Baohua Fu at CMTS, 4/4 2007

Lecture I. (Springer Maps & Stratified Mukai Flops)

nilpotent orbits / $\mathbb{C}$

motivation $\begin{cases} \text{repr theory : Springer corresp.} \\ \text{interesting examples of alg. var.} \end{cases}$

$\mathbb{P}\mathcal{O}_{min}$: contact + Fano

Conj: $\forall$ contact Fano are of the form $\mathbb{P}\mathcal{O}_{min}$ (for $\mathfrak{g}$ simple).

$\mathfrak{g}$ semi-simple Lie algebra (e.x. $sl_n, sp_n, so_n, E_6, E_7, E_8, F_4, G_2$)

G adjoint Lie gp. i.e. $\tilde{G}$ Lie gp st. Lie $\tilde{G} = \mathfrak{g}$

$G := \tilde{G}/\text{center}(\tilde{G})$. eg. PSL_n for sl_n.

$G \circlearrowright \mathfrak{g}$ adjoint representations

eg. $\underset{A}{PSL_n} \circlearrowright \underset{x}{sl_n}$ $\quad A\cdot x = AxA^{-1}$

Jordan decomposition: $\forall x \in \mathfrak{g}$, $\exists$ $x = x_s + x_n$

where x_s : semi-simple

x_n : nilpotent $\quad \leftrightarrow \quad ad_{x_s} : \mathfrak{g} \to \mathfrak{g}$, $y \mapsto [x, y]$ is semi-simple (nilp.)

$[x_s, x_n] = 0$

semi-simple orbit : $\mathcal{O}_{x_s} := G\cdot x_s$

nilpotent orbit : $\mathcal{O}_{x_n} := G\cdot x_n$ $\quad$ alg. var.

(e.x. $x \in sl_n$, x $\underset{s.s}{\text{nilp}}$ $\Leftrightarrow$ matrix x is $\underset{s.s.}{\text{nilp.}}$)

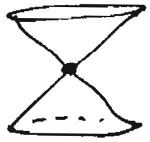
The geometry of s.s. orbits is simple:

- s.s. orbits are closed in $\mathfrak{g}$, simply connected, classfied by $\mathfrak{h}/W \xleftrightarrow{1-1} \{\text{s.s. orbits}\}$ infinitely many

Nilp. orbits are more complicated:

ex. sl_2, $x = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$. x nilp $\Leftrightarrow$ $x^2 = 0$

$\Leftrightarrow$ $\det x = 0$ $\Leftrightarrow$ $a^2 + bc = 0$



$\overline{\mathcal{O}_x} = \mathcal{O}_x \cup \{0\}$ singular cone. (A_1-sing.)

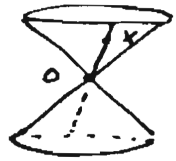
This is in fact a general fact:

Thm (Jacobson–Morosov) $\forall x \in \mathfrak{g}$ nilp. $\exists y, h \in \mathfrak{g}$ st. $\langle x, y, h \rangle \simeq sl_2$, i.e. $[h, x] = 2x$, $[h, y] = -2y$, $[x, y] =$

(in sl_2, $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$).)

$\Rightarrow$ $\forall x$ nilp, $\lambda \in \mathbb{C}^\times$ then λx is nilp. (This is simpl

$\Rightarrow$ $0 \in \overline{\mathcal{O}_x}$ $\Rightarrow$ $\overline{\mathcal{O}_x}$ is a cone, so singular.



classification:

$sl_n \ni x$ nilp $\Leftrightarrow$ x is conj. to $\begin{pmatrix} J_{d_1} & & 0 \\ & \ddots & \\ 0 & & J_{d_k} \end{pmatrix}$

where J_{d_i} is the Jordan block

$$= \left.\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}\right\} d_i$$

$\left\{ \begin{matrix} \text{nilp. orbits} \\ \text{in } sl_n \end{matrix} \right\} \longleftrightarrow \{ \text{partitions of } n \}$

i.e. $d_1 + \cdots + d_k = n$, $d_1 \geq d_2 \geq \cdots \geq d_k$.

For all other classical s.s. Lie alg, the same classificatio holds (i.e. in partitions) with extra constraints.

Rmk: for exceptional Lie alg. partitions are not enough. need to use Dynkin diagrams.

singularity of $\overline{\mathcal{O}}$:

$\overline{\mathcal{O}}$ = union of finitely many nilp. orbits

is in general non-normal, but sm in $\operatorname{codim}_{\mathbb{C}} 1$

$\overline{\mathcal{O}} = \mathcal{O} \cup \underbrace{\mathcal{O}_1 \cup \cdots \mathcal{O}_k}_{\text{symplectic}}$ $\quad \dim_{\mathbb{C}} \mathcal{O}_i$ = even

$\Rightarrow$ $\operatorname{codim} \mathcal{O}_i \geq 2$.

(yet, Thm (1980's). Kraft – Procesi:

$\overline{\mathcal{O}} \subset sl_n$ is normal!)

They also showed that $\overline{\mathcal{O}}_{\min}$ is normal for any $\mathfrak{g}$.

where $\mathcal{O}_{min}$ = the minimal non-zero nilp. orbit
= the nilp. orbit $\mathcal{O}$ st. $\forall$ nilp. orbit $\mathcal{O}' \neq \{0\}$
$\Rightarrow \bar{\mathcal{O}} \subset \bar{\mathcal{O}'}$. (this is unique).

Aim: Construct a natural resolution of $\bar{\mathcal{O}}$.
$x \in \mathcal{O} \rightsquigarrow \exists\, y, h$ st. $\langle x, y, h \rangle \simeq sl_2$
$\rightsquigarrow \mathfrak{g}$ becomes an sl_2-representation.
i.e. $x \cdot \underset{\mathfrak{g}}{\underset{\uparrow}{z}} = ad_x z = [x, z]$.

sl_2-repr theory $\underset{\text{decomp}}{\rightsquigarrow}$ $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$

where $\mathfrak{g}_i = \{ z \in \mathfrak{g} \mid [h, z] = i z \}$.

$\mathfrak{p} := \bigoplus_{i \geq 0} \mathfrak{g}_i$ (this is indep of choices of y, h
up to conjugation)
parabolic subalgebra (i.e. contains a Borel subalg)

$\mathfrak{n} := \bigoplus_{i \geq 2} \mathfrak{g}_i \ni x \qquad \Rightarrow [\mathfrak{p}, \mathfrak{n}] \subseteq \mathfrak{n}$
In the group level:
P = the subgroup of G with Lie alg. $\mathfrak{p}$
($\Rightarrow$ parabolic subgroup $\Leftrightarrow$ G/P is projective.)

$\Rightarrow P \cdot \mathfrak{n} \subseteq \mathfrak{n}$.

$$G \times^P \mathfrak{n} := G \times \mathfrak{n} / \sim \qquad (g, n) \sim (gp^{-1}, p \cdot n)$$
$\downarrow$ vector bundle, G-equivariant $\quad \forall p \in P$
G/P

Prop: $\forall$ G-equiv. vector bundle $\begin{array}{c} E \\ \downarrow \\ G/P \end{array}$ is isom to
$G \times^P E_0$ where $E_0 = \pi^{-1}(pt)$ as a P-module.

ex. $T^*(G/P) \simeq G \times^P (\mathfrak{g}/\mathfrak{p})^* \simeq G \times^P \mathfrak{p}^\perp$
$\downarrow$ since $\mathfrak{g}$ is semi-simple.
G/P

How to construct resolution:

$G \times^{P} \mathfrak{n} \xrightarrow[\mu]{} \mathfrak{g}$ well-defined.

$(g, n) \mapsto g \cdot n$

(Kostant): $\mathrm{Im}(\mu) = \overline{\mathcal{O}_x}$ (x is chosen from the beginning)

$G \times^{P} \mathfrak{n} \xrightarrow{\mu} \overline{\mathcal{O}_x}$ is a resolution of singularity:

"Pf": $G \cdot (1, x) \longrightarrow \mathcal{O}_x$

$$\begin{array}{ccc} G\cdot(1,x) & \longrightarrow & \mathcal{O}_x \\ \cong & & \cong \\ G/P^x & \xrightarrow[\text{isom}]{} & G/G^x \end{array}$$

since $P^x = G^x$ (The proof is not trivial.)

Kostant-Kirillov-Souriau sympl. form on $\mathcal{O}_x$.

$z \in \mathcal{O}_x$, $T_z \mathcal{O}_x \underset{\uparrow}{\cong} [z, \mathfrak{g}]$

since $\mathcal{O} \cong G/G^z$, $T_z \mathcal{O} \cong \mathfrak{g}/\mathfrak{g}^z \cong \mathrm{Im}(\mathrm{ad}_z)$

$\mathfrak{g}^z = \ker(\mathrm{ad}_z : \mathfrak{g} \longrightarrow \mathfrak{g})$.

$\omega_z : T_z \mathcal{O}_x \times T_z \mathcal{O}_x \longrightarrow \mathbb{C}$ via

$\omega_z([z, u_1], [z, u_2]) := K(z, [u_1, u_2])$

the Killing form

(recall $K(z_1, z_2) = \mathrm{tr}(\mathrm{ad}_{z_1} \circ \mathrm{ad}_{z_2})$.)

ω_z is a symp form $\rightsquigarrow$ ω is a symp form on $\mathcal{O}_x$

$\Rightarrow$ $\dim_{\mathbb{C}} \mathcal{O}_x$ = even, $K_{\mathcal{O}_x} \cong$ trivial. ($\cong \wedge^n \omega$)

To connect resl and symp form:

Prop: (Panyushev) '91.

$G \times^{P} \mathfrak{n} \xrightarrow{\mu} \overline{\mathcal{O}}$, $\mu^* \omega$ a symp form on $\mu^{-1}(\mathcal{O})$ extends to a holomorphic 2 form $\tilde{\omega}$ on $G \times^{P} \mathfrak{n}$.

Pf: Idea: construct a 2 form on $G \times \mathfrak{n}$ st its kernel is in the quotient direction.

$(g,n) \in G \times \mathcal{N}$, define a two form on $T_{(g,n)}(G \times \mathcal{N})$ as follows:

$\beta_{(g,n)}((u,m),(u',m')) := K([u,u'],n) + K(m',u) - K(m,u')$

$\operatorname{Ker} \beta_{(g,n)} = \{(u,[n,u]) \mid u \in \bigoplus_{i \geq -1} \mathfrak{g}_i\}$
easy to see

$T_{(g,n)} \; G \times^P \mathcal{N} \simeq \mathfrak{g} \times \mathcal{N} / \{(x,[n,x]) \mid x \in \mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}_i\}$

$\cap$ $\operatorname{Ker} \beta_{(g,n)}$

get a 2-form $\tilde{\omega}$ on $G \times^P \mathcal{N}$.

$\tilde{\omega} = \mu^* \omega$ over $\mu^{-1}(0)$ since coincides at pt $(1,x)$, both are G-inv

Rmk: $\tilde{\omega}$ is a symplectic form $\Leftrightarrow$ $\mathfrak{g}_1 = 0$. (since $\mathfrak{g}_{-1} = \mathfrak{g}_1^*$).

Cor: $\tilde{\mathcal{O}}$ = the normalization of $\bar{\mathcal{O}}$ then $\bar{\mathcal{O}}$ is a rational singularity.

(Recall, W normal with rat'l sing. $\Leftrightarrow$ $\forall$ resol. $\pi: Z \to W$, $R^i \pi_* \mathcal{O}_Z = 0$, $\forall i > 0$

there is a nice criterion due to M. Reid

$\Leftrightarrow$ on smooth part of W, $\exists\, \alpha$ regular n-form, (n = cpx dim.) nowhere vanishing (near each sing.) st. $\forall$ resol. $\pi: Z \to W$, $\pi^* \alpha$ ext to an n-form on Z.)

$\Rightarrow$ $\tilde{\mathcal{O}}$ is a "symplectic singularity".

Def'n: W is a sympl sing. if
1) W is normal
2) $\exists\, \omega$ on W_{reg} sympl. form
3) $\forall$ resol $\pi: Z \to W$ ($\exists$ one is enough) st $\pi^* \omega$ extends to a holo 2 form on Z.

ex. $\tilde{\mathcal{O}}$, $\mathbb{C}^{2n}/G$, $G \subset Sp(2n)$ finite
moduli space of sheaves on K3 surfaces ...

motivation: Sm: $\longleftrightarrow$ singular.

Calabi-Yau $\qquad$ rational sing.

$\cup$

hyperkähler $\longleftrightarrow$ sympl. sing.

Def^{n}: $\pi: Z \to W$ is a sympl resol. if $\pi^* \omega$ extends to a sympl form (i.e. non-degenerate)

(This will lead to examples of hyperkähler manifolds)

Cor. $G \times^P \mathfrak{n} \to \bar{O}$ is a sympl resol $\Leftrightarrow \mathfrak{g}_1 = 0$

i.e. O is an even orbit.

Prop: $\pi: Z \to W$ is symp resol.

$\Leftrightarrow \pi$ is crepant. i.e. $\pi^* K_W = K_Z$

$\Leftrightarrow K_Z$ trivial $\Leftrightarrow$ $\forall$ poisson str on W extends to Z.

(notice that the trivial Weil div on W is also trivial Cartier)

Thm (Kaledin):

A symplectic resol is semi-small, i.e. $\forall Z \xrightarrow{\pi} W$ symp. resol, $\forall F \subset Z$ closed subvar, we have

$2 \operatorname{codim} F \geq \operatorname{codim} \pi(F)$.

This is an important thm, but proof will not be given.

Prop: $\mathfrak{g}$ simple Lie alg. $\bar{O}_{min} \subset \mathfrak{g}$ min nilp orbit

$= O_{min} \cup \{0\}$ $\quad$ ($\Rightarrow$ isolated sing)

Then $\bar{O}_{min}$ has a sympl. resol $\Leftrightarrow \mathfrak{g} = sl_n$ (A type) for some n.

(Notice that even in sl_n, $\mathfrak{g}_1$ in fact $\neq 0$, so the resol is not the canonical one)

pf: $\Rightarrow$ $\mathfrak{g}$ not type A, $\operatorname{Pic} O_{min} = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } \mathfrak{g} \text{ of type } C \\ 0 & \text{otherwise} \end{cases}$

$= G/G^x$

$\bar{O}_{min} \simeq \mathbb{C}^{2n}/\pm 1$

$\Rightarrow$ Weil$(\bar{O}_{min})$ finite.

$\overline{O}_{min}$ is $\mathbb{Q}$-factorial, $\forall D$ Weil-div, $\exists n \in \mathbb{N}^*$
normal st. nD Cartier div.

$\Rightarrow$ $\forall$ resol. $\pi : Z \to \overline{O}_{min}$, $E := Exc(\pi) = \pi^{-1}(0)$ is of codim 1.

but if π is sympl. $\Rightarrow$ π semi-small

$\Rightarrow$ $\underbrace{2 \operatorname{codim} E}_{=2} \geq \operatorname{codim}(\pi(E)) = \dim \overline{O}_{min}$ $\times$

"$\Leftarrow$" next time. will show every nilp orbit closure in sl_n admits a sympl. resol.

Rmk: It is generally believed that if $\pi : Z \to W$ be a sympl. resol. with W having isolated sympl. sing. then $W \simeq \overline{O}_{min}$, $Z \simeq T^*\mathbb{P}^n$.
In Cho-Miyaoka-Shepherd-Barron, they gave a wrong pf.

4/5. Lecture II.

Prop. $\forall$ nilp orbit closure in sl_n admits a sympl resol.

Pf: (Explicit construction)

$$\left\{ \begin{matrix} \text{nilp. orbits} \\ \text{in } sl_n \end{matrix} \right\} \xleftrightarrow{1-1} \{ \text{partitions of } n \}$$

$$O := O_{[d_1, \cdots, d_k]}; \quad d_1 \geq \cdots \geq d_k, \quad \sum_{i=1}^{k} d_i = n$$

$$\overline{O} = \{ A \in sl_n \mid \dim \operatorname{Ker} A^i \geq \sum_{j=1}^{i} s_j \}$$

$$s_j := \# \{ l \mid d_l \geq j \} \qquad s_1 \geq s_2 \geq \cdots \geq s_k \qquad \sum s_i = n$$

the dual partition, i.e.

Young diagram  $\leftarrow d_1$

$\uparrow$ s_j

flag variety:

$$F = \{ F_1 \subset \cdots \subset F_{k'} \mid \begin{matrix} \dim F_1 = s_1 \\ \dim F_2 = s_1 + s_2 \quad \cdots \\ \dim F_{k'} = n \end{matrix} \} \simeq SL_n / P \leftarrow \text{parabolic}$$

$$T^*\mathcal{F} \simeq \{(A, F_\cdot) \mid AF_i \subset F_{i-1}\} \Rightarrow A^i F_i = 0$$

$\downarrow \pi$; $A \in \mathfrak{sl}_n$, $F_\cdot \in \mathcal{F}$; $(A,F_\cdot) \mapsto A$; $\bar{\mathcal{O}}$; $\pi^{-1}(0) = \mathcal{F}$

$\dim \operatorname{Ker} A^i \geq \dim F_i = \sum_{j=1}^{i} s_j$

$\Downarrow$

$A \in \bar{\mathcal{O}}$

claim: π is birational, $\pi^{-1}(\mathcal{O}) \to \mathcal{O}$ isom.

$\Rightarrow$ π is a sympl. resol. $F_i = \ker A^i$.

In general, $G > P$ parabolic ($\Leftrightarrow$ G/P projective)

$T^* G/P \simeq G \times^P \mathfrak{p}^\perp$, $\mathfrak{p}^\perp = \{x \in \mathfrak{g} \mid K(x,p) = 0 \ \forall p \in \mathfrak{P}\}$

$\xrightarrow{\mu}$ moment map

$\mathfrak{g} \simeq \mathfrak{g}^*$

$(g, x) \mapsto g \cdot x$ projection morphism

Richardson: Im μ is a nilpotent orbit closure, $\bar{\mathcal{O}}_P$ (say) called the Richardson orbit associated to P

$T^*(G/P) \xrightarrow{\mu} \bar{\mathcal{O}}_P$ generic finite, surj. proj morphism. called Springer map (which is a moment map)

Def'n: if μ is birat'l, it's called a Springer resol.

Thm: $\mathfrak{g}$ s.s. $\bar{\mathcal{O}} \subset \mathfrak{g}$ a nilp. orbit closure.

Suppose $\pi: Z \to \bar{\mathcal{O}}$ is a sympl. resol.

Then $\exists$ an isomorphism $Z \simeq T^*(G/P)$ for some parabolic $P < G$ and π is the moment map under this isomorphism.

Notice that Springer resol is a sympl resol clearly.

sketch of pf: $\bar{\mathcal{O}} \circlearrowleft \mathbb{C}^\times$, ie $\forall x \in \bar{\mathcal{O}}$, $\forall \lambda \in \mathbb{C}^\times$, $\lambda x \in \bar{\mathcal{O}}$

ω sympl. form on $\mathcal{O}$. $\lambda^* \omega = \lambda \omega$ $(*)$

- the G-action and $\mathbb{C}^\times$ action on $\bar{\mathcal{O}}$ lift to Z via π.
- $Z^{\mathbb{C}^\times}$ = union of smooth closed subvariety in Z

$\quad$ $\shortparallel$ fixed pts by $\mathbb{C}^\times$-action (general fact)

1) $\exists$ a G-equivariant map $p: Z \to Z^{\mathbb{C}^\times}$. (not conti.!)

Pf: $\forall x \in Z$, $\phi_x : \mathbb{C}^\times \to Z$

$\lambda \mapsto \lambda \cdot x$

$\psi_x : \mathbb{C} \to \bar{\mathcal{O}}$

$\lambda \mapsto \lambda \cdot \pi(x)$

$$\begin{tikzcd} \mathbb{C}^\times \arrow[r, "\phi_x"] \arrow[d, hook] & Z \arrow[d, "\pi"] \\ \mathbb{C} \arrow[r, "\psi_x"] \arrow[ur, "\tilde{\phi}_x"] & \bar{\mathcal{O}} \end{tikzcd}$$

π : proper $\Rightarrow$ $\exists \hat{\phi}_x : \mathbb{C} \to Z$, $p(x) := \hat{\phi}_x(0)$.

$x \in Z^{\mathbb{C}^\times}$, $T_x Z = \bigoplus_{k \in \mathbb{Z}} T_x^k Z$

$$T_x^k Z = \{ v \in T_x Z \mid \lambda * v = \lambda^k v, \ \forall \lambda \in \mathbb{C}^\times \}$$

claim: $\exists$ a component $Z_0 \subset Z^{\mathbb{C}^\times}$ st. the $\mathbb{C}^\times$-action is <u>definite</u> at pts on Z_0. $\forall z \in Z_0$, $T_{z_0}^k Z = 0$ if $k < 0$.

Pf: $\pi^{-1}(\mathcal{O}) \xrightarrow{p} p(\pi^{-1}(\mathcal{O})) \subset Z_0$

open $\quad$ irred $\exists!$

$p^{-1}(Z_0)$ open

Z_0

- ~~$\lambda^* w = \lambda w$~~ Z_0 is proper Lagrangian (sink comp.)

$\lambda^* \omega = \lambda \omega \Rightarrow T_x^k Z \simeq T_x^{1-k} Z$

$x \in Z_0 \Rightarrow T_x^k Z = 0$ if $k < 0$ $\Rightarrow T_x Z \simeq T_x^0 Z \oplus T_x^1 Z$

$Z_0 \subset \pi^{-1}(0)$ proper. $\quad \Rightarrow Z_0$ is Lagrangian

(since $T_x Z_0$ dual to $T_x^1 Z$.)

$p^{-1}(Z_0) \to Z_0$

$\quad \cong \leftarrow \omega$ is G-equiv. $N_{Z_0 / p^{-1}(Z_0)} \simeq T^* Z_0$

$T^* Z_0$

G acts on Z_0, with an open G-orbit $U \simeq G/P$, $P := G^x$

P acts on $(\mathfrak{g}/\mathfrak{p})^*$ with an open orbit. closed subgroup.

$p^{-1}(Z_0) \supset T^*(G/P) \circlearrowleft G \qquad (\mathfrak{g}/\mathfrak{p})^* \circlearrowleft P$

$\downarrow \qquad\qquad \downarrow$

$G/P \qquad\qquad x$

Thm (Knop-Richardson) $P < G$ closed subgroup. Then P has an open orbit in $(\mathfrak{g}/\mathfrak{p})^*$ $\iff$ P is parabolic.

($\Rightarrow$ is Knop, which is hard, in German. Fu says he does not

$\Rightarrow Z_0 = G/P$, $p^{-1}(Z_0) \simeq T^* G/P$.

the pf that "π is the moment map" still needs some delicate analysis. we omit them. $\square$

Cor. if $\bar{\mathcal{O}}$ admits a sympl. resol. then $\mathcal{O}$ is a Richardson orbit. (The converse is not true).

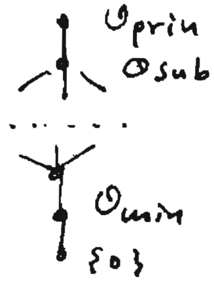
$\Rightarrow$ classification of $\mathcal{O}$ st $\bar{\mathcal{O}}$ has a symp resol.

(For more details, cf. Fu's paper on webpage)
including "extremal contr. Springer maps .... "

Ex. $\mathcal{N}$ = nilpotent cone in $\mathfrak{g}$ (union of all nilp. orbits

$\quad = \bar{\mathcal{O}}_{prin} \overset{\text{codim } 2}{\supset} \bar{\mathcal{O}}_{sub}$ ← also $\exists!$ one. but then a lot ...
↖ Kostant show that $\exists!$ one

strange tree :



← for sl_n it is complete symmetric $\updownarrow$. in general can adjust a bit to make it sym

$\mathcal{N}$ has a unique sympl. resol.

$T^* G/B \xrightarrow{\pi} \mathcal{N}$ $\quad B$ a Borel subgroup ($\exists!$)

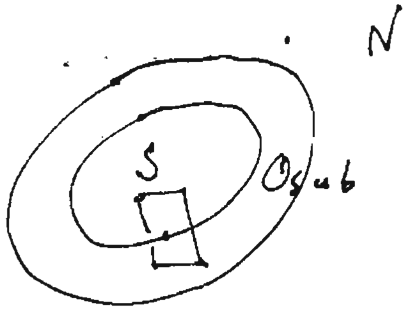
eg. Application to repr theory.
Springer correspondence $\qquad W$ = Weyl gp of G simple

$$\left\{ \begin{array}{c} \text{finite dim repr of} \\ W \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ (\mathcal{O}, \phi) \;\middle|\; \begin{array}{l} \mathcal{O} \text{ a nilp orbit in } \mathcal{N} \\ \phi \in \mathrm{Hom}(\pi_1(\mathcal{O}), \mathbb{C}^\times) \end{array} \right\}$$

$H^{top}(\pi^{-1}(x)) \curvearrowleft W \quad x \in \mathcal{O}$. $\quad$ + condition on ϕ
has many component, so W translates them

(See Ginzburg's book for more details)

Take S a slice transverse to a point $x \in \mathcal{O}_{sub}$ in $\mathcal{N}$.



S has dim 2, isolated sing.
so a rational surface sing.
i.e. A-D-E type.
Brieskorn: if $\mathfrak{g}$ is of type A (resp. D, E) then S is of the same type.
(cf. also Kronheimer's approach)

$\pi^{-1}(S) \to S$ minimal resol. sympl.

Aim: find the universal sympl. deformation of $\pi^{-1}(S)$.

Grothendieck's simultaneous resol.

idea: to do this, we deform $\pi : T^*(G/B) \longrightarrow N$.

$$\mathfrak{b} = \mathfrak{b}^{\perp} \oplus \mathfrak{h} \text{ — Cartan subalg.} \qquad T^*(G/B) \cong G \times^B \mathfrak{b}^{\perp}$$

$$\begin{tikzcd}
T^*G/B \arrow[r, hook] \arrow[d] \arrow[rr, bend right, "\pi_0"'] & G\times^B\mathfrak{b} = G\times^B(\mathfrak{b}^{\perp}\oplus\mathfrak{h}) \arrow[r, "\pi"] \arrow[d] & G\cdot\mathfrak{b} = \mathfrak{g} \circlearrowleft G \arrow[d] & N \arrow[l, hook'] \arrow[d] \\
0 \in \mathfrak{h} \arrow[r, equal] & \mathfrak{h} \arrow[r] & \mathfrak{g}/\!/G \simeq \mathfrak{h}/W & 0 \arrow[l]
\end{tikzcd}$$

$\mathfrak{g}/\!/G \simeq \mathfrak{h}/W$ — the set of semi-simple orbits in $\mathfrak{g}$

this is a quotient map, has degree
not birational yet. → Take fiber product:

$$\begin{tikzcd}
G\times^B\mathfrak{b} \arrow[r] \arrow[d] & \mathfrak{h}\times_{\mathfrak{h}/W}\mathfrak{g} \arrow[r] \arrow[d] & \mathfrak{g} \arrow[d] \\
\mathfrak{h} \arrow[r] & \mathfrak{h} \arrow[r] & \mathfrak{h}/W
\end{tikzcd}$$

π_t is an isom for t generic in $\mathfrak{h}$.

then $\tilde{S}$ = transversal slice in $\mathfrak{g}$.

Examples of sympl resol.

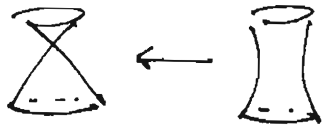
- Springer resol.
- Hilbert-Chow resol: $\mathrm{Hilb}^n(\mathbb{C}^2) \to \mathrm{Sym}^n(\mathbb{C}^2)$.

$\Gamma < SL(2)$ finite $\quad \mathbb{C}^2/\Gamma \leftarrow \widetilde{\mathbb{C}^2/\Gamma}$ minimal resol.

$\mathrm{Hilb}^n(\widetilde{\mathbb{C}^2/\Gamma}) \longrightarrow \mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$ symp resol.

These are all known examples so far.

$\mathbb{C}^2/\pm1 \leftarrow T^*\mathbb{P}^1$ $\qquad$ $\mathrm{Hilb}_0^2(T^*\mathbb{P}^1) \longrightarrow \mathrm{Sym}^2(\mathbb{C}^2/\pm 1)$



$$\begin{tikzcd} \mathbb{P}^2 \cup \Sigma \subset \mathrm{Hilb}^2(T^*\mathbb{P}^1) \arrow[rr, dashed, "\text{Mukai flop along } \mathbb{P}^2"] \arrow[dr] & & Z' \arrow[dl] \\ & \{0\} \in \mathrm{Sym}^2(\mathbb{C}^2/\pm 1) & \end{tikzcd}$$

two sympl. resol

Want to find this Z' in Springer resolution:

Eg F_6 $\quad \mathcal{O}_{[4,2]} = Sp_6 \cdot \left(\begin{array}{cccc|cc} 0 & 1 & 1 & 1 & 0 & 0 \\ & & & & 1 & 0 \\ \hline & & & 0 & 0 & 1 \\ \hline & & & 0 & & \\ & & & -1 & & \\ & & & -1 & & \end{array}\right)$ all other entries

S a transversal slice to $\mathcal{O}_{[2,2,2]}$ in $\mathfrak{g}$,

$S \cap \overline{\mathcal{O}}_{[4,2]} \cong \mathrm{Sym}^2(\mathbb{C}^2/\pm 1)$

$\overline{\mathcal{O}}_{[4,2]}$ has exactly 2 sympl. resol.

$$\begin{tikzcd} & \overline{\mathcal{O}}_{[4,2]} & \\ \overline{\mathcal{O}}_{[4,1,1]} \arrow[ur, "\text{codim } 2"', "A_1"] & & \mathcal{O}_{[3,3]} \arrow[ul] \\ & \overline{\mathcal{O}}_{[2,2,2]} \arrow[ul, "\text{codim } 2", "A_1"'] \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} Z_1 \arrow[r, hook] \arrow[ddrr, "\pi_1"] & T^*G/P_1 \arrow[dr] & & T^*G/P_2 \arrow[dl] & Z_2 \arrow[l, hook'] \arrow[ddll, "\pi_2"'] \\ & & \overline{\mathcal{O}}_{[4,2]} & & \\ & & S \cap \overline{\mathcal{O}}_{[4,2]} \arrow[u, hook] & & \end{tikzcd}$$

i.e. the Hilbert-Chow resol as a slice of a sympl. resol.

Can also do deformation.

Ginzburg shows that in general also $\exists$ sympl. deform. but only $\exists$. no explicit construction. using $\mathbb{C}^\times$-action. formal deformation, twistor construction.

Lecture III. 4/6 at NCU, CMTP

. $T^* G/P \to \bar{\mathcal{O}}_P$

Aim: study diff sympl. resol. ($\Leftrightarrow$ study parabolics)

sl_n : $\mathcal{O} = \mathcal{O}[d_1, \cdots, d_k]$ $\quad d_1 \geq \cdots \geq d_k \quad \sum d_i = n$

dual partition $s_i = \# \{ j \mid d_j \geq i \}$.

- for all $\sigma \in S_\ell$ (permutation group)

P_σ = parabolic of type $(s_{\sigma(1)}, \cdots, s_{\sigma(\ell)})$

$G/P_\sigma = \{ V_1 \subset V_2 \subset \cdots \subset V_\ell = \mathbb{C}^n \mid \dim V_i = \sum_{j=1}^{i} s_{\sigma(j)} \}$

i.e. codim $s_{\sigma(1)}$, $s_{\sigma(2)}$, $s_{\sigma(3)}$ etc.

$T^*(G/P_\sigma)$ $\sigma \in S_\ell$ gives all sympl. resol. of $\bar{\mathcal{O}}$.

let P of type $(p_1, \cdots, p_\ell)$

- P' of type $(p_1, \cdots p_j, p_{j-1} \cdots p_\ell)$ (switch $j-1, j$)

study

$$\begin{array}{ccccc} T^* G/P & \dashrightarrow & & & T^* G/P' \\ \cap & \searrow & & \swarrow & \cap \\ G/P \times \bar{\mathcal{O}} & & \bar{\mathcal{O}} & & G/P' \times \bar{\mathcal{O}} \end{array}$$

($G/P \times \mathfrak{g}$ last time)

$T^* G/P = \{ (x, V) \mid x V_i \subset V_{i-1} \}$, $x \in \bar{\mathcal{O}}$, $V \in G/P$

$$\begin{array}{ccc} T^* G/P & & T^* G/P' \\ & \searrow \quad \swarrow & \\ & X & \\ & \downarrow & \\ & \bar{\mathcal{O}} & \end{array}$$

would like to find X s.t. with simpler maps.

let $\bar{P}$ = of type $(p_1, \cdots, p_{j-2}, \underbrace{p_{j-1} + p_j}_{\text{join together}}, p_{j+1}, \cdots, p_\ell)$

$$\begin{array}{ccc} (V_1 \subset \cdots \subset V_\ell) \in G/P & & G/P' \\ \downarrow & \searrow \quad \swarrow & \\ (V_1 \subset \widehat{V_{j-1}} \subset V_j \cdots \subset V_\ell) \in G/\bar{P} & & \end{array}$$

get

$$\begin{array}{ccccc} T^* G/P \subset G/P \times \bar{\mathcal{O}} & & : & & G/P' \times \bar{\mathcal{O}} \supset T^* G/P' \\ & \overset{\phi}{\searrow} & & \overset{\phi'}{\swarrow} & \\ & & G/\bar{P} \times \bar{\mathcal{O}} & & \end{array}$$

claim: $\phi(T^* G/P) = \phi'(T^* G/P') =: X$.

$X = \left\{ (\bar{V}, x) \in G/\bar{P} \times \bar{\mathcal{O}} \;\middle|\; \begin{array}{ll} x \bar{V}_i \subset \bar{V}_{i-1} & \forall i \neq j-1 \\ x \bar{V}_{j-1} \subset \bar{V}_{j-1} & \bar{x} \in \operatorname{End}(\bar{V}_{j-1}/\bar{V}_{j-2}),\ \bar{x}^2 = 0 \end{array} \quad \operatorname{rk} \bar{x} \leq r \right\}$

$X \subset G/\bar{P} \times \bar{\mathcal{O}}$

reason: $xV_j \subset V_{j-1}$, $xV_{j-1} \subset V_{j-2}$ i.e. $x\bar{V}_{j-1} = xV_j \subset \bar{V}_{j-1}$, $x^2 V_j \subset V_{j-2} = \bar{V}_{j-2}$ which $= 0$ mod $\bar{V}_{j-2}$.

$$\begin{tikzcd} X \arrow[d, "f"'] & \ni (\bar{V}, x) \arrow[d, mapsto] \\ W = \{ & (\bar{V}, \bar{x}) \in G/P \times \operatorname{End}(\bar{V}_{j-1}/\bar{V}_{j-2}) \mid \bar{x}^2 = 0,\ \operatorname{rk}\bar{x} \le r \} \end{tikzcd}$$

claim: f is an affine bundle (so analytically a v.b. not algebraically.)

$r = \min\{p_{j-1}, p_j\}$, $m = p_{j-1} + p_j$.

$$W \xrightarrow{g} G/\bar{P} \quad \leftarrow \text{fiber} = \{ \bar{x} \in \operatorname{End}(\bar{V}_{j-1}/V_{j-2}) \mid \bar{x}^2 = 0,\ \operatorname{rk}\bar{x} \le r \}$$

$$= \bar{O}_{[2^r, 1^{m-2r}]} \subset sl_m \quad \text{i.e.}$$

$$\cong \left\{ x \text{ conj} \sim r\left\{\begin{pmatrix} \begin{smallmatrix}0&1\\0&0\end{smallmatrix} & & & \\ & \begin{smallmatrix}0&1\\0&0\end{smallmatrix} & & 0 \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix}\right. \right\}$$

(My thinking: should define stratified Mukai flop generally with base like the ordinary Mukai case)

dual partition $s_1 = m - r$
$s_2 = r$

sympl. resol.

$$\begin{tikzcd} T^*Gr(r,m) \arrow[rr, dashed] \arrow[dr] & & T^*Gr(m-r,m) \arrow[dl] \\ & \bar{O}[2^r, 1^{m-2r}] & \end{tikzcd}$$

$$\begin{tikzcd} T^*G/P \arrow[rr, dashed, "\psi"] \arrow[d] \arrow[dr] & & T^*G/P' \arrow[d] \arrow[dl] \\ Y \arrow[dr] \arrow[rr, dashed, "\psi_0" near end] & X \arrow[d, "f"] & Y' \arrow[dl] \\ & W \arrow[d] & \\ & G/\bar{P} & \end{tikzcd}$$

ψ is the pull back of ψ_0 via f.

$T^*Gr(r,m)$ bundle

$T^*Gr(m-r,m)$ bundle

$\leftarrow$ fiber are $\bar{O}_{[2^r, 1^{m-2r}]}$

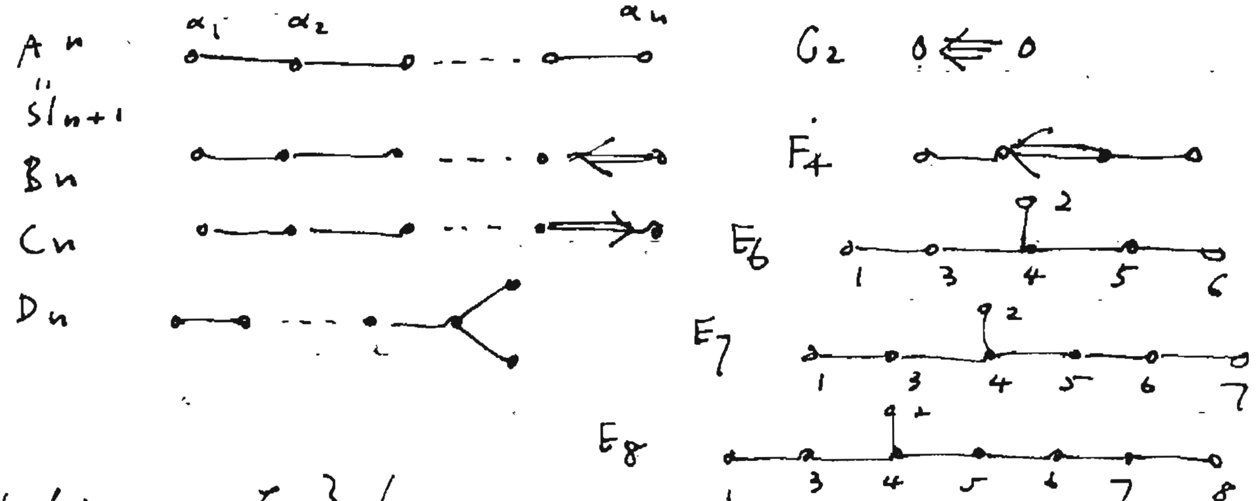
In general, need
Encoding parabolics by Dynkin Diagrams.

$\mathfrak{g}$ simple Lie alg. $n = \mathrm{rk}\,\mathfrak{g} = \dim \mathfrak{h}$

Fix $\mathfrak{h}$ Cartan subalg, $\mathfrak{b}$ = Borel, Φ^+ = positive roots

$\Delta \equiv \{\alpha_1, \cdots, \alpha_n\}$ simple roots

Dynkin diagram:



$\{\text{Parabolics in } \mathfrak{g}\}/\sim_{\text{conj}} \xleftrightarrow[1-1]{} \{\text{subset } \Gamma \text{ in } \Delta\}$

$P_\Gamma := \mathfrak{b} \oplus_{\beta \in \langle \Delta \setminus \Gamma \rangle^-} \mathfrak{g}_\beta \longleftrightarrow \Gamma$

$\longleftrightarrow$ {marked Dynkin diagram marked Γ}

eg. A_n

$\leftrightarrow$ parabolic of type $(1, n-1)$

marked this

$\leftrightarrow$ $G/P \simeq \mathbb{P}^n$

$\leftrightarrow$ type $(2, n-2)$ $\leftrightarrow$ $Gr(2, n)$

$\leftrightarrow$ type $(2, 2, 2, 1)$

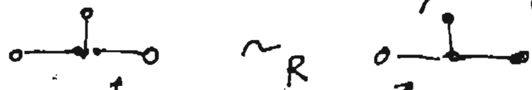
count, then remove recount

for exceptional Lie alg do not have this simple rule.

Def^{n}: $P_\Gamma \sim_R P_{\Gamma'}$ if $\mathcal{O}_{P_\Gamma} = \mathcal{O}_{P_{\Gamma'}}$ i.e. T^*G/P_Γ, $T^*G/P_{\Gamma'}$ $\to \mathfrak{g}$ have the same image.

Thm (Hirai) $\mathfrak{g}$ simple. The relation $\sim_R$ is generated by

1) In B_n or C_n, with $n = 3k-1$, $k \geq 1$, $P_{\alpha_{2k-1}} \sim_R P_{\alpha_{2k}}$
i.e. marked only one pt α_{2k-1}

2) D_4, $P_{\alpha_2} \sim_R P_{\alpha_3\alpha_4}$ 
(rank: $Pic \cong \mathbb{Z}$, $Pic \cong \mathbb{Z}^2$, so only one is birational)

3) D_n, $n = 3k+1$, $k \geq 2$, $P_{\alpha_{2k}} \sim_R P_{\alpha_{2k-1}}$

4) G_2, $P_{\alpha_1} \sim_R P_{\alpha_2}$

5) F_4; $P_{\alpha_2} \sim_R P_{\alpha_3} \sim_R P_{\alpha_1\alpha_4}$

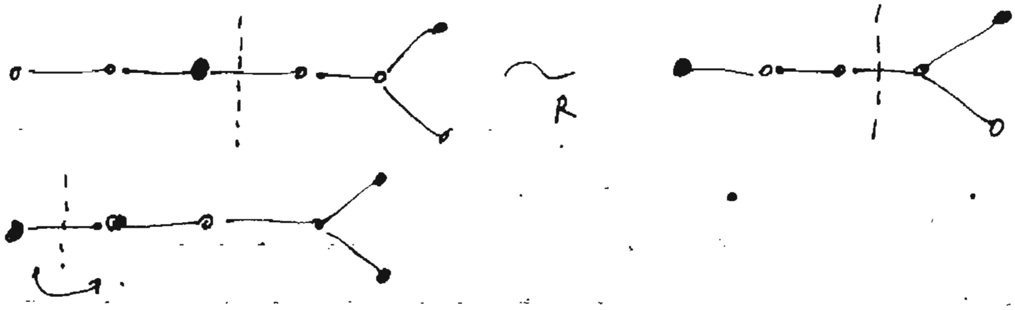
6) E_6 ..., E_8, ...

7) A_n, $P_{\alpha_i} \sim_R P_{\alpha_{n+1-i}}$ 

8) D_{2k+1} $(k \geq 2)$ $P_{\alpha_{2k}} \sim_R P_{\alpha_{2k+1}}$ 

9) E_6, $P_{\alpha_1} \sim_R P_{\alpha_6}$, $P_{\alpha_3} \sim_R P_{\alpha_5}$

(GP) (General Principle)
if $\Delta_1, \Delta_2 \subset \Delta$ disjoint, $\Gamma_i \subset \Delta_i$ two subsets, $\Gamma_1' \subset \Delta_1$ st. $P_{\Gamma_1} \sim_R P_{\Gamma_1'}$ in Δ_1 then
$P_{\Gamma_1 \cup \Gamma_2} \sim_R P_{\Gamma_1' \cup \Gamma_2}$ in Δ.

ex. 

Lemma: In (1)-(6) the 2 Springer maps cannot both be Springer resolutions.
In (7)-(9) two Springer maps are both birational, i.e. stratified Mukai flops.
(of A, D, E type)

Sure, this order is reordered by Fu.

. A : $T^*G_r^+ \dashrightarrow T^*G_r^-$

D : $T^*G_{iso}^+(k,2k) \dashrightarrow T^*G_{iso}^-(k,2k)$ k odd.

E :

they are all $\mathrm{Pic} \simeq \mathbb{Z}$ and extremal contr. to $\bar{O}$.

Goal: How $\left\{\begin{matrix}\text{degree}\\ \text{geometry}\end{matrix}\right.$ "changes under "GP" ?

Use notations in (GP):

P_{Γ_1} ~~$\Delta_1+\Gamma_1$~~ $\underset{R}{\sim}$ ~~$\Delta_1+\Gamma_1'$~~ $P_{\Gamma_1'}$

$$\begin{tikzcd}
T^*G_{\Delta_1}/P_{\Gamma_1} \arrow[dr, "\nu"] & & T^*G_{\Delta_1}/P_{\Gamma_1'} \arrow[dl, "\nu'"'] \\
& \bar{O} &
\end{tikzcd}$$

adjoint group of Δ_1

$P := P_{\Gamma_1 \cup \Gamma_2}$, $P' = P_{\Gamma_1' \cup \Gamma_2}$, $Q = P_{\Gamma_2}$

$$\begin{tikzcd}
T^*G/P \arrow[dr, "\pi"] & & T^*G/P' \arrow[dl, "\pi'"'] \\
& \bar{O} &
\end{tikzcd}$$

$U(P) = \mathfrak{p}^{\perp}$

$U(Q) = \mathfrak{q}^{\perp}$

need to find the "X".

Lemma. 1) $Q \cdot U(P) = Q \cdot U(P')$

2) $d^\circ(\pi) \cdot d^\circ(\nu') = d^\circ(\pi')\, d^\circ(\nu)$

3)
$$\begin{tikzcd}
T^*G/P \arrow[r] \arrow[dr, "\pi"'] & G\times^Q(Q\cdot U(P)) \arrow[d, "+"] & T^*G/P' \arrow[l] \arrow[dl, "\pi'"] \\
& \bar{O} &
\end{tikzcd}$$
∇

The diagram ∇ is a family of diagram given by ν, ν'.

idea of pf of 3): $\phi : T^*G/P \simeq G\times^P U(P) \simeq G\times^Q (Q\times^P U(P))$

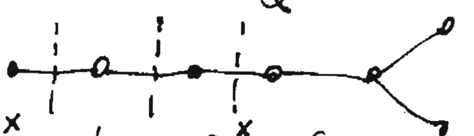
$\downarrow$ contract

$G\times^Q(Q\cdot U(P))$

need to study $Q\times^P U(P) \longrightarrow Q\cdot U(P) = L\cdot U(P)$

L : Levi factor of Q (i.e. s.s. part)

eg. Q



$\mathbb{C}^\times \times sl_2 \times \mathbb{C}^\times \times SO_8$

each disjoint Dynkin diagram has its adjoint gp.

marked $\mapsto \mathbb{C}^\times$

$\mathfrak{l} := \mathrm{Lie}\, L$.

$$\begin{tikzcd} Q \times^{P} V(P) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Q \cdot V(P) \arrow[d] \\ L \times^{L\cap P} V(\mathfrak{l}\cap \mathfrak{p}) \arrow[r] \arrow[d, equal] & L \cdot V(\mathfrak{l}\cap \mathfrak{p}) \arrow[d, equal] \\ T^* G_{S_1}/P_{\Gamma_1} \arrow[r] & \bar{\mathcal{O}}_{\Gamma_1} \end{tikzcd}$$

this is easier →

← $V(\mathfrak{p}) = V(L\cap \mathfrak{p}) \oplus V(\mathfrak{q})$ affine bundle

Theorem: If $T^*G/P \overset{\phi}{\dashrightarrow} T^*G/P'$, $T^*G/P \to \bar{\mathcal{O}} \leftarrow T^*G/P'$ have the same d°

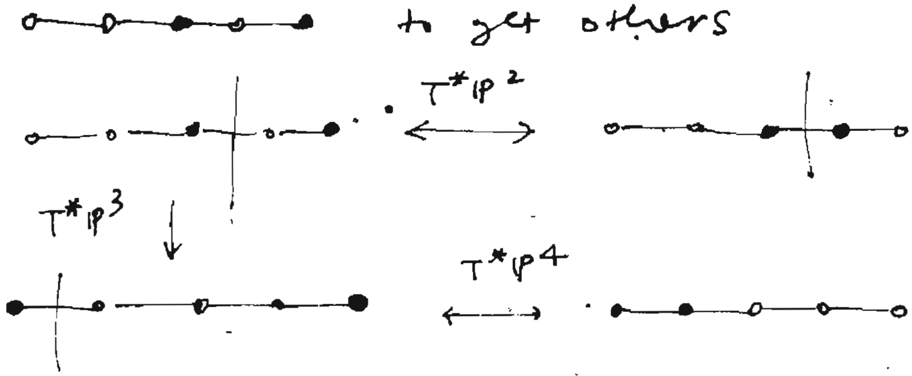
$\Rightarrow \exists\, \phi$ which decomposes into sequence of family of stratified Mukai flops.

pf. Just need to prove that we can use only (7)-(9) & (GP) to arrive P' from P.

→ case by case study. (not philosophical) $\square$

ex. $\bar{\mathcal{O}}_{[3,2,1]} \rightsquigarrow$ dual partition is also $[3,2,1]$

parabolic to get others



etc.

End