CTS 2005 Summer School in Alg Geom

K-equiv in birat'l Geom Lect I. 8/8. by Chin-Lung Wang

"motivations from 3D MMP":

Singularities: A normal var X is terminal (Canonical) if K_X is $\mathbb{Q}$-Cartier (i.e. $K_{X_{reg}}^{\otimes r}$ ext to line bundle over X for some $r \in \mathbb{N}$) and for some resol of sing $\varphi : Y \to X$ has $K_Y = \varphi^* K_X + \underbrace{\sum a_i E_i}_{\text{all exc. div}}$, $a_i > 0$ (≥ 0)

Cone Theorem (Mori, Kawamata, Shokurov, ~1980's)

let X be ter. K_X not nef, then each extremal ray $R \in \overline{NE}_{K<0}(X) := (Z_1(X)/\underset{\substack{\uparrow \\ \text{numerical equiv}}}{\sim})_{K<0}$ is spanned by a rat'l curve,

$\exists\ \psi_R : X \xrightarrow[\text{proj}]{} \bar{X}$ st $\psi_R(C') = pt \iff [C'] \in R$.

3 possibilities on $\bar{X}$:

① fiber type: $\dim \bar{X} < \dim X$, $X \underset{\text{fiber space}}{\to} \bar{X}$ ok.

② ψ_R divisorial, i.e. $\dim \operatorname{Exc} \psi_R = n-1$ ok.

③ ψ_R small, i.e. $\dim \operatorname{Exc} \psi_R \leq n-2$. Bad!

$\bar{X}$ is not $\mathbb{Q}$-Gorenstein

otherwise $K_X = \psi_R^* K_{\bar{X}}$

If $R = \mathbb{R}_+[C]$, then can't proceed. $0 > K_X . C = \psi_R^* K_{\bar{X}} . C = K_{\bar{X}} . \psi_R(C) = 0$ $*$

Def'n: X is a minimal model if X ter & K_X nef.

Classification on Singularities:

Thm: $\dim X = 2$, then ter $\iff$ sm

can $\iff$ ADE $\iff$ $\mathbb{C}^2/G$, $G \subset SL(2,\mathbb{C})$ finite.

$\dim X = 3$. then ter $\iff$ cDV/μ_r

$cDV := f(x,y,z) + t\, g(x,y,z,t) = 0$ isolated.

sketch of pf for ADE eq'n : Let X be surface

Method 1: $$\begin{tikzcd}\hat{X} \arrow[d, "\pi"] \\ X\end{tikzcd}$$ min resolution (blow down all -1 curves)

$$K_{\tilde{X}} = \pi^* K_X + \sum a_i E_i \qquad a_i \in \mathbb{Q}_+$$

If $\pi \neq$ isom. claim $a_i \equiv 0$ $\forall i$, If not, then

$$K_{\tilde{X}} \cdot \sum a_i E_i = \left(\sum a_i E_i\right)^2 < 0 \quad \text{(Hodge index)}$$

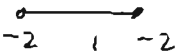
$$\Rightarrow \exists\ a_i > 0 \ \text{ st. } \ K_{\tilde{X}} \cdot E_i < 0 \ , \ \text{but } E_i^2 < 0$$

$$\Rightarrow \ 2g(E_i) - 2 = (K_{\tilde{X}} + E_i) \cdot E_i < 0$$

$$\Rightarrow \ g(E_i) = 0 \ , \ K_{\tilde{X}} \cdot E_i = -1 = E_i^2 \quad \times\!\!-\!\!-\!\!-$$

So $K_{\tilde{X}} = \pi^* K_X \Rightarrow E_i^2 = -2$, $E_i \simeq \mathbb{P}^1$ $\forall i$

number theory $\Rightarrow$ $(E_i \cdot E_j)_{i,j}$ is of type A-D-E

eg. A_2 

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}.$$

A-D-E eq'n :

A_n	$x_1^2 + x_2^2 + x_3^{n-1}$,	$n \geq 1$
D_n	$x_1^2 + x_2^2 x_3 + x_3^{n-1}$,	$n \geq 4$
E_6	$x_1^2 + x_2^3 + x_3^4$	
E_7	$x_1^2 + x_2^3 + x_2 x_3^3$	
E_8	$x_1^2 + x_2^3 + x_3^5$	

Exercise : Use explicit blow-up to get corr. int pairing.
(notice isolated cDV are always hyp surf of mult 2)

type	A_1	A_2	A_n $(n \geq 3)$	D_4	D_5	D_n $(n \geq 6)$	E_6	E_7	E_8
after blow-up	Sm	Sm	A_{n-2}	$3A_1$	$A_1 + A_3$	$A_1 + D_{n-2}$	A_5	D_6	E_7
# of exc curve	1	2	2	1	1	1	1	1	1

$\Rightarrow$ Rmk : gen hyp sect will preserv $\Big\langle$ ter / can

there can sing $\simeq \mathbb{A}^{n-2} \times$ (A-D-E) in codim 2

ter sing is codim 3.

but special hyp sect $H \ni p$ will get sing. worse !

3D flips / flops : let (K, D) log pair

D eff div /$\mathbb{Q}$. usually $d_i < 1$

$K_X + D$ (log)-flip is

$$\begin{tikzcd} X \arrow[rr, dashed, "f"] \arrow[dr, "\psi"'] & & X' \arrow[dl, "\psi'"] \\ & \bar{X} & \end{tikzcd}$$

$\psi : K_X + D$ extr contraction

i.e. $K_X + D$ ψ negative

require f isom in codim 1

$K_{X'} + D'$ is ψ' positive (ample)

$D' = f_*(D)$

- $D = 0$, f is a "flip"
- K_X is ψ-trivial , f is called a "flop"

Thm (3D MMP) (Mori 1988, Kollár-Mori 1992, Shokurov 2002)

(1) 3D log flips exists (even in families) and terminates (no ∞ sequence)

(2) Also 3D birat'l $\mathbb{Q}$-factorial minimal models are connected by sequence of flops.

Rmks : For Gorenstein terminal flops, locally

$$\begin{tikzcd} X \supset Z \arrow[dr] \arrow[d, "\psi"'] & \\ \bar{X} \ni pt = p \in U & \end{tikzcd}$$

$$\psi^{-1}(U) \setminus Z \cong U \setminus p \xrightarrow[t]{} U \setminus p \hookrightarrow \bar{X} \setminus p$$

via $x_1^2 + h(x_2, \cdots ; x_4) = 0$

$t(x_1, y_2, x_3, x_4) = (-x_1, x_2, x_3, x_4)$

X' is constructed via $f^{-1}(U) \cup_t (\bar{X} \setminus p)$.

in particular, the singularity type is not changed !

Summary of 3D MMP :

∞ : The MMP ends up with $\mathbb{Q}$-fact min model

3 : min models not unique, but all related by flops and flops are classified .

2 : $\mathrm{Def}(X) \simeq \mathrm{Def}(X')$ canonically .

1 : $H^*(X) \simeq H^*(X')$ additively .

0 : X' has the same singular type as X .

HD ? ∞ is ∞'ly hard . but 1, 2, 3. do not use it !

ring str in 1 is changed . 0 is wrong !

Idea : Use K-equivalence to replace minimal condition .

For normal $\mathbb{Q}$-Gorenstein, $X \le_K X'$ if $\varphi^* K_X \le \varphi'^* K_{X'}$

<u>Prop: (W—, 1998 JDG)</u>

let $f : X \dashrightarrow X'$ birat'l, X, X' can. sing

assume $Z \hookrightarrow X$ proper & K_X is nef along Z,

then $X \le_K X'$. If X is ter then $\operatorname{codim}_X Z \geq 2$.

<u>pf:</u> let

$$\begin{tikzcd} & Y \arrow[dl, "\varphi"'] \arrow[dr, "\varphi'"] & \\ X & & X' \end{tikzcd}$$

be a good resol of $\overline{\Gamma}_f \hookrightarrow X \times X'$

let $K_Y = \varphi^* K_X + E = \varphi'^* K_{X'} + E'$

So. $\varphi'^* K_{X'} = \varphi^* K_X + F \qquad (F := E - E')$

Will show $F_j \geq 0$. let $F = \sum_{j=0}^{n-1} F_j$, $\dim \varphi'(F_j) = j$

for $j = n-1, n-2, \cdots, 1, 0$ inductively.

Since E' is φ'-exceptional $F_{n-1} \geq 0$ (from E) is clear.

Suppose $F_j \geq 0$ $\forall j \geq k+1$, will do the case $j = k$:

consider surface $S_k := H^{n-2-k} \cdot \varphi'^* L^k$ on Y

H very ample on Y

L very ample on X'

$$\Rightarrow \varphi'^* K_{X'}|_{S_k} = \varphi^* K_X|_{S_k} + \underbrace{F \cdot H^{n-2-k} \cdot \varphi'^* L^k}_{= a - b}$$ cf. decomp of curves

b must comes from F_k since

$\sum_{j \geq k+1} F_j \geq 0$ & $L^k \cap \varphi'(F_j) = \phi$ for $j < k$

look at $\underset{\underset{0}{\parallel}}{b \cdot \varphi'^* K_{X'}} = b\,\varphi^* K_X + b \cdot a - b^2$

since $\varphi'(b) \subset L^k \cap \varphi'(F_k)$ is zero dim'l.

Since $\varphi'^* K_{X'} = \varphi^* K_X$ on $\varphi^{-1}(X \setminus Z)$, $\Rightarrow \varphi(\operatorname{Supp} F) \subset Z$.

so $b \cdot \varphi^* K_X \geq 0$

Also, $b \cdot a \geq 0$.

If $b \neq 0$, then b = combination of $\varphi'|_{S_k}$ exc. curves

$\Rightarrow b^2 < 0$ (Hodge index thm) $\nleftarrow$ $\square$

Inversion of Adjunction (Special case)

lemma: The total space of a one parameter smoothing of Gorenstein canonical sing has at most Gorenstein terminal sing.

pf:

$$\begin{tikzcd}
\mathfrak{X}' \arrow[dr, "f"] & & \\
& \mathfrak{X} \arrow[d] & \supset X \arrow[d] \\
& \Delta & \ni 0
\end{tikzcd}$$

comm. algebra $\Rightarrow \mathfrak{X}$ Gorenstein

f : log-resolution of $(\mathfrak{X}, X)$

i.e. $f^{-1}(X) = X' \cup (\cup_i E_i)$ normal crossing div in $\mathfrak{X}'$

$g := f|_{X'} : X' \to X$ resol of sing of X

$$K_{\mathfrak{X}'} = f^* K_{\mathfrak{X}} + \sum a_i E_i, \quad a_i \in \mathbb{Z}$$

$$f^* X = X' + \sum l_i E_i, \quad l_i \in \mathbb{N}$$

since $\mathfrak{X}_{sing} \subset X_{sing}$

$$\Rightarrow \quad K_{\mathfrak{X}'} + X' = f^*(K_{\mathfrak{X}} + X) + \sum (a_i - l_i) E_i =: A - F$$

Adjunction $\quad K_{X'} = g^* K_X + \underset{(a_i - l_i) \geq 0}{A|_{X'}} - \underset{(a_i - l_i) \leq -1}{F|_{X'}}$

Assumption $\Rightarrow F|_{X'} = 0$

claim: $F \neq 0 \Rightarrow F \cap X' \neq \phi$

Hence in fact $F \equiv 0$, so $a_i \geq l_i \geq 1 \; \forall i$. done.

pf of claim: $F + X'$ is connected in any nbd of a fiber of f:

$$A - (F + X') = K_{\mathfrak{X}'} + (\underbrace{-f^*(K_{\mathfrak{X}} + X)}_{f\text{-big \& } f\text{-nef}})$$

$$\Rightarrow \quad 0 \to \mathcal{O}_{\mathfrak{X}'}(A - (F + X')) \to \mathcal{O}_{\mathfrak{X}'}(A) \to \mathcal{O}_{F+X'}(A) \to \underset{\substack{\parallel \\ 0 \\ \text{K-V thm}}}{R^1 f_* \mathcal{O}_{\mathfrak{X}'}(A - (F + X'))}$$

$\Rightarrow \quad F + X'|_{f^{-1}(x)}$ is connected $\forall x \in X$.

Filling-in Problem in Dim 3

$$\begin{tikzcd} \overset{\text{projective smoothing}}{\mathfrak{X}} \arrow[d] & \supset & \overset{\text{nontrivial Gorenstein min 3-fold}}{\mathfrak{X}_0} \arrow[d] & \text{ie. terminal} \\ \Delta & \ni & 0 & \end{tikzcd}$$

Theorem (—, 1997 MRL). Let [diagram above]

then up to any finite base change of Δ,

$$\begin{tikzcd} \not\exists \quad \mathfrak{X} \arrow[rr, dashed, "f"] \arrow[dr] & & \mathfrak{X}' \arrow[dl] & \text{proj sm family of min 3-folds.} \\ & \Delta & & \end{tikzcd}$$

pf: lemma $\Rightarrow$ $\mathfrak{X}$ terminal. If $\exists$ diagram

then $\mathfrak{X} =_K \mathfrak{X}'$, isom in codim 1

$\Rightarrow$ $f_0 : \underset{\text{min}}{\mathfrak{X}_0} \dashrightarrow \underset{\text{min sm}}{\mathfrak{X}_0'}$ birational

Kollár $\Rightarrow$ $\mathfrak{X}_0$ not $\mathbb{Q}$-factorial (since $\mathfrak{X}_0$ is singular)

MMP $\Rightarrow$ $\exists$ Y $\mathbb{Q}$-fact, min, $\sim \mathfrak{X}_0'$ hence Y sm

$$\begin{tikzcd} Y \arrow[d, "\phi"'] \\ \mathfrak{X}_0 \end{tikzcd}$$

small

otherwise $K_Y = \phi^* K_{\mathfrak{X}_0} + \underset{\neq 0}{E}$

not nef

in particular

$H^k(Y) \simeq H^k(\mathfrak{X}_0') \simeq H^k(\mathfrak{X}_t') \simeq H^k(\mathfrak{X}_t)$ for $t \neq 0$

$\longrightarrow$ ~~extremal~~ transition

$$\begin{tikzcd} Y \arrow[d, "\phi"'] & & \\ \mathfrak{X}_0 \arrow[r, hook] & \mathfrak{X} & \mathfrak{X}_t \arrow[l, hook'] \end{tikzcd}$$

- If $\mathfrak{X}_0$ has only ODP's (A_1-sing) p_i, let $\phi(C_i) = p_i$

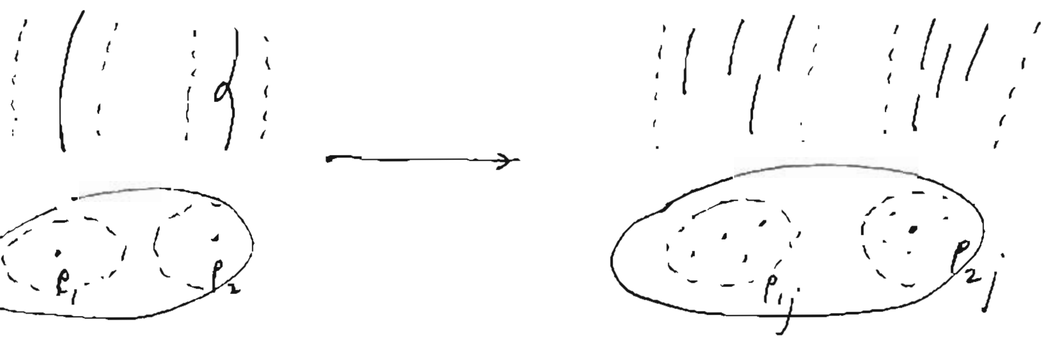
$e : \bigoplus_i \mathbb{Z}[C_i] \longrightarrow H_2(Y, \mathbb{Z})$, then $H_2(\mathfrak{X}_t) \cong \operatorname{coker}(e)$

but $H_2(Y) \simeq H_2(\mathfrak{X}_t)$ $\Rightarrow$ $\operatorname{Im}(e) = 0$ $\ast$ since Y is proj.

- In general, $\mathfrak{X}_0$ has isolated cDV pts

Friedman : If (p, V) germ of isolated cDV
$(C, U) \xrightarrow{\phi} (p,V)$ exc. curve contracted to p
then $\mathrm{Def}(C,U) \hookrightarrow \mathrm{Def}(p,V)$ sm pair of cpx spaces
for generic element in $\mathrm{Def}(C,U)$
get $(\cup \mathbb{P}^1, \tilde{U})$, $N_{\mathbb{P}^1/\tilde{U}} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$
hence the contraction deforms into $\hat{\phi}$,
$\hat{\phi}$ contracts each $\mathbb{P}^1$ down to an odp in $\tilde{V}$.

Perform this analytic process $\forall\ C_i \to p_i$ in a nbd



and patch them together in the C^∞ category
(~~locally holomorphic symplectic deformations~~).
The pf is reduced to the odp case. (The Mayer-Vietoris argument depends only on C^∞ category)

Filling-in Prob :

$$\begin{tikzcd} \mathcal{X}^\times \arrow[d] & C^\infty \text{ trivial} & \overset{?}{\Rightarrow} & \exists\ \mathcal{X} \arrow[d] & \text{sm} \\ \Delta^\times & \text{proj} & & \Delta & \text{proj} \end{tikzcd}$$

- If $\pi_1(\mathcal{X}_t) \neq \{1\}$, easy to see no.
- If $\pi_1(\mathcal{X}_t) = \{1\}$, surfaces no (Morgan ~80's)
  also $\dim X \geq 4$.
- For $\dim X = 3$, Clemens constructed examples as A_2 degeneration of quintic C-Y in $\mathbb{P}^4$.

Lecture II : Complex Elliptic Genera & K-equiv.

ϕ genus (cpx genus) with value in R

$$P_\phi(y) := \sum_{n=0}^{\infty} \phi(\mathbb{P}^n(\mathbb{C}))\, y^n$$

Want universal polynomial , let $\deg c_i = i$, $c = 1 + c_1 + c_2 + \cdots$

$K(c) = K(c_1, c_2, \cdots)$ with formal chern classes

$\quad = 1 + K_1 + K_2 + \cdots$ as variables , $\deg K_n = n$

st. $K(c(M))[M] = \phi(M)$

(if $\dim_{\mathbb{C}} M = n$, then use terms of degree n

$K_n(c_1(M), \cdots, c_n(M))[M]$)

How to get K from P_ϕ ?

Hirzebruch's procedure :

Given any $Q(x) = 1 + q_1 x + q_2 x^2 \cdots \in R[\![x]\!]$

R any arbitrary comm. ring

form $\prod_{i=1}^{n} Q(x_i) = 1 + q_1 \sum x_i + \left(q_2 \sum x_i^2 + 2 q_1 \sum_{i<j} x_i x_j\right)$

$+ \cdots$

$=: 1 + K_1(c_1) + K_2(c_1, c_2) + \cdots$

$+ K_n(c_1, \cdots, c_n) + K_{n+1}(c_1, \cdots, c_n, 0) + \cdots$

(in order to get K_N , need to use at least N products)

here c_i = i th sym. function of x_i .

since K_r is indep. of n if $n \geqslant r$, can

set $\boxed{K(c) := \prod_{i=1}^{\infty} Q(x_i)}$

• Fact : K is multiplicative . ie.

$K(c'c'') = K(c') \cdot K(c'')$

multiply and collecting terms

get $c = 1 + c_1 + c_2 + \cdots$

ie. $\phi_Q(M) := K(c(M))[M]$ is an R-genus

Notice, by definition of K,

$$K(1+x) = \prod_{\underset{\text{one product}}{\underset{\uparrow}{i=1}}}^{1} Q(x_i) = Q(x) \quad , \text{ so get}$$

Thm (Hirzebruch):

For any Q, $\exists!$ multi. sequence K st. $K(1+x) = Q(x)$.

(for the uniqueness, notice that if $c = \prod (1+x_i)$ then K is multiplicative $\Rightarrow$

$$K(c) = \prod_i K(1+x_i) = \prod_i Q(x_i)$$

this is exactly the definition of K and we think of x_i as the "chern roots".)

Thm (Hirzebruch):

Let $f(x) = x/Q(x) = x + \cdots \in R[\![x]\!]$

and $g(y) := f^{-1}(y) \in R[\![y]\!]$

then the generating function $P_{\phi_Q}(y) \equiv g'(y)$.

ie. $$g'(y) = \sum_{n=0}^{\infty} \phi_Q(\mathbb{P}^n(\mathbb{C}))\, y^n$$

- CONCLUSION :

Given an R-genus ϕ

form the generating function P_ϕ

set $g = \int P_\phi$ with $g(0) = 0$

set $f = g^{-1}$ (inverse function)

then $Q(x) := \dfrac{x}{f(x)}$ is the fund. power series

then $K = \prod_{i=1}^{\infty} Q(x_i)$ is the univ. mult. sequence

$$= 1 + K_1(c_1) + K_2(c_1, c_2) + \cdots$$

$\Omega_* := MU_* \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{P}^1, \mathbb{P}^2, \dots]$ cpx bordism ring

$Ell_* := \mathbb{Q}[\underline{\underline{x_1}}, \underline{\underline{x_2}}, \underline{\underline{x_3}}, \underline{\underline{x_4}}]$ $\deg x_i = i$

There are: $\Omega_* \longrightarrow \mathbb{Q}[x_2, x_4]$ elliptic genus (on any oriented mfd)

$\mathcal{E} : \Omega_* \longrightarrow \mathbb{Q}[x_1, x_2, x_3, x_4]$ cpx. ell. genus on weakly cpx mfd.

$\chi_y : \Omega_* \longrightarrow \mathbb{Q}[x_1, x_2]$ Hirzebruch's χ_y genus

$\chi_{yz} : \Omega_* \longrightarrow \mathbb{Q}[x_1, x_2, x_3, x_4]/(\Delta(x_2, x_3, x_4))$ Höhn's twisted χ_y genus.

with: $x_1 \longleftrightarrow \mathbb{P}^1$

$$* \begin{cases} x_2 = 24\wp \longleftrightarrow K3 \\ x_3 = \wp' \longleftrightarrow \text{Almost cpx } S^6 \\ x_4 = 6\wp^2 - \dfrac{g_2}{2} \end{cases}$$

: g_2 = Eisenstein series of wt 4.

$\longleftrightarrow$ a quadric 4-fold with non standard weakly cpx str.

$\chi_y([X]) := t^n \chi(X, \Lambda_y(T_X^*)) : MU_* \to \mathbb{Z}[t, y]$

$\chi_{yz}([X]) := t^n \chi(X, K_X^{-z} \otimes \Lambda_y(T_X^*)) : MU_* \to \mathbb{Z}[t, y, z]$

$\deg t = 1$, $\deg y = 0 = \deg z$

there are just specialization of the ~~universal one~~ (cpx ellipt. genus)

$$\begin{tikzcd}
MU_* \arrow[r, "\mathcal{E}"] \arrow[dr, "\chi_{yz}"'] & \mathbb{Q}[\overset{\mathbb{P}^1}{\underline{x_1}}, \overbrace{\underline{x_2, x_3, x_4}}^{*}] \arrow[r, hook, "\iota"] \arrow[d] & \mathbb{Q}((y))[[q, k]] \\
& \mathbb{Q}[t, y, z] &
\end{tikzcd}$$

$x_1 \longmapsto \underline{t}(1 - y + 2\underline{\underline{z}})$

$x_2 \longmapsto \underline{t}^2(2 - 20y + 2y^2)$

$x_3 \longmapsto \underline{t}^3(-y + y^2)$

$x_4 \longmapsto \underline{t}^4(-y + 4y^2 - y^3)$

- Indeed, $\varphi := \iota \circ \mathcal{E}$ is given by

$$\varphi([X]) = \chi\Big(X, K_X^{\otimes(-k)} \otimes \prod_{m \geq 1} \Lambda_{-y^{-1}q^m} T \otimes \Lambda_{-y^{-1}q^{m-1}} T^* \otimes S_{q^m} T \otimes S_{q^m} T^*\Big).$$

The corr. power series f will be given shortly.

Q: What are the most general Chern numbers (or genera) which are invariant under K-equivalence?

Theorem (Residue Thm, — JAG 2003)

Let $Z \subset X$ be a smooth pair of cpx mfds, $\phi : Y = Bl_Z X \to X$, with $E = P_Z(N_{Z/X}) \subset Y$ the exceptional div. Then for any cycle $D \subset X$, any power series $A(t) \in R[\![t]\!]$:

$$\int_{\phi^* D} A(E)\, K_Q(c(T_Y)) = \int_D A(0)\, K_Q(c(T_X)) + \int_{Z.D} \mathrm{Res}_{t=0} \frac{A(t)}{f(t) \prod_{i=1}^{r} f(n_i - t)} K_Q(c(T_Z)).$$

Here $c(N_{Z/X}) = \prod_{i=1}^{r} (1 + n_i)$.

The genera should satisfy the "change of variable" formula.

Namely, we want to find the most general $f(t)$ so that $\exists\, A(t, r)$ for each $r \in \mathbb{N}$, $A(0, r) = 1$, which leads to $\mathrm{Res}_{t=0}(\cdots) = 0 \quad \forall\, n_i$'s.

Then $A(E, r)$ is regarded as the Jacobian factor,

$K_Q(c(T_X))$ is regarded as some "measure" $d\mu_X$.

Answer:

$$f(x) = e^{(k + \zeta(z))x} \frac{\sigma(x)\, \sigma(z)}{\sigma(x+z)}$$

$$A(t, r) = e^{-(r-1)(h + \zeta(z))t} \frac{\sigma(t + rz)\, \sigma(z)}{\sigma(t+z)\, \sigma(rz)}$$

where $\sigma(z)$, $\zeta(z)$ are Weierstrass elliptic functions of some elliptic curve E_τ and $rz \not\equiv 0 \in E_\tau$.

The remarkable fact is: The original problem has nothing to do with any elliptic curves!

Solve Functional Equation: (for $r=2$) P. 5

$$\frac{1}{f(x)f(y)} = \frac{A(x)}{f(x)f(y-x)} + \frac{A(y)}{f(y)f(x-y)}$$

$$f(x) = x + f_2x^2 + \cdots \qquad A(x) = 1 + a_1x + \cdots$$

$$(f(x), A(x)) \mapsto (e^{kx}f(x), e^{-kx}A(x)) \quad \text{may let } f_2 = 0.$$

$$* \quad f(x-y)f(y-x) = A(x)f(y)f(x-y) + A(y)f(x)f(y-x)$$

$$\frac{d}{dy}\Big|_{y=0} \Rightarrow \; -f'(x)f(-x) + \cancel{f(x)f'(-x)} = A(x)f(x) + a_1f(x)f(-x) + \cancel{f(x)f'(-x)}$$

ie. $A(x)$ is det by $f(x)$ & a_1.

plug in $A(x)$, $A(y)$ into $*$ get

$$f(x)f(y)f(x-y)f(y-x)$$
$$+ f(y)^2f(x-y)\left[a_1f(x)f(-x) + f'(x)f(-x)\right]$$
$$+ f(x)^2f(y-x)\left[a_1f(y)f(-y) + f'(y)f(-y)\right] = 0$$

$$\frac{d^2}{dy^2}\Big|_{y=x} \Rightarrow \underset{①}{-2f(x)f'(x)} \quad -2\cdot f(x)\overset{2}{f(x)}\overset{\prime}{f}(0)\,\big(\underbrace{a_1f(x)f(-x)}_{②} + f'(x)f(-x)\big)$$

$$+2\cdot f(x)^2\cancel{f'}(0)\Big(\quad + a_1\underline{\big(f'(x)f(-x) - f(x)f'(-x)\big)} + \underbrace{f''(x)f(-x) - f'(x)f'(-x)}_{③}\Big) = 0$$

ie. $\underset{①}{1} + a_1\underset{②}{\big(f(x)f'(-x) + f'(x)f(-x)\big)} + \underset{③}{f'(x)f'(-x)}$

$$+\left[2\frac{f(-x)f'(x)^2}{f(x)} - f(-x)f''(x)\right] = 0$$

Also $\frac{d^3}{dy^3}\Big|_{y=0} \overset{\text{similarly}}{\Rightarrow}$

$$6f_3f(x)f(-x) + 2a_1\big(f(x)f'(-x) + f'(x)f(-x)\big) + 2f'(x)f'(-x)$$
$$+\left(2\frac{f(-x)f'(x)^2}{f(x)} - f(-x)f''(x)\right) = 0$$

Notice that by the Weierstrass' philosophy, any addition law comes from elliptic functions, from

$$\frac{-1}{f(x)f(-x)} = \frac{1}{(x+f_3x^3+\cdots)}\cdot\frac{-1}{(-x-f_3x^3\cdots)} = \frac{1}{x^2}(1-f_3x^2\cdots)(1-f_3x^2\cdots)$$
$$= \frac{1}{x^2} - 2f_3 + \cdots$$

Claim: $P(x) := \frac{-1}{f(x)f(-x)} + 2f_3$ is the Weierstrass $\wp(x)$.

let $g(x) = \dfrac{1}{f(x)}$, so $P(x) = -g(x)\,g(-x) + 2f_3$

get

$$\begin{cases} A & g(x)^2 g(-x)^2 - a_1\big(g'(x)\,g(-x) + g'(-x)\,g(x)\big) + g'(x)\,g'(-x) + \underline{g''(x)\,g(-x)} = 0 \\ B & 6f_3\, g(x)\,g(-x) - 2a_1\big(g'(x)\,g(-x) + g'(-x)\,g(x)\big) + 2g'(x)\,g'(-x) + \underline{g''(x)\,g(-x)} = 0 \end{cases}$$

in either eq'n, only last term is not sym in $x \mapsto -x$

so must have $g''(x)\,g(-x) = g''(-x)\,g(x)$

$\Rightarrow \quad g'(x)\,g(-x) + g'(-x)\,g(x) = \text{const} = \underset{\sim\sim}{6f_4}$ This simplifies A, B a lot !

(since $\frac{d}{dx} = \underline{g''(x)\,g(-x)} - \cancel{g'(x)\,g'(-x)} - \underline{g''(-x)\,g(x)} + \cancel{g'(-x)\,g'(x)} \equiv 0$)

since $\wp'^2 = 4\wp^3 - g_2\wp - g_3$

$\Rightarrow 2\wp''\wp' = 12\wp^2\wp' - g_2\wp' \Rightarrow \wp'' - 6\wp^2 = -g_2/2$

So we try to compute

$$\begin{aligned} p''(x) - 6\,p(x)^2 &= -g''(x)\,g(-x) + 2g'(x)\,g'(-x) - g(x)\,g''(-x) \\ &\quad - 6\,g(x)^2 g(-x)^2 + 24f_3\, g(x)\,g(-x) - 24f_3^2 \\ &= -6A + 4B + 12a_1 f_4 - 24f_3^2 \end{aligned}$$

ie $P(x) = \wp(x)$ with $g_2 = -24a_1 f_4 + 48f_3^2$. ☐

~~To solve f(x) explicitly~~, recall

$$\begin{cases} g'(x)\,g(-x) + g'(-x)\,g(x) = 6f_4 \\ -g'(x)\,g(-x) + g(x)\,g'(-x) = +\wp'(x) \end{cases}$$

$\Rightarrow \quad g'(x)\,g(-x) = \frac{1}{2}\big(6f_4 - \wp'(x)\big)$

Also $-g(x)\,g(-x) = \wp(x) - 2f_3$

$\Rightarrow \quad ((\log f))' = -(\log g)' = -\dfrac{g'(x)}{g(x)} = -\dfrac{1}{2}\cdot\dfrac{\wp'(x) - 6f_4}{\wp(x) - 2f_3}$

Pick z st $\wp(z) = 2f_3$, then $g(z)\,g(-z) = 0$, say $g(-z) = 0$

then $6f_4 = \wp'(z)$, so get

$$(\log f)' = -\frac{1}{2}\cdot\frac{\wp'(x) - \wp'(z)}{\wp(x) - \wp(z)} = \zeta(x) + \zeta(z) - \zeta(x+z)$$

where $\zeta(x) = -\int^x \wp = \frac{1}{x} + \cdots$

$\sigma(x) = e^{\int^x \zeta} = x + \cdots$

$\Rightarrow$ general solutions of f is

$$f(x) = e^{kx}\, e^{\zeta(z)x}\, \frac{\sigma(x)\,\sigma(z)}{\sigma(x+z)} \quad ☐$$

To make $\mathrm{Res}_{t=0} \dfrac{A(t)}{f(t)\prod_{i=1}^{r} f(n_i - t)} = 0$ $\forall\, n_i$, (any r)

we use lemma on elliptic functions :

$\prod_{j=1}^{r} \dfrac{\sigma(\lambda_j z - a_j)}{\sigma(\mu_j z - b_j)}$ is elliptic $\Leftrightarrow$ $\sum \lambda_j^2 = \sum \mu_j^2$

$\sum \lambda_j a_j = \sum \mu_j b_j$

$\sum \lambda_j \equiv \sum \mu_j \pmod 2$.

$$\mathrm{Res}_{t=0}\left(e^{-k u(N)}\, e^{(r-1)(k+\zeta(z))t}\, \frac{\sigma(t+z)}{\sigma(t)\sigma(z)} \prod_{i=1}^{r} \frac{\sigma(n_i - t + z)}{\sigma(n_i - t)\sigma(z)}\, A(t,r) \right)$$

pick $A(t,r) = e^{-(r-1)(k+\zeta(z))t}\, \dfrac{\sigma(t+rz)\,\sigma(z)}{\sigma(t+z)\,\sigma(rz)}$

then $-rz - z + \sum_{i=1}^{r}(n_i + z) = -z + \sum_{i=1}^{r} n_i$

$\Rightarrow$ the function is elliptic with only pole at $t=0$
(the part $\sigma(t+z)$ is cancelled out.) □

Conclusion: The complex elliptic genera defined by the power series $f(x)$ indeed satisfies the CVF for any smooth blow-up. By using the weak factorization theorem of birational maps (Wlodarczyk et. al.), this $\Rightarrow$ complex elliptic genera are inv. under K-equivalence.

Application: Let $I_1 := \langle [X] - [X'] \mid X \dashrightarrow X' \text{ a } \mathbb{P}^1\text{-flop} \rangle \triangleleft \Omega_*$

$I_K := \langle [X] - [X'] \mid X =_K X' \rangle \triangleleft \Omega_*$

B. Totaro (Ann. 2000) : Complex elliptic genera are the most general chern #'s inv. under $\mathbb{P}^1$-flops, ie. $\Omega_* / I_1 \simeq \mathrm{Ell}_*$

Our result then $\Rightarrow$ $I_K = I_1$

That is, up to complex cobordism, any K-equivalence $X \dashleftarrow\dashrightarrow X'$ can be decomposed into sequence of $\mathbb{P}^1$-flops. □

~~Q~~-coh & Flops

let

$$\begin{tikzcd} & Y \arrow[dl, "\varphi"'] \arrow[dr, "\varphi"] & \\ f : X \arrow[rr, dashed] & & X' \end{tikzcd}$$

Simple $\mathbb{P}^r$-flop, $N_{Z/X} = \mathcal{O}(-1)^{r+1}$

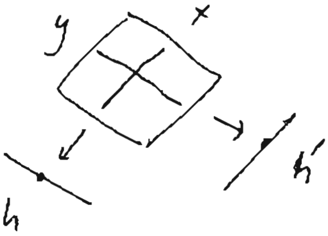
$\ell \subset \mathbb{P}^r \cong \overset{\cup}{Z}$ $\qquad$ $\overset{\cup}{Z'} \cong \mathbb{P}^r \supset \ell'$ $\qquad$ h, h' hyp. class of Z, Z'

$x = \bar{\varphi}^* h = [h \times \mathbb{P}^r]$

$y = \varphi^{-1*} h' = [\mathbb{P}^r \times h']$

in

$$\begin{tikzcd} E = \mathbb{P}^r \times \mathbb{P}^r \arrow[r, hook, "j"] \arrow[d, "\varphi"'] \arrow[dr, "\bar{\varphi}"] & Y \\ Z & Z' \end{tikzcd}$$



$T = \varphi'_* \circ \varphi^* = [\bar{\Gamma}_f]_*$

$: H^k(X) \to H^k(X')$.

<u>Thm.</u> (1) $\psi^*[h^s] = x^s y^r - x^{s+1} y^{r-1} + \cdots + (-1)^{r-s} x^r y^s$

so $T[h^s] = (-1)^{r-s}[h'^s]$, eg. $T\ell = -\ell'$.

(2) for $\alpha \in A^i(X)$, $i \le r$, let $\alpha' = T\alpha$ then

$$\varphi'^* \alpha' = \varphi^* \alpha + (\alpha . h^{r-i}) j_* \left(\frac{x^i - (-y)^i}{x+y} \right).$$

(3) Invariance of Poincaré pairing $(T\alpha . T\beta) = (\alpha, \beta)$.

(4) let $\alpha \in A^i(X)$, $\beta \in A^j(X)$, $\gamma \in A^k(X)$; $i+j+k = \dim X = 2r+1$

then $(T\alpha . T\beta . T\gamma) = (\alpha\beta\gamma) + (-1)^r (\alpha . h^{r-i})(\beta . h^{r-j})(\gamma . h^{r-k})$.

<u>pf in the $r=1$ case:</u>

(1): $0 \to N_{E/Y} \longrightarrow \bar{\varphi}^* N_{Z/X} \longrightarrow Q \longrightarrow 0$

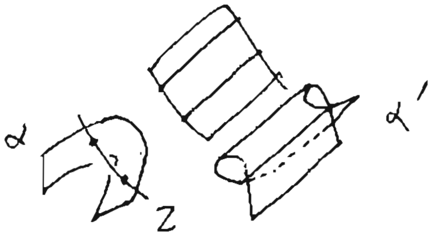
$N_{E/Y} = \bar{\varphi}^* \mathcal{O}(-1) \otimes \varphi^{-1*} \mathcal{O}(-1)$ $\qquad$ $\bar{\varphi}^* N_{Z/X} = \mathcal{O}(-1)^{\oplus 2}$

$\Rightarrow c_1(Q) = c_1(\bar{\varphi}^* \mathcal{O}(-1)^{\oplus 2}) - c_1(\bar{\varphi}^* \mathcal{O}(1) \otimes \bar{\varphi}'^* \mathcal{O}(1))$

$= -2x - (-x-y) = y - x$

$\varphi^*[h^0] = \psi^*[Z] = j_*(c_1(Q) \cap \bar{\varphi}^*[Z]) = (y-x)[E] = y - x$.

(2): $\alpha \in A^1(X)$, ie. a divisor, then for $\alpha \pitchfork Z$ get



$$\varphi'^* \alpha' = \varphi^* \alpha + (\alpha . \underset{h^0}{\uparrow}{Z}) . \underset{j_* \frac{x+y}{x+y} = j_*(1)}{\uparrow}{E}$$

(3): By sym on (2) $\Rightarrow (\alpha', Z') = -(\alpha, Z)$ but $\alpha' = T\alpha$, $Z' = -TZ$.

(4): $E^2 = E|_E = c_1(N_{E/Y}) = -(x+y)$, $E^3 = (E|_E)^2 = (-(x+y))^2 = 2$

$\Rightarrow (T\alpha)^3 = (\varphi'^* \alpha')^3 = \alpha^3 + \cancel{3(\varphi^* \alpha)^2 (\alpha . Z) E} + 3 \varphi^* \alpha . (Z\alpha)^2 E^2 + (Z\alpha)^3 E^3 = \alpha^3 - (Z\alpha)^3$ //

(Small) quantum cohomology

$$\bar{M}_{0,n}(X,\beta) \xrightarrow{e_i} X \quad ; \quad \gamma_i \in A^{d_i}(X)$$

$$\langle \gamma_1, \cdots, \gamma_n \rangle_\beta := \int_{[M_{0,n}(X,\beta)]^{vir}} e_1^* \gamma_1 \cup \cdots \cup e_n^* \gamma_n$$

non zero for $\sum d_i = v\text{-dim} = c_1(X).\beta + \dim X + n - 3$

Fund. class axiom : $\langle 1, \gamma_2, \cdots, \gamma_n \rangle_{\beta \neq 0} = \int_{[M_{0,n}(X,\beta)]^{vir}} e_2^*\gamma_2 \cdots e_n^*\gamma_n$

$$\equiv 0 \qquad \begin{tikzcd} {} \arrow[d, "\text{rel dim} = 1 > 0"] \\ {[M_{0,n-1}(X,\beta)]^{vir}} \end{tikzcd}$$

Divisor class axiom : $\gamma_1 \in A^1(X)$,

$$\langle \gamma_1, \cdots, \gamma_n \rangle_\beta = (\gamma_1.\beta) \langle \gamma_2, \cdots, \gamma_n \rangle_\beta$$

However!, in our case $\dim X \geqslant 3$.

let $T_i \in A^*(X)$ basis , $g_{ij} := (T_i, T_j)$, $(g^{ij}) = (g_{ij})^{-1}$

dual basis $T^i = \sum_j g^{ij} T_j$.

classical cup product : $T_i \cup T_j = \sum_k \langle T_i, T_j, T_k \rangle_0 \, T^k$

$$\langle T_i, T_j, T_k \rangle_0 \ \text{'''}\ (T_i, T_j, T_k)$$

small q-prod :

$$T_i *_s T_j := \sum_{\beta \in A_1(X)} \left(\sum_k \langle T_i, T_j, T_k \rangle_\beta \, T^k \right) q^\beta$$

view as formal variable q^β or $= e^{2\pi i \int_\beta \omega}$, ω complexified Kähler class

~~big~~ q-product : (still a full-q coh include all q ..)

let $\gamma = \sum t_i T_i$; t_i = parameters

$$\Phi(\gamma) = \sum_n \sum_\beta \frac{\langle \gamma^n \rangle_\beta}{n!} q^\beta \qquad \text{genus 0 pre-potential}$$

$$T_i * T_j := \sum_k \Phi_{ijk} T^k$$

$$\Phi_{ijk} = \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_k} = \sum_{n=0}^{\infty} \sum_{\beta \in A^1(X)} \frac{\langle T_i, T_j, T_k, \gamma^n \rangle_\beta}{n!} q^\beta$$

(set $t_i = 0$ $\forall i$ then get $n=0$ survives $\Rightarrow$ small)

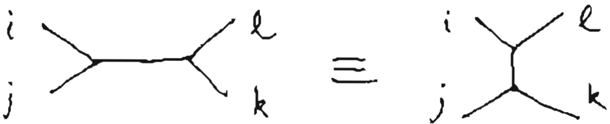
Thm (Ruan - Tian) : $QH^*(X) := \mathbb{Q}[[t, q^\beta]] \otimes A^*(X)$ is a ring.

Pf : $$(T_i * T_j) * T_k = \sum_{e,f} \Phi_{ije} g^{ef} T_f * T_k = \sum_{e,f,l} \underline{\Phi_{ije} g^{ef} \Phi_{fkl}} \, T^l$$

$$\text{''?}$$

$$T_i * (T_j * T_k) = \sum_{e,f} T_i * (\Phi_{jke} g^{ef} T_f) = \sum_{e,f,l} \underline{\Phi_{jke} g^{ef} \Phi_{ifl}} \, T^l$$

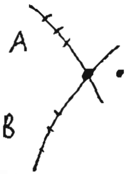
Need WDVV eq'n :
Dijkgraaf



sum over e,f .

For $A \cup B = \{1, \cdots, n\}$, $\beta_1 + \beta_2 = \beta$, define a divisor

$$\begin{tikzcd}
D(A,B;\beta_1,\beta_2) := M_{0,A\cdot}(X,\beta_1) \times_X M_{0,B\cdot}(X,\beta_2) \arrow[r, hook, "\iota"'] \arrow[d, "\alpha"] & M_{0,A\cdot} \times M_{0,B\cdot} \\
M_{0,n}(X,\beta) &
\end{tikzcd}$$



consider evaluation maps

$$\begin{tikzcd}
M \arrow[d, "\rho"] & D \arrow[l, "\alpha"'] \arrow[r, "\iota"] \arrow[d, "\eta"] \arrow[dr, phantom, "\star"] & M_A \times M_B \arrow[d, "\rho'"] \\
X^n & X^{n+1} \arrow[l, "p"] \arrow[r, "\delta"'] & X^{n+2}
\end{tikzcd}$$

diagonal map repeat "•"

splitting axiom:

$$\iota_*\alpha^*(e_1^*\gamma_1 \cup \cdots \cup e_n^*\gamma_n)$$

$$= \sum_{e,f} g^{ef} \left(\prod_{a\in A} e_a^*\gamma_a \cup e_\cdot^* T_e\right) \times \left(\prod_{b\in B} e_b^*\gamma_b \cup e_\cdot^* T_f\right).$$

pf: $\text{LHS} = \iota_* \underline{\alpha^*\rho^*}(\gamma_1 \times \cdots \times \gamma_n) = \iota_* \underline{\eta^* p^*}(\gamma_1 \times \cdots \times \gamma_n)$

$= \iota_* \underline{\eta^*}(\gamma_1 \times \cdots \times \gamma_n \times [X]) = \underline{\rho'^*}\delta_*(\gamma_1 \times \cdots \times \gamma_n \times [X])$ since $\star$ is a base change

$= \rho'^*(\gamma_1 \times \cdots \times \gamma_n \times [\Delta])$

$= \sum_{e,f} g^{ef} \rho'^*(\gamma_1 \times \cdots \times \gamma_n \times T_e \times T_f) = \text{RHS}.$ //

Notice that $\sum_{e,f} \Phi_{ije} g^{ef} \Phi_{fk\ell} = \sum_{\substack{n_1,n_2\\ e,f,\beta_1,\beta_2}} \frac{g^{ef}}{n_1!\,n_2!} \langle T_i, T_j, T_e, \gamma^{n_1}\rangle_{\beta_1} \langle T_k, T_\ell, T_f, \gamma^{n_2}\rangle_{\beta_2} \cdot q^{\beta_1+\beta_2}$

- may do WDVV for n, β fixed!

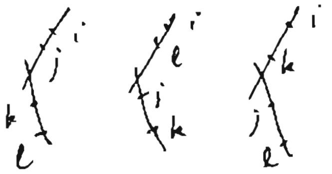
let $G(ij|k\ell) = \sum_{\substack{i,j\in A\\ k,\ell\in B\\ e,f,\ \beta_1+\beta_2=\beta}} g^{ef} \langle \prod_{a\in A} \gamma_a, T_e\rangle_{\beta_1} \langle \prod_{b\in B} \gamma_b, T_f\rangle_{\beta_2}$

by splitting axiom $= \int_{D(ij|k\ell)} e_1^*\gamma_1 \cup \cdots \cup e_n^*\gamma_n$

where $D(ij|k\ell) = \sum_{\substack{i,j\in A\quad k,\ell\in B\\ e,f,\ \beta_1+\beta_2=\beta}} D(A,B;\beta_1,\beta_2)$

But $D(ij|k\ell) \sim D(i\ell|jk)$ on $M_{0,n}(X,\beta)$ via

$M_{0,n}(X,\beta) \longrightarrow M_{0,n} \longrightarrow M_{0,\{i,j,k,\ell\}} \cong \mathbb{P}^1$)



So $G(ij|k\ell) = G(i\ell|jk)$

Now take $\gamma_i = T_i$, $\gamma_j = T_j$, $\gamma_k = T_k$, $\gamma_\ell = T_\ell$, $\gamma_s \equiv \gamma$ for other s $\square$

quantum corrections by extremal rays

$f : X \dashrightarrow X'$ simple $\mathbb{P}^r$- flop, for small q-coh,

since $(T\alpha . T\beta) = (\alpha, \beta)$, only need to compare

$$\langle \alpha, \beta, \gamma \rangle := \langle \alpha, \beta, \gamma \rangle_0 + \sum_{d \in \mathbb{N}} \langle \alpha, \beta, \gamma \rangle_{d\ell}\, q^{d\ell} + \sum_{\Gamma \neq d\ell} \langle \alpha, \beta, \gamma \rangle_\Gamma\, q^\Gamma$$

For extremal ray $d\ell$, $(g, k) = (0, 3)$:

$$\text{v-dim } M_{0,k}(X, d\ell) = (\cancel{c_1(X). d\ell}) + \dim X (1-\cancel{g}) + 3\cancel{g} - 3 + k$$

$$= \dim X = 2r + 1$$

So only need

$$\alpha \in A^i(X),\ \beta \in A^j(X),\ \gamma \in A^k(X) \ ; \quad i + j + k = 2r + 1$$

Thm (H.W. Lin, W—; 2004) Generalized multiple cover formula

$$\langle \alpha, \beta, \gamma \rangle_{d\ell} \equiv \int_{[M_{0,3}(X, d\ell)]^{virt}} e_1^* \alpha . e_2^* \beta . e_3^* \gamma$$

$$= (-1)^{(d-1)(r+1)} (\alpha . h^{r-i})(\beta . h^{r-j})(\gamma . h^{r-k}) .$$

Cor: The small q-ring modulo non-exceptional classes is invariant under simple ordinary flops .

Pf: $(T\alpha . h'^{(r-i)}) = (-1)^i (T\alpha, T h^{r-i}) = (-1)^i (\alpha, h^{r-i})$

$$\Rightarrow \langle T\alpha, T\beta, T\gamma \rangle - \langle \alpha, \beta, \gamma \rangle = (-1)^r (\alpha . h^{r-i})(\beta . h^{r-j})(\gamma . h^{r-k})$$

d=1 is, >0

$$\underset{\uparrow}{-} (\alpha . h^{r-i})(\beta . h^{r-j})(\gamma . h^{r-k}) \left(\frac{q^{\ell'}}{1 + (-1)^r q^{\ell'}} \underset{\uparrow}{+} \frac{q^\ell}{1 + (-1)^r q^\ell} \right)$$

$-(-1)^{2r+1} = +1$

under T, identify ℓ' by $-\ell$, get

$$\frac{1}{q^\ell + (-1)^r} + \frac{q^\ell}{1 + (-1)^r q^\ell} = (-1)^r$$

\#

Remark:

$\omega = A + Bi$, $A \in H^{1,1}_{\mathbb{R}}(X)$, $B \in \mathcal{K}_X$ (Kähler cone)

Then $q^\Gamma = e^{2\pi i (\omega, \Gamma)} = e^{2\pi i (A, \Gamma)} . e^{-2\pi (B, \Gamma)}$

modulo non-exceptional classes,

$\langle \alpha, \beta, \gamma \rangle^X$ converges in $H^{1,1}_+ = \{ \omega \mid (B . \ell) > 0 \} \supset H^{1,1}_{\mathbb{R}} \times \mathcal{K}_X$

now T identifies $H^{1,1}$, $H^{n-1,n-1}$ & (.)

$\langle T\alpha, T\beta, T\gamma \rangle^{X'}$ converges in $H^{1,1}_- = \{ \omega \mid (B, \ell') > 0 \} \supset H^{1,1}_{\mathbb{R}} \times \mathcal{K}_{X'}$

i.e. $(B, \ell) < 0$

so Cor holds in the sense of analytic continuations from $H^{1,1}_+$ to $H^{1,1}_-$.

For general base S. Is this some modularity on S ?

Conjecturally, $\langle \alpha, \beta, \gamma \rangle^X$ conv for $B \in \mathcal{K}_X$, at least for $B \gg 0$ and the large radius limit $= (\alpha, \beta, \gamma)$.

Also $\mathcal{K}_X$ among K-equiv models form a chamber structure .

Pf of Thm via Euler data of Lian-Liu-Yau :

virtual fund class : $H^1(C, f^*T_X) = H^1(C, f^*T_{\mathbb{P}^r} \oplus f^*N_{Z/X})$

$\Rightarrow$ the obstruction is given by eg $C \cong \mathbb{P}^1$

positive (convex, no H^1) ; $N_{Z/X} = \overset{r+1}{\oplus} \mathcal{O}(-1)$ (concave, no H^0)

$\dim H^1(C, f^*\mathcal{O}(-1)^{r+1}) = (r+1)\, h^1(\mathbb{P}^1, \mathcal{O}(-d))$
$= (r+1)(d-1)$

From
$$\begin{tikzcd} M_{04}(\mathbb{P}^r,d) \arrow[r, "e_4"'] \arrow[d, "\rho"] & \mathbb{P}^r \\ M_{03}(\mathbb{P}^r,d) & \end{tikzcd}$$
univ. curve

; $U_d = R^1\rho_*(e_4^*\, N_{Z/X})$

Equiv to prove $i+j+k = 2r+1$

$$\int_{M_{03}(\mathbb{P}^r,d)} e_1^*h^i . e_2^*h^j . e_3^*h^k . e(U_d) = (-1)^{(d-1)(r+1)}.$$

More generally, $\int_{M_{0k}(\mathbb{P}^r,d)} e_1^*h^{d_1} \cdots\cdots e_k^*h^{d_k} . e(U_d)$ — exists on all $M_{0k}(\mathbb{P}^r,d)$ say $U_{d,k}$

with $d_1 + \cdots + d_k = 2r+1+(k-3)$.

let $\phi = \sum_{i=⓪}^{r} t_i h^i$ 1 is enough by fund. class's axiom, $e^k(\phi) = e_1^*\phi \cdots\cdots e_k^*\phi$

Fact : both $e^k(\phi)$, $e(U_d)$ are gluing sequences

let $b_d^k = e^k(\phi) . e(U_d)$

$Q_d^k = \zeta_*^k\, \pi^*(b_d^k)$

where $\zeta^k : M_d^k \longrightarrow N_d$
non-linear σ-models — linear σ-models

$$\begin{tikzcd} & M_d^3 \arrow[r, "\pi"] \arrow[d, "\zeta_3"'] & M_{03}(\mathbb{P}^r,d) \arrow[d] \arrow[dddl, "\psi"'] \arrow[ddddr, "e_j"] & \\ & M_d^2 \arrow[r, "\pi"] \arrow[d, "\zeta_2"'] & M_{02}(\mathbb{P}^r,d) \arrow[d] & \\ & M_d^1 \arrow[r, "\pi"] \arrow[d, "\zeta_1"'] & M_{01}(\mathbb{P}^r,d) \arrow[d] & \\ M_{00}(\mathbb{P}^1\times\mathbb{P}^r,(1,d)) =: & M_d \arrow[r, "\pi"] \arrow[d, "\varphi"'] & M_{00}(\mathbb{P}^r,d) & \\ \mathbb{P}^{(r+1)(d+1)-1} \cong & N_d \arrow[rr, dashed, "e_i"'] & & \mathbb{P}^r \end{tikzcd}$$

φ, ψ birational morphism

N_d is the naïve compactification of $\mathbb{P}^1 \xrightarrow{f} \mathbb{P}^r$

$(w_0 : w_1) \mapsto (f^0(w_0,w_1) : \cdots : f^r(w_0,w_1))$

$= (z_{00}, \cdots, z_{0d}, z_{10}, \cdots z_{1d}, \cdots, z_{r0} \cdots z_{rd})$

$p_{is} = (0, \cdots 0, z_{is} w_0^s w_1^{d-s}, 0 \cdots 0)$

- $\mathbb{C}^\times$ action on $\mathbb{P}^1$, $t\cdot(w_0,w_1) = (tw_0 : w_1)$ call weight α

$T = (\mathbb{C}^\times)^{r+1}$ action on $\mathbb{P}^r$ (standard) weights $\lambda_0, \cdots, \lambda_r$

induced $G = \mathbb{C}^\times \times T$ action on all moduli spaces on LHS

eg. weight at $p_{is} = \lambda_i + s\alpha$.

Fact : all maps on LHS are G-equivariant. (J. Li)

$Q_d^k \in H^*_G(N_d) = \mathbb{Q}[\alpha,\lambda][K] / f(K) = \prod_{i,s} (K - (\lambda_i + s\alpha))$

Rule: $i_{is} : P_{is} \longrightarrow N_d$, $i_{is}^* \omega = \omega(\lambda_i + s\alpha) \in \mathbb{Q}[\alpha, \lambda]$.

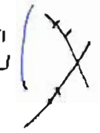
Thm: The sequence $Q_d := \sum_{k=0}^{\infty} \frac{Q_d^k}{k!} T^k$ is an $\Omega := e_T(N)^{-1}$ Euler data, ie. $\forall\, s = 0, \cdots, d$; $i = 0 \ldots r$, have $Q_0 = \Omega$,

$$i^*_{p_i}(\Omega) . \, i^*_{p_{is}}(Q_d) = \overline{i^*_{p_{i0}}(Q_s)} . \, i^*_{p_{i0}}(Q_{d-s}) .$$

bar means $\alpha \mapsto -\alpha$, $\lambda_i \mapsto \lambda_i$

Sketch-pf: ($k=0$ case is in LLY)

- Atiyah - Bott localization
- splitting axiom apply to b_d^k
  equivariant normal splits (as in Zhou's lecture)
- write back to 2 products by localization again. $\square$

" Σ() = (Σ )(Σ ) "

in Zhou, sum over g. here sum over k.

Two Ω-Euler data are linked if $\{P_d\}$, $\{Q_d\}$ have

$$i^*_{p_{j0}} P_d = i^*_{p_{j0}} Q_d \quad \text{at} \quad \alpha = (\lambda_j - \lambda_i)/d \quad \forall\, i \neq j, \ d > 0 .$$

Uniqueness Thm (LLY): if $\deg_\alpha i^*_{p_{i0}}(P_d - Q_d) \leq (r+1)d - 2$ for all i, d, then $P_d \equiv Q_d$.

Reconstruction Thm:

$\{Q_d' := Q_d \mod T^4\}$ is linked to, in fact equal to

$$P_d' = \sum_{k=0}^{3} \Big(\sum_{i=0}^{r} t_i K^i \Big)^3 \frac{T^k}{k!} \prod_{m=1}^{d-1} (-K + m\alpha)^{r+1}$$

$$\equiv e^{\sum_{i=0}^{r} t_i K^i T} \prod_{m=1}^{d-1} (-K + m\alpha)^{r+1} .$$

Idea of pf: $\left(i^*_{p_{j0}} Q_d^k \ \text{at} \ \alpha = \frac{\lambda_j - \lambda_i}{d} \right) = Q_d^k \Big(\overset{K}{\lambda_j + \cancel{0} \cdot \alpha} ; \overset{\alpha}{\frac{\lambda_j - \lambda_i}{d}} \Big)$

which $= Q_d^k(K, \alpha)$ restrict to the smooth point

$$P_{ij} = (0, \cdots 0, \underset{\substack{\uparrow \\ \text{in } i\text{-th}}}{w_0^d}, \cdots, \underset{\substack{\uparrow \\ j\text{-th}}}{w_1^d} \cdots\cdots 0) \in N_d \xleftarrow{\varphi} \in M_d \xleftarrow{\psi} \in M_{03}(\mathbb{P}^r, d)$$

coor of $\mathbb{P}^r$

ie. d-multiple cover of $\overline{P_i P_j} \cong \mathbb{P}^1 \subset \mathbb{P}^r$ fixed under $G' \subset G$ st $\alpha = (\lambda_j - \lambda_i)/d$.

All can be handled as in LLY, except that we need to do it on $M_{03}(\mathbb{P}^r, d) \ni (\varphi \circ \psi)^{-1}(P_{ij})$.

This is OK since $\varphi \circ \psi$ is isom near P_{ij}. Also, we simply replace $e_i^* h$ by K everywhere. $\square$

Finally, $\left. \frac{\partial^3 Q_d'}{\partial t_i \partial t_j \partial t_k} \right|_{\alpha=0} = (-1)^{(r+1)(d-1)} K^{i+j+K+(r+1)(d-1)} T^3 \pmod{T^4}$ ⁂

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