VCTS/TPE lecture.

Birational geom in positive char p

Birkar lect 1 9/4 2013

Main tools of birat'l geom in char 0 $k=\bar{k}$

- resol of sing if X is an var/k then $\exists$ sm proj var Y birat'l to X (Hironaka)
- Kodaira vanishing if X sm proj, A ample div $\Rightarrow H^i(-A)=0$ if $i<\dim X$

How these tools are used? Let X var/k

eg a div $L=m(K_X+B)$, and we like to show L is free
div ≥ 0

The trick is first resolve sing and assume (X,B) is as simple as possible. Try to write

$$L=m(K_X+B)=K_X+S+\text{ample}$$ then
primitive

$$0\to \mathcal{O}_X(L-S)\to \mathcal{O}_X(L)\to \mathcal{O}_S(L|_S)\to 0$$ and

$$H^0(L)\to H^0(L|_S)\to H^1(L-S)\to$$

$$\overset{\shortparallel}{H^1}(K_X+\text{ample})\cong H^{\dim-1}(-\text{ample})^*=0$$

So if $L|_S$ is free on S then L is free near S


X

How about char $p>0$?

- resol of sing known in dim ≤ 3
- Kodaira vanishing simply fails (even in dim 2)

There are other bad things

Generic smoothness char $k=0$

$f:X\to Y$ morphism, X sm $\Rightarrow \exists\, U\subseteq Y$ open st fibers of f over pts in U are sm

failure of generic smoothness in char $p>0$

let $X=V(y^2+x^p+t)\subseteq \mathbb{A}^3$

$Y=\mathbb{A}^1=\operatorname{Spec} k[t]$ $f:X\to Y$ projection onto Y
$(a,b,c)\mapsto c$

Fiber over $c\in\mathbb{A}^1$ is the curve in $\mathbb{A}^2$ by

$g=y^2+x^p+c$, $\frac{\partial g}{\partial x}=0$ $\Rightarrow \exists$ sing pt $\forall c$

$\frac{\partial g}{\partial y}=2y=0$ if $y=0$

Even worse, non-reduced fibers

$X = Y = \mathbb{A}^1 = \mathrm{Spec}[t]$, $f: X \to Y$ $a \mapsto a^p$

i.e. $k[t] \to k[t]$, $t \mapsto t^p$

assume $a \in Y$ and $b^p = a$, then fiber is given by $(t-b)^p$

But char p also has advantages

The Frobenius : X scheme $/K$ F $X \to X$ is given by $\begin{cases} \text{identity on points} \\ f \mapsto f^p \text{ on the structure sheaf } \mathcal{O}_X. \end{cases}$

This is the <u>absolute</u> Frobenius

Also have <u>geometric</u> Frobenius

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr] \arrow[d] & & X \arrow[d] \\ & X^{(p)} \arrow[ur] \arrow[dl] & \\ \mathrm{Spec}\,k \arrow[rr, "F"] & & \mathrm{Spec}\,k \end{tikzcd}$$

$X^{(p)} = \mathrm{Spec}\,k \times_{\mathrm{Spec}\,k} X$

The morphism $X \to X^{(p)}$ is called the geometric Frob

As a scheme $X^{(p)}$ is just X but its structure $/k$ is very different

Example $X = \mathbb{A}^1 = \mathrm{Spec}\,k[t]$, corr to the diag we have

$$\begin{tikzcd} k[t] & & k[t] \arrow[ll, "f^p \leftmapsto f"'] \arrow[dl] \\ & k[t] \arrow[ul, "t^p \leftmapsto t"'] & \\ k \arrow[uu] \arrow[ur] & & k \arrow[ll, "a^p \leftmapsto a"] \arrow[uu] \end{tikzcd}$$

$\sum a_i t^i \mapsto \sum a_i^p t^i$

i.e. $X \to X^{(p)}$ is just the map change the coor $x \mapsto x^p$

Frobenius can be used to deal with <u>thickening</u> of subschemes X sch$/k$, D an effective Cartier div (D can be considered as closed subscheme of X)

The pull back $F^*D = pD$, thicker then D $\quad X \xrightarrow{F} X \supset D$

suppose L is also a Cartier div $F^*L = pL$

we have the diagram

$$\begin{tikzcd} D \arrow[r, hook] \arrow[dr, "F"'] & pD \arrow[r, hook] \arrow[d, "\pi"] & X \arrow[d, "F"] & pL \\ & D \arrow[r, hook] & X & \end{tikzcd}$$

suppose $L|_D$ is free (§6.5). π induces a map

$H^0(L|_D) \to H^0(pL|_{pD}) \Rightarrow pL|_{pD}$ is also free.

More generally: if Z, Z' closed subschemes st

$Z \subset Z'$ and $Z_{red} = Z'_{red}$

Then $\exists e > 0$ st we have a factorization

$$\begin{tikzcd} Z \arrow[rr, "F^e"] \arrow[dr, hook] & & Z \\ & Z' \arrow[ur] & \end{tikzcd}$$

So if $L|_Z$ is free, then $\Rightarrow p^e L|_{Z'}$ is also free

Back to X sm proj var$/k$, char $k = 0$

Conjecture X is covered by rational curves (image of $\mathbb{P}^1$)

$\Leftrightarrow \kappa(X) < 0$
ie $\forall m > 0$, $H^0(mK_X) = 0$

(known in dim ≤ 3)

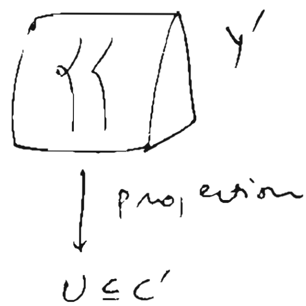
This is NOT true in char $p > 0$

Example $p = 3$ Let $C \subseteq \mathbb{P}^2$ be an elliptic curve given st $C' = C \cap \mathbb{A}^2$ is $\underset{g=}{s^2 - t^3 + t} = 0$

Let $Y' = V(\underset{f}{y^2 z + x^3 + t z^3}) \subseteq \mathbb{P}^2_{C'} = \mathbb{P}^2 \times C'$

$\exists\, U \underset{\text{open}}{\subseteq} C'$ st if $y \in Y'$ is mapped into U then y is sm (as a point of Y'), and every fiber over pts in U has a singular point

The fiber over $(a,b) \in U$ is given by $y^2 z + x^3 + b z^3$



if $z = 1$ $\begin{cases} \frac{\partial}{\partial x} = 0 \\ \frac{\partial}{\partial y} = 2y = 0 \text{ if } y = 0 \end{cases}$

To see inverse of U in Y' is smooth use the fact where $Y' \subseteq \mathbb{P}^2 \times \mathbb{A}^2$ given by f, g

Let $Y \subseteq \mathbb{P}^2 \times C$ be the closure of Y'

Let $X \to Y$ be a resol of sing st it does not modify sm points of Y ($\Rightarrow X \to Y$ isom over U)

Let $\pi : X \to C$ be the induced morphism

if $u \in U \Rightarrow$ fiber of $\pi/_u$ is a singular rat'l curve (arithmetic genus of fiber $= 1$, hence rat'l)

If G is a general fiber of $\pi \Rightarrow K_G \sim 0$ because G is defined $\subseteq \mathbb{P}^2$ by eq^n of deg 3 then by adjunction.

p.4 One can show that $\kappa(X) \geq 0$

ie $\exists m$ $H^0(mK_X) \neq 0$

- can use classification
- or use Iitaka $\kappa(X) \geq \kappa(K_G) + \kappa(C) \geq 0$

[Yifei Chen, Zheng]

Lect 2. Birkar . 9/6 2013 p.5

Surfaces MMP & classification $k=\bar{k}$ char $p>0$

Pair (X,B) consists of normal var$/k$: X
& $\mathbb{Q}$-div $B=\sum b_i B_i$, $b_i\in\mathbb{Q}\cap[0,1]$ st K_X+B $\mathbb{Q}$-Cartier

we say (X,B) is log canonical if $\forall$ bir $Y\xrightarrow{f}X$ normal proper and write
$$K_Y+B_Y=f^*(K_X+B)$$
then coefficients of B_Y to be ≤ 1

eg. $\mathbb{A}^2$ lc, not lc. not lc

Adjunction formula (X,B) a pair, S comp of B with coeff $=1$

classical adjunction: let $\dim X=2$

X sm S sm $\Rightarrow K_S=(K_X+S)|_S$

more generally, we can take a resolution

$Y\xrightarrow{f}X$ st

$$\begin{array}{ccc} S^\nu & \hookrightarrow & Y \\ {\scriptstyle\text{normalization}}\downarrow & & \downarrow \\ S & \hookrightarrow & X \end{array}$$

st if $K_Y+B_Y=f^*(K_X+B)$ $\Rightarrow B_Y\geq 0$

Now $(K_Y+B_Y)|_{S^\nu}=(K_Y+S^\nu+B_Y-S^\nu)|_{S^\nu}$
$$=K_{S^\nu}+(B_Y-S^\nu)|_{S^\nu}=:K_{S^\nu}+B_{S^\nu}$$
with $B_{S^\nu}\geq 0$

Rmk If (X,B) lc $\Rightarrow(S^\nu,B_{S^\nu})$ is lc

Now $K_{S^\nu}+B_{S^\nu}$ = pull back of K_X+B

Kodaira dim. X normal proj, D Weil div
the Kodaira dim of D is the largest number $\kappa(D)$
st $\limsup_{m\to\infty}\frac{h^0(mD)}{m^{\kappa(D)}}>0$ $(\Rightarrow\kappa(D)\in\{-\infty,0,1,\dots,\dim X\})$

Minimal model and Mori fiber space

(X,B) a proj pair, suppose φ $X\dashrightarrow Y$
is a birational map (Y normal), $B_Y=\varphi_*B$

Assume (Y,B_Y) is a pair.

We say that (Y,B_Y) is a minimal model if
K_Y+B_Y is "positive" (i.e nef)

a ~~Mori fiber space~~ if (Mfs)

$\exists$ fib $Y \to Z$ st $K_X + B_Y$ is "negative" on the fibers

The goal of birat'l geom is to look for such special models (fib means $\dim Y > \dim Z$)

Thm (MMP for surfaces) (X, B) proj, of dim 2 for simplicity assume X sm Then $\exists$ MMP which ends up with a minimal model or Mfs (Y, B_Y)

Sketch If $K_X + B$ is nef $\Rightarrow$ done

Assume $K_X + B$ is not nef, ie $\exists$ curve C st $(K_X+B) C < 0$ Pick an ample div A

let t be the smallest number st $\underset{L}{K_X + B + tA}$ is nef

Actually $t \in \mathbb{Q}$ (see lect 5)

<u>Assume</u> L is <u>semi-ample</u> (ie mL is g.b.g.s for some $m>0$)

So $\exists$ morphism $f: X \to X'$ st $L = f^*(\text{ample})$

- L is big $\Leftrightarrow L^2 > 0 \Leftrightarrow$ f is birat'l
  in this case, put $B' = f_* B$, continue with (X', B')
- L is not big $\Leftrightarrow L^2 = 0 \Leftrightarrow$ f is a fibration
  in this case put $Z = X'$, so get Mfs $X \to Z$
  since $K_X + B$ is negative on curves in fibers

So the main question is to show semi-ampleness :

Assume L is big ' need

Keel's semi-ample thm (see lect 3)

X proj scheme /k & L $\mathbb{Q}$-Cartier div, nef

Define $\mathbb{E}(L) = \bigcup_{L|_V \text{ is not big}} V$, $V \subseteq X$ integral, $\dim > 0$

L is semi-ample $\Leftrightarrow L|_{\mathbb{E}(L)}$ is semi-ample

Rmk Will see $\mathbb{E}(L)$ is in fact a closed set /

This holds only in char $p > 0$ fail for char 0

pf really uses Frobenius and thickening technique.

Back to the pf X is of dim 2, L is big

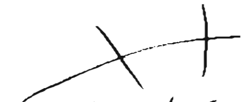
so $\mathbb{E}(L)$ is a finite collection of curves

enough to show $L|_{\mathbb{E}(L)}$ is semi-ample

we can change the setting st we could assume

the connected components of $\mathbb{E}(L)$ are irreducible

Pick a component C of $\mathbb{E}(L)$



L is big $\Rightarrow$ we can write

$L \sim_{\mathbb{Q}} D \geq 0$ st C is a comp of D — need some trick to get rid of 1 by 1

Now let $\alpha \geq 0$ be the number st

the coeff of C in $\underbrace{B + tA + \alpha D}_{\Delta}$ to be 1

So by adjunction, $\exists\ \Delta_{C^\vee} \geq 0$ st

pull back of $K_X + \Delta$ to $C^\vee$ is $K_{C^\vee} + \Delta_{C^\vee}$

Since $C \in \mathbb{E}(L) \Rightarrow L.C = 0 \Rightarrow (K_X+\Delta)\,C = 0$

$\Rightarrow \deg(K_{C^\vee} + \Delta_{C^\vee}) = 0 \Rightarrow \deg K_{C^\vee} < 0 \Rightarrow C^\vee \simeq \mathbb{P}^1$

Even more $\quad C^\vee \xrightarrow{f} C$

if $P_a(C) > 0 \Rightarrow \deg \Delta_{C^\vee} \geq 2$, which is not possible

$\Rightarrow C^\vee \to C$ isom $\Rightarrow C \simeq \mathbb{P}^1 \Rightarrow L|_{C=\mathbb{P}^1}$ is semi-ample (torsion) $(\equiv 0)$

$\Rightarrow L$ is semi-ample

if $\kappa(L) = 1 \Rightarrow \exists$ fib $X \xrightarrow[\pi]{} Z$ st $L \sim_{\mathbb{Q}} \pi^* M \Rightarrow \pi$ is a Mfs

if $\kappa(L) = 0 \Rightarrow$ need to use quite different method

Finally need to show $\kappa(L) \neq -\infty$

classification (Mumford & Bombieri)

(X, B) proj of dim 2 Assume (Y, B_Y) is a minimal model or Mfs

$\kappa(K_X + B) = -\infty = \kappa(K_Y + B_Y) \Rightarrow \exists$ Mfs $Y \xrightarrow{\text{fib}} Z$

(in classical case, if X = sm, $B = 0 \Rightarrow$

$Z = pt \Rightarrow Y = \mathbb{P}^2$

Z = curve $\Rightarrow Y$ $\mathbb{P}^1$-bundle over Z)

8 $\kappa(K_X+B) = 0 = \kappa(K_Y+B_Y) \Rightarrow K_X+B_Y \underset{\mathbb{Q}}{\sim} 0$
torsion

(X sm, $B=0$, Y can be an ab var K3 surface)

$\kappa(K_X+B) = 1 = \kappa(K_Y+B_Y) \Rightarrow \exists$ fib $Y \to S$
with fibers of $p_a = 1$ (ie elliptic curve or singular rat'l curve)

$\kappa(K_X+B) = 2 = \kappa(K_Y+B_Y) \Rightarrow$

$\exists$ morphism $Y \underset{\text{bir'l}}{\to} Y'$ st K_Y+B_Y = pull back of ample

In last 3 cases, K_Y+B_Y is semi-ample
(the abundance thm)

$k=\bar{k}$, char $=p>0$ Keel's basept free thm and application in cone thm

Rmk Suppose X is a proj var$/k$ & L is a big Cartier div ($h^0(mL)$ grows like $m^{\dim X}$, $m\to\infty$)

Suppose $S \subsetneq X$ is a closed subscheme, consider

$$0\to \mathcal{I}_S \to \mathcal{O}_X \to \mathcal{O}_S \to 0$$

$$0\to \mathcal{I}_S(mL)\to \mathcal{O}_X(mL)\to \mathcal{O}_S(mL)\to 0 \quad \text{get}$$

$$0\to H^0(\mathcal{I}_S(mL))\to H^0(\mathcal{O}_X(mL))\to H^0(\mathcal{O}_S(mL))$$

$\Rightarrow$ dim $H^0(\mathcal{I}_S(mL))$ very large $m\gg 0$ (can grow at most like $m^{\dim S}$)

so $\exists$ many sections vanishes on S

Keel's thm L semi-ample $\Leftrightarrow$ $L|_{\mathbb{E}(L)}$ is semi-ample

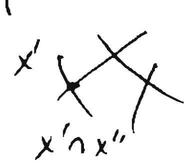
if instead, L is nef then

$$\mathbb{E}(L)=\bigcup_{L|_V \text{ not big}} V, \quad V \text{ integral} \subseteq X$$

pf: (simplified, weaker, version of Cascini-McKernan-Mustata)

- If L is semi-ample $\Rightarrow$ $L|_{\mathbb{E}(L)}$ is
- Now suppose $L|_{\mathbb{E}(L)}$ is s.a. by lect 1 may assume X is reduced.
- Suppose first that $L|$ any comp of X is big
- pick X' of X, $X''=$ union of other comp

by Noetherian induction, may assume the theorem holds for proper closed subscheme $\subseteq X$ By induction on # of components



may assume $\exists\ \alpha'\in H^0(mL-A|_{X'})$ st $0=\alpha'|_{X'\cap X''}=\alpha''|_{X'\cap X''}=0$

$\alpha''\in H^0(mL-A|_{X''})$ A ample

st α' & α'' don't vanish on any ~~other~~ whole comp of X' & X''

we then use $0\to \mathcal{O}_X\to \mathcal{O}_{X'}\oplus \mathcal{O}_{X''}\to \mathcal{O}_{X'\cap X''}\to 0$

& $0\to \mathcal{O}_X(mL)\to \mathcal{O}_{X'}(mL-A)\oplus \mathcal{O}_{X''}(mL-A)\to \mathcal{O}_{X'\cap X''}(mL-A)\to 0$

$\Rightarrow \exists \alpha \in H^0(\mathcal{O}_X(mL-A))$ st $\alpha|_{X'} = \alpha'$, $\alpha|_{X''} = \alpha''$

Moreover, α does not vanish on comp of X

α corr to some morphism $\mathcal{O}_X \to \mathcal{O}_X(mL-A)$ get

$\mathcal{O}_X(-mL+A) \to \mathcal{O}_X$

Also X is reduced, $\mathcal{O}_X(-mL+A) \to \mathcal{O}_X$ is injective

so $\mathcal{O}_X(-mL+A)$ is an ideal sheaf defining a eff div

Cartier div D (D is the zero scheme of α)

$D \sim mL - A$, A ample $\Rightarrow \mathbb{E}(L) \subseteq D$

By Noetherian induction, $L|_D$ s a.

if $V \nsubseteq D \Rightarrow L|_V \sim A|_V + D|_V$ big

For $n > 0$ we have $0 \to \mathcal{O}_X(nmL - nD) \to \mathcal{O}_X(nmL) \to \mathcal{O}_{nD}(nmL) \to 0$

Since $mL \sim A + D$, $nmL \sim nA + nD$

$\Rightarrow nmL - nD \sim nA$

So if $n \gg 0$, by Serre vanishing $H^1(\mathcal{O}_X(nmL - nD)) = 0$

Amazing part: By Fujita vanishing, we also have $H^1(\mathcal{O}_X(\ell L + nmL - nD)) = 0$ $(n \gg 0, \ell > 0)$

i.e. Serre vanishing can be added by a nef div

$\Rightarrow$ exact sequence

$H^0(\mathcal{O}_X(\ell L + nmL)) \to H^0(\mathcal{O}_{nD}(\ell L + nmL)) \to H^1(\mathcal{O}_X(\ell L + nmL - nD))$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{0}$

Since $L|_D \Rightarrow L|_{nD}$ is semi-ample, we can lift the semi-ampleness to X

$\Rightarrow L$ is semi-ample #

Now assume $L|$ some comp of X is not big

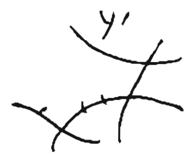
let $Y' =$ union of such comps

let $Y'' =$ union of other comp

If $Y'' = \phi \Rightarrow$ then already holds because

$\mathbb{E}(L) = Y' = X$, So can assume $Y'' \neq \phi$

Similar to argument before, $\exists\ \beta' \in H^0(mL - A|_{Y'})$ $\overset{P}{\underset{0}{\|}}$
$\exists\ \beta'' \in H^0(mL - A|_{Y''})$ st
$\beta'|_{Y' \cap Y''} = \beta''|_{Y' \cap Y''}$
β'' does not vanish on any comp of Y''
$\Rightarrow$ we get $\beta \in H^0(mL - A)$ st
$\beta|_{Y'} = \beta'$ & $\beta|_{Y''} = \beta''$ & β does not vanish on any comp of Y''
Let $Z \subseteq X$ be the zero subscheme of β
by assumption, $Y' \subseteq Z$,



we know $\mathbb{E}(L) \subseteq Z$
by induction, $L|_Z$ is s a

Let $\gamma \in H^0(tL|_Z)$, let $\gamma' = \gamma|_{Y'}$. Now
$Z \cap Y''$ is an eff Cartier div $M \subseteq Y''$
Now $\gamma^{\otimes s}|_{nM}$ can be lifted to Y'' for some $n > 0$
By the vanishing thm discussed, if $n \gg 0$
we can lift $\gamma^{\otimes s}$ to an element $\gamma'' \in H^0(sL|_{nM})$.
Now we can glue γ'' & $\gamma'^{\otimes s}$
$\Rightarrow$ get a section $\lambda \in H^0(sL)$
Since $L|_Z$ is s a & L is s a outside Z
$\Rightarrow$ L is semi-ample $\square$

Rmk: Suppose $k = \overline{\mathbb{F}_p}$, X proj $/k$, L Cartier div st $L \equiv 0$ (numerical trivial ie $L.C = 0$ for any curve $C \subseteq X$)
Usual difficulty is to show L is torsion
this usually fails but for finite field can do great thing
Would like to show L is s.a, $\Leftrightarrow$ L torsion
can consider $L \in \mathrm{Pic}^0(X)$, $\exists$ finite subfield $k' \subset k$ st. X and L are both defined over k'
So, $\exists$ proj sch X'/k' & L'/k' st. X & L are obtained by extension from k' to k

'12 So enough to show L' is torsion

$L' \in Pic^0(X)$. Now $Pic^0(X)$ is of finite type $/k'$

$\Rightarrow$ it has only finitely many k' rat-l pts

L' is one of these pts $\Rightarrow L'$ is torsion $\Rightarrow L$ torsion

Example Suppose X is proj surf $/k = \bar{\mathbb{F}}_p$ &

L a nef Cartier div If $L \equiv 0$, then $\Rightarrow L$ s a

But in fact if L is big $\Rightarrow L$ is semi-ample !!

L is big, $E(L) \subsetneq X \Rightarrow E(L) = \phi$ or $E(L)$ has dim 1

If $E(L) = \phi \Rightarrow L$ is ample

If $\dim E(L) = 1 \Rightarrow L|_{E(L)} \equiv 0$

$\Rightarrow L|_{E(L)}$ is s a $\Rightarrow L$ is s a.

(If $L \not\equiv 0$ & L is not big then L may not be s. a

eg [Tantaro])

Thm (3-folds) Suppose (X, B) is a pair of dim 3 over $k = \bar{\mathbb{F}}_p$. Assume $L = K_X + B + A$ is nef & big where A is ample ($\mathbb{Q}$-div), Then L is s. a.

(not known if $k \neq \bar{\mathbb{F}}_p$ or $k \neq \mathbb{C}$, but it's expected)

pf L big $\Rightarrow mL \sim G + D$ with $D \geq 0$ and $G \geq 0$ ample.

Now $E(L) \subseteq Supp(D)$, So it's enough to show that $L|_D$ is s a

Let $D = D' + D'' + D'''$ st restriction of L to any comp S of D' (resp D'') (resp D'''), $L|_S \equiv 0$ ($L|_S \not\equiv 0$ not big) ($L|_S$ big)

by the previous, $L|_{D'}$ & $L|_{D'''}$ are both s. a

Let S be a comp of D'', Take $\alpha \geq 0$, st coeff of S in $\underbrace{B + A + \alpha G + \alpha D}_{\Delta =}$ is $= 1$

On the normalization S^ν of S, we can apply adjunction

$K_{S^\nu} + \Delta_{S^\nu} =$ pull back of $K_X + \Delta$

$\Delta_{S^\nu} \geq$ pullback of A

On some resolution of S^ν apply RR

we get lots of sections of pull back of $n(K_{S^\nu} + \Delta_{S^\nu})$

$\Rightarrow$ pull back on the rest is semi-ample for easy reasons. Since $k = \bar{\mathbb{F}}_p$, we can glue all these semi-ampleness to get semi-ampleness of $L|_{D''}$ & $L|_{D'}$

Notice that in general field can't glue s a'ness even for $\mathbb{P}^1 \wr \mathbb{P}^1$ $L|$ each $\mathbb{P}^1$ trivial but not on $\wr$ $\not\exists$

Assume $k = \bar{\mathbb{F}}_p$, and (X, B) a pair, proj

Assume $K_X + B$ is not nef, we can take an ample A st. $\underset{L}{=}K_X + B + A$ is nef but $L - \varepsilon A$ is not for any $\varepsilon > 0$

if L s.a $\Rightarrow$ we get non-trivial morphism $X \to X'$

if dim $X = 3$, and if L is big $(\kappa(K_X+B) \geq 0)$

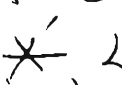
By thm L is s a $\Rightarrow$ get $X \to X'$ can continue to work with X'

'14 lect 4 (Birkar)

F singularities & rel with char 0

log can & klt:

(X,B) a pair, it is log can (klt) if $\forall$ proj birat'l morphism $f: Y \to X$ & if we write
$K_Y + B_Y = f^*(K_X+B)$, the coeff of $B_Y \le 1$ (<1)
(if $\exists$ resol of sing then just need to check one)

Eg (X,B), $X = \mathbb{A}^2$, lc for $B =$  , 
not lc for $B =$ 
indeed, X sm, B SNC $\Rightarrow$ lc (assuming resol of sing)

Example: $C \subseteq \mathbb{P}^n$ sm proj
let X be cone over C — cone  $0 \in \mathbb{A}^{n+1}$ $\downarrow$ $\mathbb{P}^n$ C
it is defined in $\mathbb{A}^{n+1}$ by the same equations as C
We can blow up the vertex
$Y \to X$, we get the exceptional div $E \simeq C$ (Hartshorne)
if $g(C) \ge 2 \Rightarrow (X, B=0)$ is not lc in fact not even a pair since K_X is not $\mathbb{Q}$-Cartier
If $g(C) = 1 \Rightarrow$ if we write $K_Y + B_Y = f^*K_X \Rightarrow$ coeff of E in $B_Y = 1$ ($B_Y = E$, by adjunction, $K_Y+E|_E \sim K_E \sim 0$)
$\Rightarrow (X,0)$ is lc, but not klt.
if $g(C) = 0 \Rightarrow (X,0)$ is klt

Now go to char $p > 0$ F-singularities klt

Frobenius splitting X normal var, $F: X \to X$
the absolute Frobenius $F^e = F \circ F \circ \cdots \circ F$ e times
when the natural morphism
$$\mathcal{O}_X \to F^e_* \mathcal{O}_X$$
splits globally? ie $\exists$? $\mathcal{O}_X \longrightarrow F^e_*\mathcal{O}_X \to \mathcal{O}_X$, composite $= \mathrm{id}$

Example: $X = \mathbb{P}^1$, $F^e_*\mathcal{O}_X = \mathcal{O}_X(n_1) \oplus \cdots \oplus \mathcal{O}_X(n_{p^e})$
since $H^0(F^e_*\mathcal{O}_X) = k$, can assume $n_1 = 0$, $n_i < 0$ $(i>1)$
since $H^1(F^e_*\mathcal{O}_X) = 0 \Rightarrow F^e_*\mathcal{O}_X = \mathcal{O}_X \oplus \mathcal{O}_X(-1)^{p^e-1}$, splits

Example: X supersingular elliptic curve;
i.e. $H^1(\mathcal{O}_X) \to H^1(F_*\mathcal{O}_X)$ is zero (see [Hartshorne])
$\Rightarrow$ $\mathcal{O}_X \to F_*\mathcal{O}_X$ does not split

Example X sm proj curve $g(X) \geq 2$, $x \in X$ closed
pf if $\mathcal{O}_X \to F^e_*\mathcal{O}_X$ splits, then
$\mathcal{O}_X(x) \to (F^e_*\mathcal{O}_X)(x) = F^e_*\mathcal{O}_X(p^e x)$ splits
$\Rightarrow$ $H^1(\mathcal{O}_X(x)) \to H^1(F^e_*\mathcal{O}_X(p^e x))$ is injective
R.R shows $H^1(\mathcal{O}_X(x)) \neq 0$. but $H^1(F^e_*\mathcal{O}_X(p^e x)) = 0$ for e large
hence $\mathcal{O}_X \to F^e_*\mathcal{O}_X$ does not split if $e \gg 0$

Thm (Kunz) X var then X sm $\Leftrightarrow$ $F^e_*\mathcal{O}_X$ is loc. free $\forall e > 0$
($\Rightarrow$ is easy since F^e is then flat).

F-singularities (X, B) a pair
we have a morphism $\mathcal{O}_X \xrightarrow{\varphi} F^e_*\mathcal{O}_X(\lceil (p^e-1)B \rceil)$
we say that (X,B) is <u>sharply F-pure</u> at x if $\exists e$
and if φ splits locally at x, i.e. split when localize at x
we say (X,B) is <u>strongly F-regular</u> if $\forall E \geq 0$
$\exists e > 0$ st the morphism
$$\mathcal{O}_X \to F^e_*\mathcal{O}_X(\lceil (p^e-1)B \rceil + E) \text{ splits at } x$$

Example $X = \mathbb{A}^1 = \operatorname{Spec} k[t]$, $B = 0$
The morphism $\mathcal{O}_X \to F_*\mathcal{O}_X$ corr to $k[t] \to k[t]$, $f \mapsto f^p$
we have a diagram

$$\begin{tikzcd} k[t] \arrow[rr, "f \mapsto f^p"] \arrow[dr, "\alpha"] & & k[t] \\ & k[t] \arrow[ur, "\beta"] & \end{tikzcd}$$

$\sum a_i t^i \mapsto \sum a_i^p t^i$, $t \mapsto t^p$

α is a bi-jection and β makes $k[t]$ a free module$/k[t]$ with basis $1, t, \dots, t^{p-1}$

$\Rightarrow$ β splits $\Rightarrow$ f splits $\Rightarrow$ $\mathbb{A}^1$ is sharply F-pure

More interesting example X sm, fix $x \in X$ closed
we want to show X is sharply F-pure at x
Enough to show $(\mathcal{O}_X)_x \to (F^e_*\mathcal{O}_X)_x$ and in turn it is enough to show $\mathcal{O}_X \to F^e_*\mathcal{O}_X$ splits
Put $R = \mathcal{O}_{X,x} := (\mathcal{O}_X)_x$ a local regular ring of dim $= \dim X$

p.16 we have the exact sequence

$$0 \to R \to F^e_* R \to N \to 0$$

enough to show N is a free R-module
let $t \in R$ be an element of a system of local params
at x, Now $R/(x)$ is a reg local ring of $\dim = \dim X - 1$
we can use the induction and Nakayama lemma
a basis of N/tN lifts to a basis of N
(R is an UFD) $\Rightarrow$ R is Frobenius splits $\Rightarrow$
X is sharply F-pure at $x \in X$

Rem X normal var, $B \geq 0$ some $\mathbb{Q}$-~~Cartier~~ div
st $(K_X + B)$ is $\mathbb{Q}$-Cartier
If (X, B) is sharply F-pure $\Rightarrow$ coeff of $B \leq 1$:

By def'n $\mathcal{O}_X \to F^e_* \mathcal{O}_X(\lceil (p^e - 1) B \rceil)$ splits at every pt
Assume $\exists$ comp of S of B with coeff $b > 1$
could assume $B = bS$, and X smooth
after replacing e with some multiple, can
assume $(p^e - 1) b \geq p^e \Rightarrow \lceil (p^e - 1) b S|_B \rceil \geq p^e S$
$\Rightarrow$ We have a diagram

$$\begin{array}{ccc} \mathcal{O}_X & \to & F^e_* \mathcal{O}_X(\lceil (p^e - 1) B \rceil) \\ \downarrow & \text{splits locally} & \uparrow \\ \mathcal{O}_X(S) & \to & F^e_* \mathcal{O}_X(p^e S) \end{array}$$

notice $p^e S = (F^e)^* S$

$\Rightarrow \mathcal{O}_X \to \mathcal{O}_X(S)$
splits locally A simple local calculation shows $S = 0$ $*$

Thm (Hara - Watanabe) (X, B) a pair
If (X, B) is sharply F-pure $\Rightarrow (X, B)$ is lc

(but converse not true, eg X = cone over
supersingular elliptic curve)

Duality X normal var. $D \geq 0$ integral div
remember $\mathcal{O}_X \xrightarrow{\beta} F^e_* \mathcal{O}_X(D)$
get $\mathrm{Hom}(F^e_* \mathcal{O}_X(D), \mathcal{O}_X) \xrightarrow{\alpha} \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X)$

For simplicity assume X sm (can be avoided)

By duality for finite maps

$\mathcal{H}om(F^e_* \mathcal{O}_X(D), \mathcal{O}) = F^e_* \mathcal{H}om(\mathcal{O}_X(D), \omega_{X/X})$

Here $\omega_{X/X} = \mathcal{O}_X(K_X - p^e K_X) = \mathcal{O}_X(-(p^e-1)K_X)$ $\quad X \xrightarrow{F^e} X$

$= F^e_* \mathcal{O}_X(-(p^e-1)K_X - D)$ $\quad$ the origin of the "p^e-1".

If $D = \lceil (p^e-1)B \rceil$, then we get

$F^e_* \mathcal{O}_X(-\lceil (p^e-1)(K_X+B) \rceil)$

The Frob β locally splits $\Leftrightarrow$ α is surjective.

If (X,B) is sharply pure & $Y \to X$ birational morphism and $(p^e-1)(K_X+B)$ Cartier, then we can "pull back" α to a surjective normal morphism

$F^e_* \mathcal{O}_Y(-\lceil (p^e-1)(K_X+B) \rceil) \to \mathcal{O}_Y(\cdots)$ $\qquad K_Y + B_Y = f^*(K_X+B)$

restrict to $U = Y \setminus \mathrm{supp}$ negative coeff of B_Y

$\Rightarrow$ sharply F-pure on U $\Rightarrow$ every coeff of $B_Y \leq 1$

(Y, B_Y)

P18 Birkar lect 5.

Numerical properties of adjoint divisors

$k=\bar{k}$, char $= p > 0$

Recall Suppose $(X, A+B)$ is klt, over $\mathbb{C}$, proj; $A, B \geq 0$, A ample, Assume $L = K_X + A + B$ is nef then L is semi-ample

Remark We expect the same over k (char $p>0$)

Thm (Cascini-Tanaka-Xu)
Suppose $(X, A+B)$ is proj, strongly F-reg, $A, B \geq 0$
A ample, $L = K_X + A + B$ strictly nef (ie $D \cdot C > 0$ $\forall C$)
Then L is semi-ample ($\Leftrightarrow$ ample in this case)

Ref's [CTX, Tanaka, Freeness & very ampleness]

Cor Suppose (X, B) is proj strongly F-reg, G is ample Cartier, B: $\mathbb{Q}$-div
let $\lambda = \inf\{t \geq 0 \mid K_X + B + tG \text{ is nef}\}$
Then $\lambda \in \mathbb{Q}$ & if $\lambda > 0 \Rightarrow \exists C$ st $(K_X + B + \lambda G) \cdot C = 0$

Pf May assume $\lambda > 0$
If $\exists C$ st $(K_X + B + \lambda G) C = 0 \Rightarrow \lambda \in \mathbb{Q}$ and we are done
If $\nexists$ such C, $\Rightarrow$ $L = K_X + B + \lambda G$ is strictly nef
by thm $\Rightarrow$ L is ample $\Rightarrow$ contradicts def'n of λ

Singularities in linear systems

X normal proj, $x \in X$ closed and M a Cartier di
$|M|$ is the set of $D \geq 0$ st $D \sim M$.

Roughly, if $|M|$ is large (ie $h^0(M)$ is large)
$\Rightarrow \exists D$ st D is very singular at x

eg $X = \mathbb{P}^2$, $M = lH$, H hyperplane, $l \gg 0$

let $m_x \subseteq \mathcal{O}_x$ be the max ideal
have a natural map

$$H^0(M) \to \mathcal{O}_x$$

Now fix i, then m^i / m^{i+1} is a k v s of finite dimension

$$m_x^i \subseteq m_x^{i-1} \subseteq \cdots \subseteq m_x \subseteq \mathcal{O}_x$$

if $H^0(M)$ is large $\Rightarrow \exists\, \alpha \in H^0(M)$ st
$\alpha \mapsto$ into $m_x^i \Rightarrow D = \mathrm{Div}(\alpha) + M$ is very singular at x

Creating singularities at a pt $x \in X$ normal proj

L nef Cartier, $\neq 0$, A Ample Cartier

R-R: $h^0(mL+A) \to \infty$ if $m \to \infty$

hence, $\exists$ div $D \sim mL + A$ with high mult at x

assume $\exists\, C \ni x$ st $L.C = 0$, Then $D.C = A.C$

if $C \not\subseteq D \Rightarrow D.C$ is much larger than $A.C \Rightarrow C \subseteq D$

But if $\nexists$ such C (eg if L is strictly nef)

$\Rightarrow \exists D$ as above, avoiding any given subvariety $V \ni x$

in particular, $\exists D_1, \dots, D_n$, $D_i \sim mL + A$ st

D_i have large mult at x, but not any pts near x

For example, if we work over $\mathbb{C}$, if $t =$ lc threshold at x

$\Rightarrow (X, t(D_1 + \dots + D_n))$ is exactly lc at x but klt nearby

sketch of pf of thm:

Recall: If (X, Δ) is a pair, we say D sharply F-pure at x if the trace map

$$F^e_* \mathcal{O}_X(\lceil -(p^e-1)(K_X+\Delta)\rceil) \xrightarrow{\text{surj at } x} \mathcal{O}_X$$

$$\mathcal{H}om(F^e_* \mathcal{O}_X(\lceil (p^e-1)\Delta \rceil), \mathcal{O}_X) \to \mathcal{H}om(\mathcal{O}_X, \mathcal{O}_X)$$

(X, Δ) is strongly F-reg if $\forall\, \varepsilon > 0$, $\exists\, e$ st the map $F^e_* \mathcal{O}_X(\lceil -(p^e-1)(K_X+\Delta) \rceil - E) \to \mathcal{O}_X$ surj at x

For simplicity, we assume A, B, L are $\mathbb{Q}$-div

$0 \le D_i \sim_{\mathbb{Q}} (mL + A)$, $m \gg 0$

st. D_i very singular at x but "not at other pts nearby"

Let $W = D_1 \cap \dots \cap D_n$, Can assume $\dim W = 0$

we could change B st $p \nmid$ index of $K_X + B$

ie $\exists\, e$ st. $(p^e-1)(K_X+B)$ is Cartier

choose such $e \gg 0$

Put $N = -(p^e-1)(K_X + B + t(D_1 + \dots + D_n))$ Cartier

> 0 then we have exact sequence

$$0 \to \mathcal{I}_W \to \mathcal{O}_X \to \mathcal{O}_W \to 0$$

$$\begin{array}{ccccccccc} 0 \to & F^e_* \mathcal{O}_X(N) \otimes \mathcal{I}_W & \xrightarrow{\theta} & F^e_* \mathcal{O}_X(N) & \to & F^e_* \mathcal{O}_W(N) & \to 0 \\ & \downarrow \alpha & & \downarrow \text{trace map} & & \downarrow \beta & \\ 0 \to & m_x & \to & \mathcal{O}_X & \to & k(x) & \to 0 \end{array}$$

Here is the main part of char $p > 0$

Can assume the trace map surjective. (by choosing t taking t largest possible

By construction, "there is no more room to add subschemes to keep the trace map surj"

$\Rightarrow$ θ can not be surj if t is maximal (to keep the trace map surj)

$\Rightarrow$ $\operatorname{Im}\theta \subseteq m_x \Rightarrow \exists\, \alpha, \beta$ making the diag commu

Since β = composition of 2 surj maps, β is surj

$\Rightarrow H^0(F^e_* \mathcal{O}_W(N)) \to H^0(k(x))$ since W is affine

consider

$$\begin{array}{ccc} H^0(F^e_* \mathcal{O}_X(N + np^e L)) & \xrightarrow{\rho} & H^0(F^e_* \mathcal{O}_W(N + np^e L)) \\ \downarrow & & \downarrow \text{surj} \\ H^0(\mathcal{O}_X(nL)) & \xrightarrow{\gamma} & H^0(k(x) \otimes \mathcal{O}_X(nL)) \end{array}$$

if ρ is surj $\Rightarrow$ γ is surj $\Rightarrow$ $x \notin$ base locus of nL

$\Leftrightarrow$ nL is free at x ($\Rightarrow$ L is semi-ample at x)

We only need to show $H^1(F^e_* \mathcal{O}_X(N + np^e L) \otimes \mathcal{I}_W) = 0$

Remember $N + np^e L = \underbrace{-(p^e - 1)(K_X + B) + np^e L}_{\text{"quite" ample}} - (p^e - 1)(t D_1 + \cdots + D_n))$

By construction

t is relatively "small"

$\Rightarrow$ $N + p^e nL$ is ample enough to get the required vanishing $H^1 = 0$

Minimal models and flips for 3-folds in char p ($k = \bar{k}$) ~~> 0~~

Ref : char 0 (Shokurov)
char p [Hacon - Xu]

Recall (MMP) (X, B) proj pair, remember we choose ample div A, Take $\lambda = \inf\{t \geq 0 \mid K_X + B + tA$ nef$\}$
$\lambda = 0 \Rightarrow (X, B)$ minimal model
if $\lambda > 0 \Rightarrow$ we like to show $K_X + B + \lambda A$ is semi-ample
$\Rightarrow$ in the case we get a contraction $f\ X \to Z$
There are 3 diff kind of contractions
- divisorial f is birat'l, contracts some div
- flipping f is birat'l, but does not contradicts divisors (eg $\dim X = 3$, f contra curves)
- fiber type : $\dim X > \dim Z$
$\Rightarrow$ done get a Mori fiber space

if divisorial then continue
If f is flipping $\Rightarrow$ sing of Z are too bad
eg we could not define int numbers on Z
mK_Z not Cartier $\forall m$

The idea is to find a diagram
$$\begin{tikzcd} X \arrow[rr, dashed, "\text{bir}", "\varphi"'] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Z & \end{tikzcd}$$
f^+ does not contract div, but
$K_{X^+} + B^+ := \varphi_*(K_X + B)$ ample $/ Z$
continue with X^+
and so on This process is called LMMP
this works if we have all the necessary contractions flips and termination
Known cases (char $p > 0$).
- dim 2
- dim 3, $k = \bar{\mathbb{F}}_p$ $(p > 5)$, X smooth, $B = 0$

Flips and finite generation
(X, B), $X \to Z$ is a flipping contraction
To find X^+ is a local problem on Z, so we could assume Z affine, say $Z = \operatorname{Spec} A$

p. 22

Define $R = \bigoplus_{m \geq 0} H^0(m(K_X+B))$ a graded A-alg

$A = \deg 0$ piece.

The flips exists $\iff$ R is f.g. A-algebra

If the flip exists $\Rightarrow$ $R \simeq \bigoplus H^0(m(K_{X^+}+B^+))$

$\Rightarrow$ R is f.g. because $K_{X^+}+B^+$ is ample

Conversely if R is f.g. $\Rightarrow$ put $X^+ = \mathrm{Proj}\, R$

Example 1 let $Z = V(xy - zt) \subset \mathbb{A}^4$ Any dimension Z

Z has a sing pt $a := (0,0,0,0)$

we can blow up a to get a smooth $Y \longrightarrow Z$

contracts only one div $E = (xy - zt) \subset \mathbb{P}^3$, $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$

we can contracts E in 2 diff directions

$$\begin{tikzcd} & Y \supseteq E \arrow[dl] \arrow[dr] & \\ X \arrow[dr] \arrow[rr, dashed] & & X^+ \arrow[dl] \\ & Z & \end{tikzcd}$$

this is a "flop"

Take B^+ on X^+ which is ample, $B := \varphi^{-1}B^+$, Then K_X+B is negative over Z, But $K_{X^+}+B^+$ is positive over Z (note $K_X \sim 0$, $K_{X^+} \sim 0$)

Idea of Shokurov Reduce to PL flip

How to make some induction possible?

(X,B), $X \to Z$ flipping contr Assume (X,B) klt

suppose $\exists$ log resolution $Y \xrightarrow{g} X$ ($\exists$ in dim 3 even for $p>0$)

let $B_Y = B^\sim + E$, $B^\sim$ = birat'l trans of B

E = reduced exceptional div + components of $M_0 \sim K_X+B$

we can write

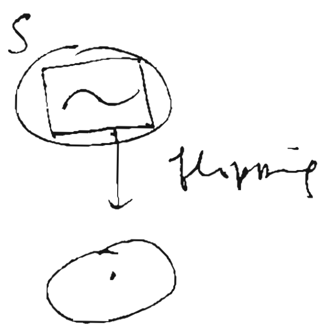
$K_Y + B_Y = g^*(K_X+B) + F$, (X,B) klt $\Rightarrow$ $F \geq 0$ exceptional

we try to run the LMMP on K_Y+B_Y over Z

$\Rightarrow$ in every step, the curves involved (the extl ray) intersect negatively some component B_Y with coeff $= 1$

$\Rightarrow$ We can do induction

$K_Y + B_Y|_S = K_S + B_S$



$\Rightarrow$ we will need only pl-flips.

and the end, after removing some div we get to X^+

pl flips. $(X, B+S)$ S not a comp of B

Assume $X \to Z$ is a flipping contr for $(X, B+S)$

Also assume S is negative over Z

Then we say we have a pl flipping contraction

Thm (Hacon-Xu) char $k = p > 5$

Assume $(X, B+S)$ is plt of dim 3. B has coefficients in $\{1 - \frac{1}{n}\}_{n \in \mathbb{N}}$. Then pl flip exists for $(X, B+S)$

($\Rightarrow$ flips exist for similar pairs)

(plt means klt with one comp of coeff $= 1$)

In particular if X is sm, $k = \overline{\mathbb{F}_p}$, $B = 0$ $\Rightarrow$
we could run a LMMP which ends with a minimal model or Mori fiber space.

Some idea of proof of existence of flips

we could assume $Z = \operatorname{Spec} A$.

we need to show that $R = \bigoplus H^0(m(K_X + B + S))$ is f.g.

we could assume S is not a comp of $K_X + B + S$

(by chang repr of K_X say)

we have a restriction map

$$R \longrightarrow \bigoplus_{m \geq 0} H^0(m(K_X + B + S)|_S)$$

Let's call the image of the map $R|_S$

claim: R f.g. $\Leftrightarrow$ $R|_S$ f.g.

$\Rightarrow$ clear, $\Leftarrow$ Since $K_X + B + S$ & S are both negative over Z and (assume $X \to Z$ is extreme), $\exists\, a, b \in \mathbb{N}$ st. $a(K_X + B + S) \sim bS$ (assume B $\mathbb{Q}$-div) otherwise by perturbation

simple alg R is f.g $\Leftrightarrow$ $\bigoplus_{m \geq 0} H^0(mS)$ is f g

choose $D \sim S$ (S not comp of D) ($\Rightarrow \bigoplus H^0(mD)$ is f.g

Now we have the exact sequence

$$0 \to \bigoplus H^0(mD - S) \to \bigoplus_{m \geq 0} H^0(mD) \longrightarrow \bigoplus_{m \geq 0} H^0(mD)|_S \longrightarrow 0$$

$$\bigoplus H^0((m-1)D) \cong \bigoplus H^0(mD - S)$$

same obj with deg shift

p.24 If $\alpha \in \bigoplus H^0(mD)$, we can write it as a poly in finitely many fixed elements

If you are very lucky, then

$R|_S = \bigoplus H^0(m(K_X+B+S)|_S)$ (usually only $\subset$)

By induction, done

In general case, we try to find $T \xrightarrow{\text{bir'tl}} S$

and B_T on T st $R|_T = \bigoplus H^0(m(K_T+B_T))$,

This is really the breakthrough idea of Shokurov

This is essentially an extension problem (to ambient space)

in char 0 we use vanishing thm to identify $R|_S$

(Kodaira, Kawamata-Viehweg, Nadel)

In char $p>0$; Use "S^0 sections"

Schwede (X, Δ) pair in char p

Assume $(p^e-1)(K_X+\Delta)$ is Cartier ($\exists e \Leftrightarrow p \nmid \text{index}(K_X+\Delta)$)

Remember we have a trace map let M Cartier.

$\mathcal{L}_{ne} = F_*^{ne} \mathcal{O}_X(-(p^e-1)(K_X+\Delta)) \to \mathcal{O}_X(M)$

$$\Rightarrow H^0(\mathcal{L}_{ne}) \xrightarrow{\varphi_{ne}} H^0(\mathcal{O}_X(M))$$

$+p^e M$

we look at only those sections from $H^0(\mathcal{L}_{ne})$

Define $S^0(\sigma_\Delta \otimes M) := \bigcap_{n>0} \mathrm{Im}(\varphi_{ne})$

Now if $M = K_X+\Delta+\text{ample}$ & if T is a comp of D with coeff 1 then $\exists$ natural surjective map

$$S^0(\sigma_\Delta \otimes M) \to S^0(\sigma_{\Delta_T} \otimes M|_T)$$

That's the idea!

End