The First

NCTS Summer School on Algebraic Geometry

July 19 — 30, 1999

Professor Ching-Li Chai
University of Pennsylvania

(Notes by Chin-Lung Wang)

Langlands Program and Non-Abelian Class Field Theory

NCTS Summer School in Alg. Geom.
Prof. Ching-Li Chai
Lecture I. 7/20 at Wu-Ling Farm.

General Langlands' Program ~
non abelian C.F.T. (class Field Theory)

Review : classical C.F.T.

F - global field or local field

Galois side : Analytic side :

$$\left\{ \begin{array}{l} \mathrm{Gal}(F^{sep}/F) \to \mathbb{C}^\times \\ \text{characters of} \\ \text{finite order} \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{l} \mathbb{A}_F^\times / F^\times \longrightarrow \mathbb{C}^\times \\ F^\times \longrightarrow \mathbb{C} \end{array} \right\}$$

characters of finite order

$$\mathrm{res}_F : \mathbb{A}_F^\times / F^\times \longrightarrow \mathrm{Gal}(F^{sep}/F) \quad \text{reciprocity norm map}$$

This is Artin's formulation.

Another formulation : L- functions

abelian Hecke L-functions (Dirichlet) $\overset{1-1}{\longleftrightarrow}$ Hecke's L-function

Outline of the shape of non-abelian generalization

Analytic Side	Galois Side
automorphic representation (generalization of modular form + consider all levels simultaneously)	are parametrized by <u>parameters</u>
Auto. repr of $G(\mathbb{A}_F)$	$L_F \longrightarrow {}^L G$ "L-group"
irred. adm. repr of $G(F)$	$\cup$ $Gal(F^{sep}/F)$
G = conn. reductive alg. gp $/F$.	

"extension of $Gal(F^{sep}/F)$ by a cpx reductive Lie group"

Remark: one parameter may correspond to several repr of LHS
(L-indistinguishable / stable conj class etc)

Serious Prob: when F is a global field, don't really know what L_F is!

But the situation in the local case is a lot better
eg. Local Langlands' Conj for GL_n has been proved (M. Harris, R. Taylor, Henniart)

When F is a local field,

$$L_F = W_F \times SL_2(\mathbb{C})$$

W_F the Weil group:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I & \xrightarrow{\text{inertial gp}} & Gal(F^{sep}/F) & \longrightarrow & Gal(k^{sep}/k) & \to & 1 \\ & & \| & & \cup & & \cong \; \hat{\mathbb{Z}} & & \text{residue field} \\ 1 & \longrightarrow & I & \longrightarrow & W_F & \longrightarrow & \cup \; \mathbb{Z} & \to & 1 \end{array}$$

gen. by frob. $= \mathbb{Z}$

${}^L G = \hat{G} \rtimes W_F$

via simple co-characters:

dual root system determine dual groups

exchange characters $\longleftrightarrow$ co characters.

$\hat{G}$ has root system dual to that of G.

${}^L G$ is defined up to inner automorphism.

eg. $G = GL_n$, $\hat{G} = GL_n$

G = type C_n, $\hat{G}$ = type B_n etc.

From the number theoretic point of view:

when $G = GL_n$, $L_F \longrightarrow {}^L G$ is very close to

what we want the ("Galois repr'n")

$\pi = \bigotimes_{v \in \Sigma_F} \pi_v$

φ_v : parameter of π_v

It looks like we get a family of local Galois repr'n.

this reminds the notion of compatible system of ℓ-adic repr'n.

A substantial part of number theory are talking about this. $\rightarrow$

eg. $X/_F \rightsquigarrow H^*(X \otimes_F \bar{F}, \bar{\mathbb{Q}}_\ell)$

ℓ-adic representations.

Prophecy / expectation:

If $\pi = \otimes_v \pi_v$ is an <u>algebraic</u> auto. repr'n,

then π "should" correspond to a "motive".

For simplicity, let $G = GL_n$:

Algebraic: 2 aspects

- archimedean parameters $\pi_\infty = \otimes_i \pi_{\infty_i}$ should be algebraic:

  Aside: $F \cong \mathbb{R}$, $1 \to \bar{F}^\times \to W_F \to \mathrm{Gal}(\bar{F}/F) \to 1$

  $F \cong \mathbb{C}$, $W_F := F^\times$ the only non-trivial ext.

  i.e. $W_F = \bar{F}^\times \amalg j\bar{F}^\times$ with $jzj^{-1} = \bar{z}$.

  when we talk about homs $W_F \longrightarrow \cdots$ this is the cpx Hodge structure in alg. geom.

  i.e. Hodge str $/\mathbb{R}$ with value in $\mathbb{C}$.

  Now algebraic in the sense that the restriction of φ_∞ to $\mathbb{C}^\times$ are algebraic ("$z^{m_1}\bar{z}^{m_2}$")

  (Weil: Grothen characters)

- $\mathbb{Q}(\pi_f)$ = field of moduli of π_f is algebraic

  finite Adelic component

  i.e. the fixer is an alg. number field.

Given such a repr. try to construct "motive" in the sense of L-functions. In good case, may use Shimura varieties, e.g. for GL_2, Deligne has done this.

Remark: In function field case, the Hodge part is totally different. That's the problem.

$X_{/F}$ alg. v. over number field F.

X smooth. proj. say.

$\rightsquigarrow$ $H^*(X \otimes_F \bar{F}, \bar{\mathbb{Q}}_\ell)$

[From the point of L-functions:

Identify $\{\text{auto. L-fun}\} \overset{?}{\longleftrightarrow} \{\text{motivic L-func}\}$

Auto L-functions:

$\pi = \otimes_v \pi_v$, need $r: {}^LG \longrightarrow GL(V)$

get $L(s, \pi, r)$

should be meromorphic, has functional eq'n

$$L(s, \pi, r) = \varepsilon(s, \pi, r) \cdot L(1-s, \pi, \tilde{r})$$

If one admits $L_F \xrightarrow{\varphi} {}^LG$ (finite collection)

$$\begin{tikzcd} L_F \arrow[r, "\varphi"] \arrow[dr] & {}^LG \arrow[d, "\alpha"] \\ & {}^LH \end{tikzcd}$$

Langlands' functoriality

α: {packets for G} $\longrightarrow$ {packets for H}.]

Expect:

- $L(s, H^*(\bar{X}, \bar{\mathbb{Q}}_\ell))$ should be automorphic L-function of GL_n, $n = \dim(H^*(\bar{X}, \bar{\mathbb{Q}}_\ell))$.

  But what we want is the smallest such gp.

Question: what is the "smallest" gp H st. this

$L(s, H^*(\bar{X}, \bar{\mathbb{Q}}_\ell))$ is $L(s, \underbrace{\otimes_v \pi_v}_{\text{auto. repr. of } H}, r)$?

Take an archimedean place ∞_1 of F:

natural Hodge str. $\rightsquigarrow$ $H^*(X_{\infty_1}(\mathbb{C}), \mathbb{Q})$

$\rightsquigarrow$ "$\hat{G}$" $MT(\cdots)$ conn. reductive gp / $\mathbb{Q}$.

Optimistically. via Tannakian category.

To be continued

$F \begin{cases} \text{local} \\ \text{global} \end{cases}$ fields G : reductive conn alg. gp $/F$

Analytic Side:

- irreducible admissible repr of $G(F)$, local
- irred. automorphic repr "of $G(\mathbb{A}_F)$". global

Galois Side:

<u>admissible</u> homomorphism:

$$L_F \longrightarrow {}^L G$$

- local: W_F' or $W_F \times SL_2(\mathbb{C})$ $(SU_2(\mathbb{R}))$
- global: ?? (unknown yet)

1. Weil groups:

Explicitly:

F: local, non archimedean:

$$\begin{array}{ccccccccc} 1 \to & I & \to & Gal(F^{sep}/F) & \to & Gal(K^{sep}/K) & \to 1 \\ & \| & & \cup\!\vert & & \cup & \\ 1 \to & I & \to & W_F & \longrightarrow & \{\mathrm{Frob}^{\mathbb{Z}}\} & \to 1 \end{array}$$

discrete top.

topology: from I.

use W_F to get more characters.

$F \cong \mathbb{C}$, $W_F \cong F^\times \cong \mathbb{C}^\times$

$F \cong \mathbb{R}$: $1 \to \bar{F}^\times \to W_{\mathbb{R}} \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1$

non-split

$j \longmapsto \sigma$, $jzj^{-1} = \bar{z}$

$\mathbb{R}^\times \xrightarrow{\sim} W_{\mathbb{R}}/W_{\mathbb{R}}^c$: reciprocity law map

$-1 \longmapsto j\,W_{\mathbb{R}}^c$

$x \longmapsto \sqrt{x}\cdot W_{\mathbb{R}}^c$

typical example in class field theory

In general : $F^\times$

$$\mathbb{A}_F^\times/F \xrightarrow{\sim} W_F^{ab} \quad \text{reciprocity law map.}$$

This is what we want for Weil gp.

$$W_F \longrightarrow \mathrm{Gal}(F^{sep}/F) \quad \text{with dense image}$$

(i.e. capture all top. information and get more repr'n. for local F, we completely know them.)

Functoriality :

$F \longrightarrow {}^{\sigma}F$ (by Frob.) get

$W_F \longrightarrow W_{\sigma F}$

global - local compatibility : let F global

W_F

$\uparrow$

W_{F_v}

and just like C.F.T.

for E/F finite separable ext. get

$$\begin{tikzcd}
W_E^{ab} \arrow[r] \arrow[d, shift left] & E^\times,\ \mathbb{A}_E^\times/E^\times \arrow[r] \arrow[d, shift left] & \mathrm{Gal}(E^{ab}/E) \arrow[d, shift left] \\
W_F^{ab} \arrow[r] \arrow[u, shift left] & F^\times,\ \mathbb{A}_F^\times/F^\times \arrow[r] \arrow[u, shift left] & \mathrm{Gal}(F^{ab}/F) \arrow[u, shift left]
\end{tikzcd}$$

one is inclusion

one is transfer

In Short : Weil gp are suitable version of "Galois group".

2. Langlands dual groups :

G/F

(1). Root data for G / F^{sep} :

T : max. torus

B : Borel subgroup (over F^{sep})

(minimal parabolic subgp. e.g. 's).

get $X^*(T)$, Δ^* : simple roots, and dual :

$(X^*(T), \Delta^*, X_*(T), \Delta_*)$

To be canonical, take $\varprojlim$ (of these)

Now, when G/F, has action of $\Gamma_F = \mathrm{Gal}(F^{sep}/F)$.

on $(X^*, \Delta^*, X_*, \Delta_*)$.

Now, take the dual root data $(X_*, \Delta_*, X^*, \Delta^*)$

which is by def an "abstract root system"

general theory $\to$ can produce reductive gp

over any alg. closed field.

say $\mathbb{C}$.

call it $\hat{G}$. $/\mathbb{C}$.

Next thing : to produce (Weil version of it)

$$\begin{tikzcd} 1 \arrow[r] & \hat{G} \arrow[r] & {}^L G \arrow[r] & W_F \arrow[r] \arrow[l, bend left, "\text{split}"] & 1 \end{tikzcd}$$

then will get an action of W_F on $\hat{G}$. hence get

$\hat{G} \rtimes W_F$ (choosing a splitting)

(There are also finite Galois versions.)

(trivial) example:

1) GL_n : $(\mathbb{Z}^n, e_i - e_j, \mathbb{Z}^{n*}, x_i - x_j)$

$\widehat{GL_n} = GL_n$

2) SL_n : (Fact: dual of simply connected semi-simple gp get adjoint gp)

$\widehat{SL_n} = PSL_n$

3) Forgetting the center:

$A_n \longleftrightarrow A_n$; $B_n \longleftrightarrow C_n$; $D_n \longleftrightarrow D_n$

(we do not bother exc. gp here)

4) $PSp_{2n} \longleftarrow Spin_{2n+1}$

5) GSp_{2n} : $\widehat{GSp_{2n}}^{der}$ is simply connected!

6) E/F finite extension,

$Res_{E/F}(GL_2/E)$ (Weil restriction)

L-dual group $= GL_2/\mathbb{C} \rtimes W_F$

action of $W_F \twoheadrightarrow Gal(F^{Gal}_{null}/F)$ on the root system

$\simeq$ action $Gal(F^{sep}/F)$ on $Hom(E, F^{sep})$.

4. $\varphi : L_F \longrightarrow {}^L G$ is admissible (homomorphism)

if (roughly):

- semi-simple
- (locally) relevant

Wait,

3. L_F : F local.

If $F \cong \mathbb{R}$ or $\mathbb{C}$, take $L_F \cong W_F$

If F is local non-archemedean, then one has to enlarge W_F (Deligne):

Two Versions: (Deligne-Weil gp)

- $W_F' = W_F \ltimes \mathbb{G}_a$

commutation relation $\underset{\mathbb{G}_a}{w \overset{\frown}{x}} w^{-1} = |w| x$ via the reciprocity law map.

- Jacobson-Morosov : $W_F' \rightsquigarrow W_F \times SL_2(\mathbb{C})$.

for the equivalence, we want homomorphism which is holomorphic (alg) in $SL_2(\mathbb{C})$ factor.

Motivation:

Grothendieck's semi-stable reduction theorem:
Not an alg. notion

$\hookrightarrow \rho_\ell : Gal(F^{sep}/F) \to GL(V_{/\overline{\mathbb{Q}_\ell}})$ $\quad \ell \neq char(K)$ residue field

$\rho_\ell|_U$ is unipotent
$U \subset I$ finite index

$N = \log \rho_\ell|_U \in Hom(V(1), V)$ Tate twist

(Arithmetic analogue of top'l quasi-unipotency thm)

But, one can "take away" the monodromy to make this into an algebraic notion.

$$1 \to I \to W_F \to Gal(K^{sep}/K) \to 1$$

$$\Phi \mapsto Frob.$$

$$\rho_\ell(\Phi^n \underset{\substack{\cap \\ I}}{\sigma}) = \underset{\substack{\uparrow \\ \text{repr of } W_F'}}{\rho'}(\Phi^n\sigma)\exp(t_\ell(\sigma)N)$$

Galois repr.

$t_\ell : I \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \xrightarrow{\sim} \mathbb{Q}_\ell(1)$

the point is, $\rho'(W_F)$ is finite, hence algebraic

eg. can do conjugation. etc.

4. (continue).

- semi-simple: easy via 2nd version

  $W_F \times SL_2(\mathbb{C}) \to GL_n$. semi-simple in the usual sense (direct sum of irreducibles)

  In general, $\varphi : W_F \times SL_2(\mathbb{C}) \longrightarrow {}^L G$ is semi-simple if:

$$\begin{tikzcd} W_F \times SL_2(\mathbb{C}) \arrow[r] \arrow[dr] & {}^L G \arrow[d] \\ & GL(V) \end{tikzcd}$$

- relevant:

$$\underset{\substack{\circlearrowleft \\ \Gamma_F}}{\left\{\begin{array}{l}\text{conjugacy class} \\ \text{of parabolic} \\ \text{of } G/F^{sep}\end{array}\right\}} \xleftrightarrow[1-1]{} \left\{\begin{array}{l}\text{conj. classes} \\ \text{of parabolics} \\ \text{of } \hat{G}\end{array}\right\}$$

Real problem: If a conj class is fixed by Γ_F, it may or may not come from a parabolic of G rational over F.

$\varphi : L_F \to {}^L G$ is relevant if,

If $\varphi(L_F) \subset$ Levi of a F-ratl conj class of a parabolic

then this parabolic is relevant.

Ex. B: quaternion division alg.

$\rightsquigarrow B^\times$

Say, $W_F \xrightarrow[\varphi]{} {}^L G = GL_2 \times W_F$, then $\varphi = \mu \oplus \nu$ is not allowed for μ, ν 2 characters.

5. Analytic Side:

F: local Archemedean,

admissible = H-C modules (Harish-Chandra)

G: Lie group, usually consider just continuous representations V, (say 50 years ago).

"Algebraic / essential" part:

but "same repr" may have many diff top. ver's!

$K \subseteq G$ max. cpt.

take the space of K-finite vectors

(picking out a subspace which is dense)

$$V_{K\text{-finite}} \begin{array}{l} \circlearrowleft U(\mathfrak{g} = \mathrm{Lie}(G)) \\ \circlearrowleft K \end{array} \Big\rangle \text{ compatible}$$

Def: An admissible repn of G is a $(U(\mathfrak{g}), K)$ module (or equivalently $(\mathfrak{g}, K)$-mod) s.t. each irred. repn of K occurs with $< \infty$ multiplicities.

$\to$ Irred. admissible $(\mathfrak{g}, K)$-module.

F: local non-archemedean: $G(F)$.

Def: An admissible repr of $G(F)$ is one which is smooth (stabilizer are open) and $\forall$ cpt open K. (as in previous).

Automorphic Representations :

Idea : (Consider only sub quotient of very explicit representations)

$$\left\{ f : G(F)\backslash G(\mathbb{A}_F) \longrightarrow \mathbb{C} \;\middle|\; \begin{array}{l} \text{smooth / Schwartz} \\ \text{which grows slowly} \\ \text{for every place (or just} \\ \text{for one place)} \end{array} \right\}$$

Number field case :

- $Z(U(\mathfrak{g}))$ = finite
- growth condition at ∞

function field case :

- equiv. to condi. at one place.

these functions = " automorphic forms " $\circlearrowleft G(\mathbb{A}_F)$.

Now we want to think about subquotient of this.

to be continued

$\{$ automorphic forms on $G(F)\backslash G(\mathbb{A}_F) \}$ $*$

$\cup\!|$

$\{$ cuspidal auto. forms on " $\}$ $**$

$f: G(F)\backslash G(\mathbb{A}_F) \to \mathbb{C}$

$\int_{N(F)\backslash N(\mathbb{A}_F)} f(nx)\,dn = 0$ $\forall$ N: unipotent radical of a F-rat'l parabolic of G

(with this. then $L^2 \equiv$ bounded $\equiv$ decay very fast)

irred. auto. repr means those which appears in $*$

cuspidal repr " in $**$.

Fact: π automorphic repr $\rightsquigarrow$ $\pi = \bigotimes'_{v \in \Sigma_v} \pi_v$

restrict product w.r.t. $(\mathcal{H}_v, K_v)$

$\mathcal{H}_v$: loc. const. measure on $G(F_v)$, v non-arch (inv. by K_v)

K_v: G/F_v is un-ram. quasi-split and split over an un-ram. ext'n of F_v

$\rightsquigarrow$ K_v = hyperspecial max'l cpt.

For almost all v, π_v will be unramified

$\Rightarrow$ $V_v^{K_v}$ is 1-dim'l

V_v: the repr'n space

$G/F \rightsquigarrow G/\mathcal{O}_F \rightsquigarrow G(\mathcal{O}_v)$.

Basic Question:

Given admissible repr $\pi = \bigotimes'_{v \in \Sigma_v} \pi_v$

is it a (cuspidal) auto. repr'n?

e.g. $X_{/F} \rightsquigarrow \{ H^i_e(X, \overline{\mathbb{Q}_\ell}) \}_\ell$ it is admissible, in general it's very easy to get admi. ones, but very hard to get cuspidal one.

π	π_v
• L-functions	local L-factors
• functional Eq'n	functional Eq'n
• constant for F.E.	local const. for F.E.

$$\prod_v \text{local} \begin{pmatrix} L\text{-factor} \\ \varepsilon\text{-factor} \end{pmatrix} = \text{global} \begin{matrix} L \\ \varepsilon \end{matrix} \text{ factors}$$

Point: can be read off from the parameters.

Recipe: $\pi_v \rightsquigarrow r \circ \varphi_v$; $L_F \xrightarrow{\rho} GL(V)$, $L_F \to {}^LG \xrightarrow{r} GL(V)$

(local)

$L_F = W_F'$ or $W_F \times SU_2(\mathbb{R})$

omitting some additive character

$\rho_v' = (\rho, N)$ Nilp. op. from Lie algebra.

$$L(\rho_v', s) = \det\left(Id - q_v^{-s} \Phi_v \Big|_{V_N^{I_v}} \right)^{-1}$$

fixed by inertial gp

killed by monodromy

$$\varepsilon(\rho_v', s) = \varepsilon(V)\, q^{-a(V)s}$$

- $a(V) = a(\rho) + \dim(V^I) - \dim(V_N^I)$
- $\varepsilon(V) = \varepsilon(\rho) \cdot \det\left(-\Phi \big| (V^I / V_N^I)\right)$

local const. for ρ (Dwork, Deligne ...)

$L(\pi, s, r) = \prod_v L(\pi_v, s, r)$

expect: Analytic continuation

+ F.E. $L(\pi, s, r) = \varepsilon(\pi, s)\, L(\pi, 1-s, \tilde{r})$.

Properties of Representation and Parameters.

$\pi_v \longleftrightarrow \varphi_v$

spherical repr (unramified principal series)	<u>unramified</u>	I_v operates trivial + $N=0$
( )	<u>supercuspidal</u> (non-arche)	irred. (& $N=0$) (say for GL_n)
essentially L^2 (L^2 after modified by an central char)	(essentially) <u>discrete series</u>	$\operatorname{Im}\varphi_v \not\subset$ any proper F-rat'l Levi
(idea: unitarily induced from discrete series)	<u>essentially tempered</u>	essentially bounded (modified by a char)

⋮

EXAMPLES: GL2

Easy part: $L_{F_v} = W_{F_v} \times SL_2(\mathbb{C}) \xrightarrow[\text{semi-simple}]{\varphi_v} GL_2(\mathbb{C})$

i. irreducible and $SL_2(\mathbb{C})$ operates trivially

ii. $SL_2(\mathbb{C})$ operates irreducibly already

$\Rightarrow W_{F_v}$ acts via an abelian character iii • $\mu \oplus \nu$, $SL_2(\mathbb{C})$ operate trivially.

(i. ⊇ dihedral type say.)

Back. from parameters $\longrightarrow$ repr'n.

iii. $\mu \oplus \nu$: $\mu, \nu : F_v^\times \longrightarrow \mathbb{C}^\times$ (by C.F.T.)

induction : $B : \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \xrightarrow[\mu,\nu]{} \mathbb{C}^\times$

$$\rho(\mu,\nu) = \mathrm{Ind}_B^G(\mu \oplus \nu) = \left\{ \begin{array}{l} f : G \to \mathbb{C} \\ \text{smooth} \end{array} \middle| \; f\left(\begin{pmatrix} u & * \\ 0 & v \end{pmatrix} \cdot x\right) = \mu(u)\cdot\nu(v) \cdot \left\|\frac{u}{v}\right\|^{1/2} \cdot f(x) \right\}$$

∞-dim'l repr of G, admissible. quasi-Haar measure $\|\rho_G\|^{1/2}$

- For non-archemedean.

most of the time $\rho(\mu,\nu)$ is irreducible

$\rightsquigarrow$ take $\pi(\mu,\nu) = \rho(\mu,\nu)$

If reducible $\rightsquigarrow$ 2 irreducible subquotient

one ∞-dim'l

one finite dim'l $= \pi(\mu,\nu)$,

- Non-arche. $\rho(\mu,\nu)$ reducible $\Leftrightarrow \mu\nu^{-1} = |\cdot|_F^{\pm 1}$

In this case

$\sigma(\mu,\nu) =$ up to twist by 1-dim char

the (a) special repr'n.

ii. $SL_2(\mathbb{C})$. has a special repr $sp_2(\rho, N)$

e_0, e_1

W : $e_i \mapsto \|w\|^i e_0$, $e_0 \xrightarrow{N} e_1 \longrightarrow 0$

General fact for $G = GL_n$:

$$\det(\varphi_v) \xleftrightarrow[CFT]{} \text{central character.}$$

$\mu\nu^{-1} = |\cdot|^{\pm 1}$ unramified

$\Rightarrow$ $\pi(\mu,\nu) = $ 1-dim'l

$\pi(\|\cdot\|^{\frac{1}{2}}, \|\cdot\|^{\frac{1}{2}}) = $ trivial repr of $GL_2(F_v)$.

For ii) only occur when non-arch.
so the only important thing is for i. 2 dihedral :

$\rightarrow$ <u>Weil representation</u>

K/F sep. quad. $\theta : K^\times \longrightarrow \mathbb{C}^\times$

$$\begin{tikzcd} K^\times \arrow[r] \arrow[d, "S"'] & \mathbb{C}^\times \\ W_K^{ab} \arrow[ur] & \end{tikzcd}$$

auxiliary $\psi : (F_v, +) \to \mathbb{C}^\times$ non-trivial

$\rightsquigarrow$ $\pi(\mathrm{Ind}_{W_K}^{W_F} \theta)$ constructed using the "Weil repr"

$\{ f : K^\times \to \mathbb{C} \mid f(t^{-1}x) = \theta(t) f(x) \ \forall t, x \in K^\times \}$

$\rightsquigarrow r(\theta,\psi)$, $G_1 \circlearrowleft$

$\cap$ index 2

$G(F_v)$

$\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} f \right)(x)$

$= \psi(u) N_{K/F}(x) f(x)$

$\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f \right)(x)$

$=$ unital part of $\varepsilon(\hat{\sigma}, s)$ $\int_K f(y) \psi_{K/F}(x y^\sigma) \, d_\psi y$

$$\pi(\mathrm{Ind}_{W_K}^{W_F} \theta) := \mathrm{Ind}_{G_1}^{G(F_v)} (r(\theta,\psi)).$$

· irred. if $\theta \neq {}^\sigma\theta$ by an auto σ of order 2.

This is roughly the <u>matching process</u> goes.

i). called supercuspidal, $\supseteq$ dihedral

ii). called special Steinberg "=" unless char $K = 2$.

Global Situation: $GL_2/\mathbb{Q}$ say.

- classical holomorphic modular forms
- Maaß forms (Non-holomorphic)

Representation theoretic way:

$\mathcal{F} := \{$ functions on adelic lattices in $\mathbb{C}\} =$

$$GL_2(\mathbb{Q}) \backslash GL_{\mathbb{R}}(\mathbb{C}) \times \prod_p' GL_2(\mathbb{A}_{\mathbb{Q},f})$$

$GL_{\mathbb{R}}(\mathbb{C}) \cong GL_2(\mathbb{R})$

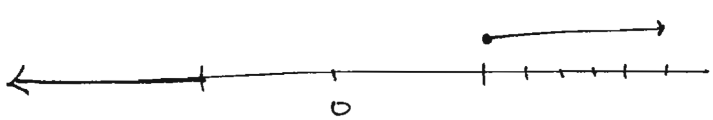
$$\mathrm{Hom}_{GL_2(\mathbb{R})}(\pi_{\mathbb{R}}, \mathcal{F}) = \begin{cases} \text{holo. modular forms.} \\ \text{Maaß forms.} \end{cases}$$

canonically, intertwinding operator evaluating at a specific generator of $\pi_{\mathbb{R}}$.

In $\mathcal{F}$, if $\pi = \pi_{\mathbb{R}} \otimes \pi_f$ occurs $\rightsquigarrow$ get holo. mod. forms / Maaß forms.

$\pi_f = \otimes' \pi_p$

Roughly: For $\pi_{\mathbb{R}} =$ a discrete series repr of $GL_2(\mathbb{R})$ with a specific H-C module D_{k-1} with a basis indexed by a subset of $\mathbb{Z}$



$$GL_2(\mathbb{R}) \downarrow \mathbb{C} - \mathbb{R} = \mathbb{H}^{\pm}$$

<ins>To be continued.</ins>

Prof. Ching-Li Chai. 7/30.

Lecture IV. (Final Lecture of Summer School)

parameters continued.

$$\Phi_G = \left\{ \begin{array}{l} \text{(locally) relevant "admissible"} \\ L\text{-homomorphisms } L_F \longrightarrow {}^L G \end{array} \right\} / \mathrm{Inn}(\hat{G})$$

{unramified repr of $G(F)$, F local field}

$\xleftrightarrow[1:1]{}$ {unramified parameters in Φ_G}

(in general many to 1)

classically in the case of Galois repr'n, these are conjugacy classes of Frob. at unramified places.

ie.

$$\begin{array}{ccc} X & \subseteq & \mathfrak{X} \\ \text{proper smooth} \downarrow & & \downarrow \text{smooth} \\ \operatorname{Spec} F & \hookrightarrow & \operatorname{Spec} \mathcal{O}_{F,S} \end{array}$$

F: number field
(or global field)

ie. Example from alg. geom. are unramified repr. by Conjecture, should be automorphic forms.

unramified parameters:

$$\begin{array}{ccc} L_F & \longrightarrow & {}^L G \\ \searrow & & \nearrow \\ & W_F \twoheadrightarrow \{\mathrm{Frob}_F^{\mathbb{Z}}\} & \end{array}$$

1. Factor through W_F
2. Acts trivially for inertial

Taking a generator σ

$\{$unram. parameters$\} \leftrightarrow ({}^LG \rtimes \sigma)_{ss} / \mathrm{Inn}(\hat{G})$

$\leftrightarrow$ (expressible in terms of the maximally F-split torus T_F and its Weyl group.) $*$

On the other hand, have Satake homomorphism:

$$\mathcal{H} \xrightarrow{\sim} \mathbb{C}[T_d(F)/_F W] = \mathbb{C}[X_*(T_d)]^{FW}$$

$$\overset{\Psi}{f} \mapsto \left(t \mapsto \delta(t)^{1/2} \int_{u \in U(F)} f(tu)\,du = \delta(t)^{1/2} \int_{u \in U(F)} f(ut)\,du \right)$$

where U = unipotent radical of a maximal F-rational parabolic subgroup.

$\mathcal{H} = \mathcal{H}(G; K)$; K: good max. cpt. sub gp. $\mathcal{H}$: Hecke algebra

T_d : (split part ? of) max torus T.

$\delta(t)$ sum of positive roots. $|\rho_G|$.

this is the dual version of $*$. So, taking dual : art

$* \leftrightarrow$ 1-dim'l characters $\mathcal{H} \to \mathbb{C}$.

on the other hand. (spherical functions (repr'ns))

$\mathcal{H} \longrightarrow \mathbb{C}$ $\qquad \mathrm{Ind}_{P(F)}^{G(F)}(\chi)$ max. parabolic

$\updownarrow \qquad\qquad \uparrow$

$\chi : T(F) \longrightarrow \mathbb{C}^\times$ $\qquad$ irreducible "most of the time"

or $T_d(F)$

This 1-1 correspondence was solved only for local case in case $G = GL_n$. (Recently)

"Importance"

$\pi = \otimes_v \pi_v$ automorphic $\rightsquigarrow \exists S$ finite st. $v \notin S$
π_v is unramified

$\{\pi_v\}_{v\notin S} \longleftrightarrow \{\varphi_v\}_{v\notin S}$ for GL_n, there is only one way to fill in others.

"Functoriality Principle": for packets

$$\begin{tikzcd} {}^L G \arrow[rr, "\alpha"] \arrow[dr] & & {}^L H \arrow[dl] \\ & W_F & \end{tikzcd}$$

homo.

$\pi = \otimes_v \pi_v$ auto. repr of G

$\{\varphi_v\}_{v\notin S} \rightsquigarrow \{\alpha\circ\varphi_v\}_{v\notin S}$

unramified by construction

Question (Weak form): $\exists ?$ $\tilde{\pi} = \otimes_v \tilde{\pi}_v$ auto. repr of H st. $\tilde{\varphi}_v = \alpha\circ\varphi_v$ for almost all places v?

An example of local Langlands classification at arch. places:

$F = \mathbb{C}$, max cpt sp has smaller rank $\left(= \frac{1}{2}\,\mathrm{rk}\right)$

H-C. Thm $\Rightarrow$ no discrete series.

each $\varphi_v \longleftrightarrow$ one repr. a parameter is just

$\mathbb{C}^\times \longrightarrow \hat{G}$

Remark: characters of $\mathbb{C}^\times$ are of the form

$z \mapsto z^\lambda \bar{z}^\mu$, $\lambda - \mu \in \mathbb{Z}$. In fact

$$\begin{tikzcd} \mathbb{C}^\times \arrow[r] \arrow[dr] & \hat{G} \arrow[d, phantom, "\cup"] \\ & \hat{T} \end{tikzcd}$$

$\longleftrightarrow$ $z \mapsto z^\lambda \bar{z}^\mu$, $\lambda, \mu \in X^*(T)\otimes\mathbb{C}$
$\lambda - \mu \in X^*(T)$

$\hat{T}$ max. torus

let P_1 = parabolic sub group of G (standard) containing T st.

$|\hat{\varphi}|\big|_{\mathfrak{a}_{P_1}} \in$ open Weyl chamber.

get a parameter $\varphi_1 : \mathbb{C}^\times \longrightarrow L_1$ (Levi for P_1)

this corr. to a tempered (repr) of L_1
parameter

then use induction procedure from L_1 to ...

This is Historically how Langlands begin his work.

Global Consideration:

Shimura Varieties

$G/\mathbb{Q}$ connected reductive $/\mathbb{Q}$

$\mathbb{C}^\times = \underset{\sim}{\mathbb{C}}^\times(\mathbb{R})$
as alg. gp over $\mathbb{R}$

consider homomorphism: $h : \underset{\sim}{\mathbb{C}}^\times \xrightarrow[/\mathbb{R}]{} G$ st.

$\mathrm{Ad}(h) : \underset{\sim}{\mathbb{C}}^\times \longrightarrow \mathrm{Aut}(\mathfrak{g}_{\mathbb{R}})$ require a real Hodge str.

condition: Hodge type $(-1,1), (0,0), (1,-1)$.

(and some other conditions)

$G(\mathbb{R})$-conjugacy class X of h "is" a finite union of bounded symmetric domains

(not all gp have this property)

data (G, X):

$$Sh(G,X) := \text{"}\varprojlim_K\text{"}\ G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K$$

open cpt subgp of $G(\mathbb{A}_f)$.

has canonical structure as alg. v. / $\mathbb{C}$

Thm (...) $Sh(G,X)$ has a natural structure as a (pro) algebraic variety over a number field (reflex field) $E = E(G,X) \hookrightarrow \mathbb{C}$

$Gal(\bar{\mathbb{Q}}/E)$ = the fixer of the $G/\mathbb{C}$ – conjugacy class of μ_h, any $h \in X$.

Example: $G = GL_2$, $X = \mathbb{C} - \mathbb{R}$, we get projective system of modular curves.

$Sh(G,X) \circlearrowleft G(\mathbb{A}_f)$ almost like a homog. space.
but never has an action of alg. gp
only finite Adelic pts $G(\mathbb{A}_f) = \prod_p{}' \mathbb{Q}_p$.

(does not preserve the finite level, only the whole system is preserved).

$Sh(G,X) \circlearrowleft G(\mathbb{A}_f)$ gives rise to Hecke corr.
$/E = E(G,X)$ on each $Sh(G,X)/K$.

$\rightsquigarrow$ Can use these symmetry to study these alg. v.

For number theory: Pick $H^\cdot$ any coh. theory

$$\circlearrowright H^\cdot(Sh(G,X)) := \varinjlim H(Sh(G,X)/K)$$

$G(\mathbb{A}_f) \leftarrow$ big gp action

$\rightarrow$ decomposition according to repr of $G(\mathbb{A}_f)$.

$$= \bigoplus_{\pi_f} \cdot W_{\pi_f} \otimes V_{\pi_f}$$

finite dim — repr. space of π_f.

For instance, if $H^\cdot = H^\cdot(\ , \mathbb{Q}_\ell)$

then have $\mathrm{Gal}(\bar{\mathbb{Q}}/E) \circlearrowright W_{\pi_f}$ and

$G(\mathbb{A}_f) \circlearrowright V_{\pi_f}$.

Hodge structures appear in W_{π_f}.

How to get information of the cohomology?

Look at arch places: try to "compute"

take $IH^\cdot(Sh(G,X)_{/\mathbb{C}}, V_\tau)$ for $\tau : G \longrightarrow GL(V_\tau)$

$\uparrow S$ — cpx version

thm of Looijenga, Saper-Stern

$H^\cdot_{(2)}(Sh(G,X)(\mathbb{C}), V_\tau)$

this can be done for such inductive system because those maps are (finite) étale.

essential piece of L^2-cochains:

$$\left(\Lambda^\cdot(\mathfrak{g}/\mathfrak{k}) \otimes L^2(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes V_\tau\right)^{K_\infty}$$

take cohomology $= H^\cdot((\mathfrak{g}, K_\infty), V_\tau \otimes L^2(G(\mathbb{Q})\backslash G(\mathbb{A})))$

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})) = \bigoplus_\pi m_\pi V_\pi$$

the question reduces to compute

$$H^\cdot(\mathfrak{g}, K; V_\tau \otimes \pi_{\mathbb{R}} \otimes \pi_f)$$ can taken out

to compute $H^\cdot(\mathfrak{g}, K; V_\tau \otimes \pi_{\mathbb{R}})$ is purely Hodge-th:

(Kuga) $\square = \tau(C) - \pi(C)$, C Casimir op.

Laplacian $= dd^* + d^*d$

so $H^\cdot(\cdots) = 0$ unless $\square = 0$. but $\square = 0$ then get

$$H^\cdot(\cdots) = (\Lambda^\cdot(\mathfrak{g}/K) \otimes V_\tau \otimes L^2(\cdots))^K.$$

Importance of these:

Get Galois Repr that we know something about. (via automorphic repr) eg. Eichler-Shimura relations through Hecke relations.

"secret thoughts":

try to { search for Galois repr among these (Taniyama-Shimura ...)
Build up (approximate) Galois repr to get a "given" one. (or just a piece)

- There is a recipe for computing the IH / Lie alg. coh. of a real packet.

$$\begin{tikzcd} -\mu_h \arrow[r, squiggly] & {}^L\hat{G} \arrow[r, "``\mu"''"] & \mathrm{Aut}(V_\mu) \\ & W_{\mathbb{R}}^\times \times SL(2,\mathbb{C}) \arrow[u, "\psi", "\leftarrow \text{using } \tau"'] \arrow[ur] & \end{tikzcd}$$

both can be read off combinatorially.

↗ will give the all like Hodge, Lefschetz structures.

<u>END</u>