2 Quantum Lefschetz (Coates - Givental)

Thm 1 X toric Fano mfd (may work for simplicial case but still quite unclear even in H.H. Tseng's thesis)

$$G_X(t) = \sum_{k \in \mathbb{L} \cap (\text{chamber})} t^{-K k} \frac{1}{(D_1 \cdot k)! \cdots (D_r \cdot k)!}$$

(due to Givental)

Thm 2 (Quantum Lefschetz)

$\mathbb{F}$ toric Fano mfd, $L_1, \dots, L_c$ nef line bdle on $\mathbb{F}$

$A = (-K_F + \sum L_i) \in (\text{chamber})$, set $X = \text{c.i.} \subset \mathbb{F}$ as $s_i = 0$ (general sections in L_i)

$$F_X(t) = \sum_{k \in \mathbb{L} \cap (\text{chamber})^*} t^{A k} \frac{(L_1 \cdot k)! \cdots (L_c \cdot k)!}{(D_1 \cdot k)! \cdots (D_r \cdot k)}$$

write $F_X(t) = 1 + a_1 t + \cdots$

$\Rightarrow G_X(t) = \exp(-a_1 t) F_X(t)$

Rmk: May relax to $-K_{\mathbb{F}}$ nef, A nef, but the statement becomes seriously more complicated

Example: $b_2 = 3$ (No 10 in Mori-Mukai's list)

X = <u>Blow up</u> of $Q^3 \subset \mathbb{P}^4$ with center = disjoint union of 2 conics

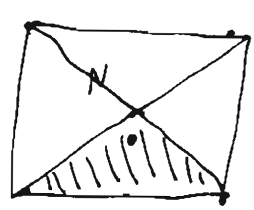
but this is NOT the ideal way to present it in order to calculate QH

consider $\mathbb{L}^* = \mathbb{Z}^3$, $D : \mathbb{Z}^7 \to \mathbb{Z}^3$

s_0	s_1	t_2	t_3	x	y	x_4	
1	1	0	0	-1	0	0	L
0	0	1	1	0	-1	0	M
0	0	0	0	1	1	1	N

consider the toric variety $\mathbb{F}$ with ample $\mathbb{F} = \langle L, M, N \rangle$

picture of the 2nd fan (in 3 dim space) just draw slice.



$-K_{\mathbb{F}} = 2L + 2M + (N-L) + (N-M) + N$
$= L + M + 3N$ $\Rightarrow$ $\mathbb{F}$ is Fano <u>mfd</u> (check)

$X \in |2N|$ on $\mathbb{F}$ (4 fold, $7-3$)

$-K_X = (-K_{\mathbb{F}} - X) = L + M + N|_X \in \text{Amp}\,\mathbb{F}$

to match with Mori-Mukai, consider $\mathbb{F} \to \mathbb{P}^4$ $(s_0 x, s_1 x, t_2 y, t_3 y, x_4) \leftarrow (x_0, x_1, x_2, x_3, x_4)$

Thm 2 $\Rightarrow$ $$G_X(t) = e^{-2t} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=\max(\ell,m)}^{\infty} \frac{t^{\ell+m+n}\,(2n)!}{(\ell!)^2 (m!)^2 (n-\ell)!\,(n-m)!\,n!}$$

Generalization of thm 1 & 2 to

G, 2 repr A, V, $\rho : G \to GL(V)$

$\mathbb{F} = A /\!\!/ G$

$X = (s=0) \subset \mathbb{F}$, $s \in \Gamma(\mathbb{F}, \mathbb{F} \times_\rho V)$

Abelian-nonabelian correspondence

(established for G of type A)

Corti lecture 3. 4/17

[ACGK] : Minkowski poly & mutations

§1 $^\pi f(t)$ satisfies P-F eq^n L_f (general theory OK GKZ)

§2 Minkowski poly

§3 mutations, main result for 3-folds

§4. the extremal properties for MP's in 3 variables

Minkowski poly (idea: vast generalization of Batyrev's constr. of MS)

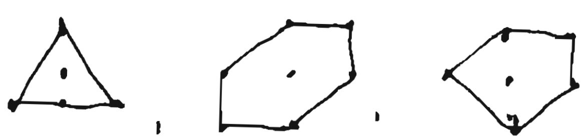
Def^n A lattice polytope P is reflexive if

(1) $0 \in \operatorname{Int} P$, and $\supset \mathbb{R}^n$

(2) $P^* = \{ f \in \mathbb{R}^{n*} \mid \langle f, v \rangle \geq -1 \ \forall v \in P \} \subset \mathbb{R}^{n*}$ is also a lattice polytope.

Goal: To define a class of Laurent poly f with Newt(f) reflexive (in ≤ 3 variables)

Rmk $\exists$ 16 lattice polygons



etc

$\exists$ 4319 3d reflexive polytopes

4d are also classified, $\exists \geq 473$ million reflexive poly (Kreuzer - Skarke). Need 3 months to write the hard disk

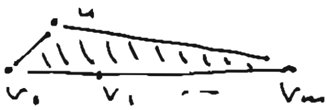
$P \subset \mathbb{R}^n$ lattice polytope (possibly degenerate)

$\Rightarrow P \cap \mathbb{Z}^n$ generates an affine lattice whose underlying lattice denoted by $L(P)$

Def^n A lattice Minkowski decomposition (LMD) of a lattice polytope Q is a Minkowski sum decomp $Q = \sum Q_i$ where Q_i are lattice polytopes & $L(Q) = \sum L(Q_i)$

Let $Q \subset \mathbb{R}^n$ be a lattice polytope, $n=3$

- Q is a segment of length 1 if $Q \cap \mathbb{Z}^3 = \{v, w\}$
  set $f_Q = x^v + x^w \in \mathbb{C}[\mathbb{Z}^n]$.
- Q is an A_m triangle if $Q \cap \mathbb{Z}^3 = \{u, v_0, \dots, v_m\}$ & $v_0, \dots, v_m$ are consecutive lattice element on segment $\overline{v_0 v_m}$; ie
  set $f_Q = x^u + \sum \binom{m}{k} x^{v_k} \in \mathbb{C}[\mathbb{Z}^n]$



Def^n: Let $Q \subset \mathbb{R}^3$ be a 2-dim lattice polygon
A LMD $Q = \sum Q_i$ is admissible if each Q_i is either a segment of length 1, or an A_m triangle
set $f_{Q, \{Q_i\}} = \prod_i f_{Q_i}$.

Def^n. Let $P \subset \mathbb{R}^3$ be a reflexive polytope,
$f \in \mathbb{C}[\mathbb{Z}^3]$ is a Minkowski polynomial supported on P
if $f = \sum_{v \in P \cap \mathbb{Z}^3} a_v x^v$ with

① $a_0 = 0$
② if $F \subset P$ is a facet, then
$f_F = \sum_{r \in F \cap \mathbb{Z}^3} a_r x^r = f_{F, \{F_i\}}$ for some admissible LMD : $F = \sum F_i$

Note : Newt $f = P$

Facts
1294 of the 3d reflexives support NO Minkowski poly's
The remaining 3025 support exactly 3747 distinct MPs
(since some facet can have >1 LMD)

Thm: These generate only 165 period sequences (classical period fns)

of these, 98 are of manifold type
($\Leftrightarrow$ monodromy is unipotent around 0)
& they are in 1-1 correspondence with Fano 3 folds with 98 families with $-K_X$ very ample

"Explanation" (Vaguely) via toric degeneration

← general Fano with $-K_X$ v.a

↑ Gorenstein toric Fano

proves very simplest case are done by Batyrev

3 Mutations

Lemma: Suppose $f(x,y,z) = z^{-1}a(x,y) + b(x,y) + z\,c(x,y)$, a, b, c Laurent poly. Consider the transformation

$$x \mapsto x, \quad y \mapsto y, \quad z \mapsto h(x,y)\,z, \quad \text{where } h \mid a$$

then $f \mapsto z^{-1}\frac{a}{h} + b + zhc := g$ & $\pi_g(t) = \pi_f(t)$.

pf: purely change of variable formula.

key point: $\varphi: \mathbb{C}^{\times 3} \circlearrowright$ is vol preserving

that is, $\varphi^{-1}\dfrac{dx\,dy\,dz}{xyz} = \dfrac{dx\,dy\,dz}{xyz}$.

Striking!! there is no conceptual pf yet in fact 3 mutations are enough!

We proved (by computer):

If f, g are Minkowski poly, then $\pi_f = \pi_g \iff$ f & g differs by a chain of these kind of transformations (& possibly change coord in torus, i.e. $SL_3(\mathbb{Z})$))

What is going on: for each of these 165 Minkowski periods, there is a unique "generalized cluster variety"

$f: Y \to \mathbb{C}$

different LP with same periods are f in different patches on Y.

4. Extremal properties:

A local system $\mathbb{V}$ on $\mathbb{P}^1 \setminus S$ ($\overset{j}{\hookrightarrow} \mathbb{P}^1$) is extremal if $\mathbb{V}$ is irreducible, nontrivial, & $\chi(\mathbb{P}^1, j_*\mathbb{V}) = -h^1(j_*\mathbb{V}) = 0$. (i.e. no invariants)

this is always true since & PD

Euler formula

$$-\chi(\mathbb{P}^1, j_*\mathbb{V}) = \underbrace{\sum_{s \in S} \dim\left(\mathbb{V}_\eta / \mathbb{V}_\eta^{T_s}\right)}_{\substack{\shortparallel \\ \text{ramification } \mathrm{rf}(\mathbb{V})}} - 2\mathrm{rk}(\mathbb{V})$$

monodromy around all S

So for $\mathbb{V}$ non-trivial, irreducible, $\mathrm{rf}(\mathbb{V}) \geq 2\mathrm{rk}(\mathbb{V})$ and "=" $\Leftrightarrow$ $\mathbb{V}$ extremal.

If f is a MP, then $\mathbb{V} = \underline{\mathrm{Sol}}\, L_f$ is extremal

Rmk: for 2d, see a book of Beauville, only 4 singular pts case there are 6 such examples (unipotent monodromy)

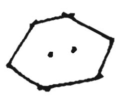
the 3d analogue is the extremal case

Do we have the complete list of extremal Laurent poly?

Can ask a lot of question in this fashion, eg

$K3 \longrightarrow \mathbb{P}^1$

<u>Rmk</u>: for orbifold surface, eg $\frac{1}{3}(1,1)$



then should use lattice polytope with more interior int points!

<u>orbifold</u> QH

Minkowski poly

same cat

For del Pezzo surface (orbifold), this may be possible.

For 3 fold, even orbifold QH ... cases not be able to computed at current technology!

<u>End</u>