Calabi-Yau and G_2-geometry

NCTS. 12/26

梁琲聰

§1. Holonomy

$M, g \rightsquigarrow \nabla g = 0$ Levi-Civita on TM

cpt. $\quad \mathrm{Tor}\,\nabla = 0$

Holonomy $\mathrm{Hol}(g) \subseteq U(n) \leftrightarrow$ Kähler manifold

i.e. J cpx str, $\nabla J = 0$

$\omega(u,v) := g(Ju, v)$, so $\nabla\omega = 0$ ($\Rightarrow d\omega = 0$).

Next stage: $\mathrm{Hol}(g) \subseteq SU(n)$

i.e. preserves holo. vol. form $\nabla\Omega = 0$

$|\Omega|^2 = 1$, $\Omega \overset{\text{loc.}}{=} f(z)\, dz^1 \wedge \cdots \wedge dz^n \in \Omega^{n,0}(M)$

$\bar\partial f(z) = 0$

These are Calabi-Yau manifolds.

Next stage: $\mathrm{Hol}(g) \subseteq Sp(n)$ Hyperkähler mfds

I, J, K cpx str. $\nabla I = \nabla J = \nabla K = 0$

$I^2 = J^2 = K^2 = IJK = -\mathrm{id}$ on TM.

RmK: One more slightly general

$\mathrm{Hol}(g) \subseteq Sp(n)Sp(1)$, Quaternic Kähler (but not Kähler at all).

Next stage: Caley: $\mathrm{Hol}(g) \subset G_2$, M^7

And another one M^8, $\mathrm{Hol}(g) \subset Spin(7)$

Together with $O(n)$, $SO(n)$, the above consist of the complete list of Berger's list.

Today's viewpoint, let $Hol(g) = G =$

$SO(n)$	$SO(4)$	$U(n)$	$SU(n)$	G_2	$Sp(n)$
M^n	M^4	M^{2n}	M^{2n} C-Y	M^7 G_2	M^{4n} H.K.
			1. All Einstein 2. $\pi_1(M)$ finite 3. Moduli space is finite dim'l and smooth for each comp. (BTT, Joyce)		

Yu's observation: Simply connectedness of G affects geometry of M?

§2. Gauge Theory: (1st order)

$\mathbb{C}^r \to E \to (M, g)$, D_A connection

$D_A(*F_A) = 0$ (2nd order PDE)

this is E-L eq'n for $\int_M |F_A|^2$.

eg. M^4 $(SO(4))$: $*F_A = \pm F_A$ (anti-) self-dual

Donaldson theory

$\mathcal{M} := \{ *F_A = F_A \} / \text{symmetry}$.

Donaldson, later Uhlenbeck-Yau: this solution space can be given by alg-geom in Kähler case.

M^{2n} $(U(n))$, $\Omega^2(M, \mathbb{C}) = \Omega^{2,0} + \Omega^{1,1} + \Omega^{0,2}$

$$\begin{cases} F_A \wedge \dfrac{\omega^{n-1}}{(n-1)!} = \mu \dfrac{\omega^n}{n!} & \text{one eq'n (HYM)} \\ F_A^{0,2} = 0 & \text{many eq'n, corr to integrability} \end{cases}$$

i.e. $\longleftrightarrow$ E holo. str.

D, U-Y: HYM eq'n $\Leftrightarrow$ Mumford stable bundles

Rmk: for M^4 ($SU(2)$), $*F = -F \Leftrightarrow$ HYM
possible to describe $\mathcal{M}$.

How about M^n ($SO(n)$), $n \neq 4$?

G does not reduce, can only ask for F_A reduce:

$F_A = 0$, $\mathcal{M} = \{\text{flat connections}\}/\sim$

$= \mathrm{Hom}(\pi_1(M), U(r)) /$ conjugation

Potentially useful for

i) M^3 (Casson invariants etc.)

ii) M^{2n} ($U(n)$): Non-abelian Hodge theory,
means $\mathbb{C}^\times$-action on $\mathrm{Hom}(\pi_1(M), GL(n, \mathbb{C}))$.
from Kähler condition by
Bochner principle.

§3. Cycles:

CY mfds, M^{2n}, $SU(n)$,

g, J, $\omega \in \Omega^2$, $\mathrm{Re}\,\Omega \in \Omega^n$, $\mathrm{Im}\,\Omega = *\mathrm{Re}\,\Omega \in \Omega^n$.

notice: $\mathrm{Re}\,\Omega$ completely determines J

ω determines the symplectic str.

complex (B-) geometry	Symplectic (A-) geometry
$C \hookrightarrow M$ cpx submanifolds	$L^n \subset M^{2n}$ Lagrangian submfd i.e. $\omega\|_L = 0$
Gauge Theory: $\begin{cases} F_A \wedge \omega^{n-1} = \mu \\ F_A^{0,2} = 0 \end{cases}$	$\begin{array}{c} \mathbb{C}^r \to E \\ \downarrow \\ L^n \end{array}$ s.t. $F_A = 0$. (the only non-trivial Gauge theory)

Hyperkähler case M^{4n}, g $\quad \omega_I = \omega, \ \omega_J, \ \omega_K$

$\Omega := \omega_J + i\,\omega_K \in \Omega^{2,0}$ (fix a cpx str I)

holomorphically symplectic (i.e. $T \xrightarrow{\lrcorner\Omega} T^*$)

cycles are: $C^r \longrightarrow E$ + HYM

$\downarrow$

cpx Lagr: $L \hookrightarrow M$.

Summary:	$C^n: SO(n)$,	$C^4: SO(4)$,	$C^{2n}: U(n)$
look at Gauge	$F_A = 0$	$*F_A = \pm F_A$	$F_A^{0,2} = 0$, $F_A \wedge \omega^{n-1} = c$
	$\cap$ $\omega\vert_C = 0$, $\mathrm{Im}\,\Omega\vert_C = 0$	$\cap$ $\Omega\vert_C = 0$	$\cap$
	M^{2n}: CY $(SU(n))$,	$M^7: G_2$,	M^{4n}: HK $(Sp(n))$

These are called supersymmetric cycles.

§4. G_2-manifold:

M^7, let $\mathbb{R}^7$ = Im. part of Cayley:

$\times : \mathbb{R}^7 \times \mathbb{R}^7 \longrightarrow \mathbb{R}^7$

$\Omega(u, v, w) := \langle u \times v, w \rangle$ skew-symmetric

notice G_2 = automorphism of Cayley

M^7, $\Omega \in \Omega^3_+(M, \mathbb{R})$

$\nabla \Omega = 0$ (suffices $d\Omega = 0 = d(*\Omega)$ i.e. harmonic.)

At a point, $\Omega = dx^{123} - dx^1(dy^{23} + dy^{14}) - dx^2(dy^{31} + dy^{24}) - dx^3(dy^{12} + dy^{34})$.

means "positive".

reason: $u, v \in T_M$

$g(u,v) := \iota_u \Omega \wedge \iota_v \Omega \wedge \Omega \, / \pm \text{ori}$

if g non-degenerate (generic)

then g must has signature $(7,0)$ or $(4,3)$.

there are also inv. theoretic point of view.

Ω parallel, in particular, $\Theta := * \Omega$ also parallel

cycles in a G_2-manifold: there are 2 types

- C-cycle (co-associative):

(C, D_E), $C^4 \hookrightarrow M^7$ st. calibrated, i.e.

$\Theta|_C = dv_C$ ($\Rightarrow C \hookrightarrow M$ absolute min. sub-mfd)

and D_E: ASD connections on C.

$\mathcal{M} = \{C\text{-cycles}\}$.

- $T\mathcal{M}$ at (C, D_E) is parametrized by

for D_E part: $H^1(C, \mathrm{ad} E)$

1st coh. of cpx

for C part: $0 \to \Omega^0(C, \mathrm{ad} E) \to \Omega^1(C, \mathrm{ad} E) \to \Omega^2_+(-) \to 0$

eg. $C = \{x^1 = x^2 = x^3 = 0\}$ in $\mathbb{R}^7$ is co-ass:

we find $* \Omega = dy^{1234} - 0$!

Fact: for C' any co-ass 4-plane in $\mathbb{R}^7$,

$\exists\ C' = g \cdot C$, $g \in G_2 \subset SO(7)$.

so $\{\text{co-asso planes}\} = G_2 / SO(4)$.

Cor: $C^4 \hookrightarrow M^7$, G_2-mfd, co-ass.

$\Rightarrow N_{C/M} \cong \Lambda^2_+(C)$.

the pf is just pointwise: $v \mapsto \iota_v \Omega|_C$.

Infinitesimal deformation $= H^2_+(C)$.

All together get $TM_{(C,D_E)} \cong H^2_+(C) \oplus H^1(C, \mathrm{ad}\,E)$.

M has a natural 3-form: Ω_M

for $\alpha, \beta, \gamma \in H^2_+(C)$

$\varphi, \eta, \rho \in H^1(C, \mathrm{ad}\,E)$

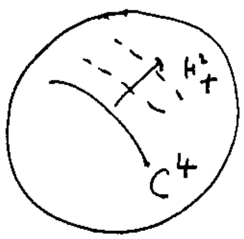
$$\Omega_M(\alpha,\beta,\gamma) = \int \langle [\alpha,\beta]_{H^2_+}, \gamma \rangle$$

$$\Omega_M(\alpha,\varphi,\eta) = \pm \int \mathrm{Tr}\, \alpha \wedge \varphi \wedge \eta$$

other components $= 0$.

Rmk 1: Relation to Donaldson theory:

$M \ni (C, D_E)$



M fiber = Donaldson moduli sp. $\downarrow$

$\{\text{Co-ass}\} \ni [C]$

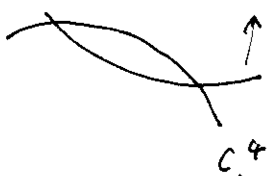
From Donaldson theory: expect M to have isolated reducible ASD connections.

PROBLEM: Count $(M)^{red}$.

Rmk 2: Relation to C-Y geometry: Joyce: counting SL. S^3_Q.

take $M = X^6 \times S^1$
$SU(3)$

PROBLEM:



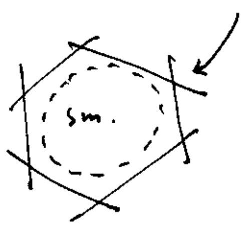
unobstructed (McLean)

take 2, intersect $\Rightarrow$

$\dim C_1^+ \cap C_2^+ = 1$ or 2

($\neq 4$, $C_1 = C_2$).

say 1 case, i.e. circle:



$b_2^+ \to b_2^+ + 1$.

to be continued.