The First

NCTS Summer School on Algebraic Geometry

July 19 — 30, 1999

Invited Speakers

Professor Yujiro Kawamata
Tokyo University

Professor Eckart Viehweg
University of Essen

Professor H’el`ene Esnault
University of Essen

Professor Ching-Li Chai
University of Pennsylvania

Organizers

Jing Yu — NCTS and Academia Sinica

Chin-Lung Wang — National Taiwan University

Sponsors

Mathematics Division, National Center of Theoretic Sciences (NCTS)
Institute of Mathematics, Academia Sinica

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NCTS Summer School on Algebraic Geometry

Speaker & Title

Speaker	Title
Professor Ching-Li Chai (U. Penn)	Introduction to Langlands' Program
Professor Helene Esnault (U. Essen)	(1) Introduction to Classical Chern-Weil, Chern-Simons and Cheeger-Simons Theory (2) The Algebraic Theory (3) Riemann-Roch for Regular Connections (4) Riemann-Roch for Irregular Rank One Connections and Prospectives for the Higher Rank Case: New Non-Commutative Invariants
Professor Yujiro Kawamata (Tokyo U.)	Algebraic Fiber Space, Semipositivity Theorem and Adjunction of Canonical Divisors (These are Related and Explained in a Series)
Prefessor Eckart Viehweg (U. Essen)	Four Lectures on " Families of Projective Manifolds ", in Particular Curves and Surfaces

Schedule

Date	Time	Speaker	Place
July 19 (Monday)	10:00-11:30	Professor Yujiro Kawamata	Institute of Mathematics, Academia Sinica
	13:00-14:30	Professor Helene Esnault	
	15:00-16:30	Prefessor Eckart Viehweg	
July 20-July 22		Tour to Wu-Ling Farm	Wu-Ling Farm
July 21 (Wednesday)	19:00-20:30	Professor Ching-Li Chai	
July 23 (Friday)	10:00-11:30	Professor Yujiro Kawamata	Institute of Mathematics, Academia Sinica
	13:00-14:30	Professor Helene Esnault	
	15:00-16:30	Prefessor Eckart Viehweg	
July 26 (Monday)	10:10-12:00	Professor Yujiro Kawamata	Institute of Mathematics, Academia Sinica
	13:10-14:30	Professor Helene Esnault	
	15:15-16:45	Professor Ching-Li Chai	
July 27 (Tuesday)	10:10-12:00	Prefessor Eckart Viehweg	Institute of Mathematics, Academia Sinica
	14:10-16:00	Professor Ching-Li Chai	
July 29 (Thursday)	10:10-12:00	Professor Yujiro Kawamata	Institute of Mathematics, Academia Sinica
	14:10-16:00	Professor Helene Esnault	
July 30 (Friday)	10:10-12:00	Prefessor Eckart Viehweg	Institute of Mathematics, Academia Sinica
	14:10-16:00	Professor Ching-Li Chai	

The First

NCTS Summer School on Algebraic Geometry

July 19 — 30, 1999

Professor Yujiro Kawamata
Tokyo University

(Notes by Chin-Lung Wang)

Lecture I — 7/19, p.1

Algebraic Fiber Spaces

Lecture II — 7/23, p.11

Hodge Theory for Algebraic Fiber Spaces

Lecture III — 7/26, p.20

Adjunction Theory

Lecture IV — 7/29, p.29

Fano Manifolds

NCTS Summer School in Algebraic Geometry (1999)

Prof. Yujiro Kawamata

Lecture I. 7/19 at Academia Sinica

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* Algebraic Fiber Spaces, overview

complex algebraic varieties :
reduced, irreducible, finite type scheme / $\mathbb{C}$

Relative situation : alg. fiber space

$$\begin{array}{ccc} f: X & \longrightarrow & Y \\ \cup & & \Psi \\ X_\eta & \longrightarrow & \eta \end{array}$$

morphism

generic point

generic fiber / $\mathbb{C}(Y)$ = rat'l function field of Y

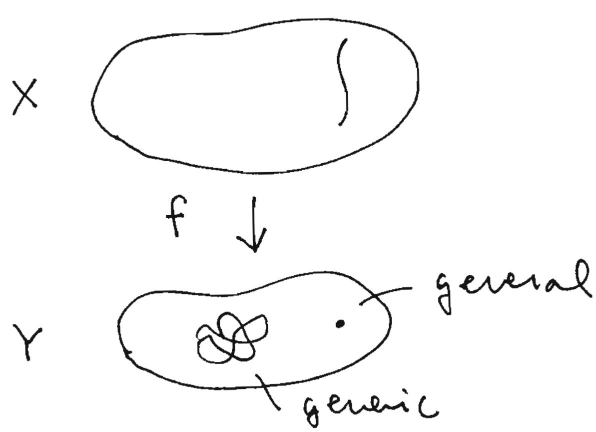
$\mathbb{C}(Y) \subset K$ (alg. closed) eg. $K = \overline{\mathbb{C}(Y)}$

$X_{\bar\eta}$ geometric generic fiber

Def : f alg- fiber space

$\Leftrightarrow$ $X_{\bar\eta}$ algebraic variety

$\Leftrightarrow$ general fiber are alg. v. / $\mathbb{C}$



$Y \supset U \ni {}^\forall p$
open

$X_p = f^{-1}(p) = \text{variety}$

morphism (for complete varieties)
- alg. fiber space
- finite morphism
  - normalization (simplest singularities)
  - covering with ramification
  - closed immersion

We consider only alg. fiber space.

fiber bundle $\Rightarrow$ no degeneration

good model of alg. f. space $\ni$ controlled degeneration

Very important example:

Elliptic surface (Kodaira theory)

$f : X \to Y \quad \dim X = 2, \ \dim Y = 1$

X_p generic fiber $=$ elliptic curve

good model $\leftrightarrow$ X, Y smooth, no (-1) curve.

good model for a variety "$=$" smooth

$Z \subset X$, X var. $Z =$ closed subset

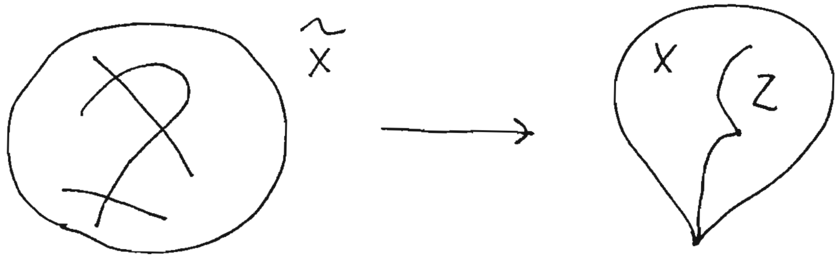
Theorem: $\exists\ \mu : \tilde{X} \to X$ birational morphism

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X \\ \cup & & \cup \text{ open} \\ \tilde{U} & \xrightarrow{\sim} & U \end{array}$$

$\tilde{X}$ smooth, $\tilde{Z} = \mu^{-1}(Z) =$ normal crossing divisor

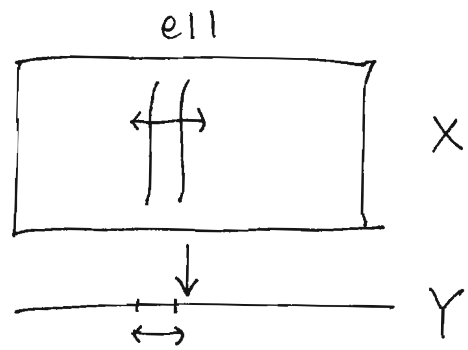
$(\tilde{X}, \tilde{Z})$ smooth pair.

locally $(\tilde{X}, \tilde{Z})$ looks like $\left(\mathbb{C}^n, \text{union of coor. hyperplanes } x_1 x_2 \cdots x_r = 0\right)$.



This is "birational geometry" since
$\mathbb{C}(X) \cong \mathbb{C}(\tilde{X})$.

For elliptic surface



① moduli for general fibers

$X_p \cong \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$

$J : Y \longrightarrow \mathbb{P}^1 \setminus \{\infty\}$, J function

$\longleftrightarrow$ positivity of curvature (Griffiths)

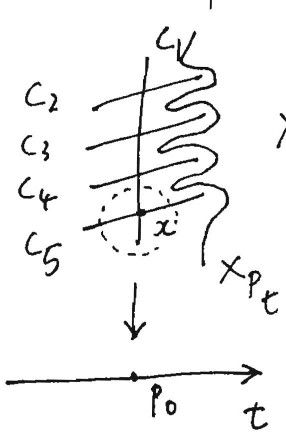
② Degenerate fibers (singular fibers)
completely classified by Kodaira

(-1)-curves : $C \subset X$, $C \cong \mathbb{P}^1$, $N_{C/X} \cong \mathcal{O}(-1)$

$\rightsquigarrow$ contractible to a smooth point

now we assume no such curves.

One typical case:



$C_i \cong \mathbb{P}^1$

$X_{p_0} = F = 2C_1 + C_2 + C_3 + C_4 + C_5$ — scheme-theoretic fiber

$f^* t = x^2 y$

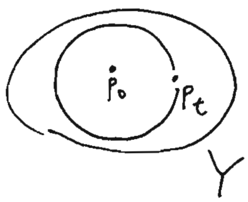
$x = 0 \to C_1$

$y = 0 \to C_5$

X_{p_t} = general fiber = smooth elliptic curve.

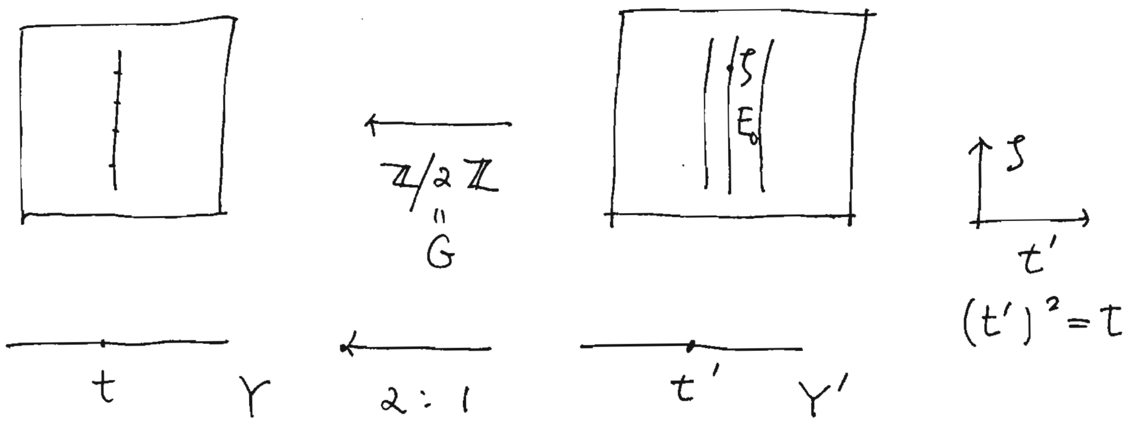
: monodromy

$$H^1(X_{p_t}, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$$



$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

this can be constructed by



$E_0/G \cong \mathbb{P}^1$, $\mathbb{C}^2$ $\left\{ \begin{array}{l} \zeta \to -\zeta \\ t' \to -t \end{array} \right.$

degenerate fibers $\to$ $\left\{ \begin{array}{l} \text{monodromy} \\ \text{multiple fiber} \\ \infty\text{-moduli point} \end{array} \right.$

multiple fiber: $F = mF_0$, F_0 elliptic

$m \in \mathbb{N}$, multiplicities

∞-moduli $J(p_0) = \infty$

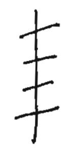
$F =$  nodal $\mathbb{P}^1$,  chain of $\mathbb{P}^1$'s.

Kodaira's canonical bundle formula :

$$K_X = f^*(K_Y + \text{positive } \mathbb{Q}\text{-divisor } \Theta)$$

$\Theta = \sum a_i P_i$, $f^{-1}(P_i)$ singular fiber , $a_i \in \mathbb{Q}^+$

eg. F :



mF_0

$a = 1/2$ $\qquad$ $a = \frac{1}{12}$ $\qquad$ $a = \frac{m-1}{m}$

K = canonical divisor
= div (differential n-form) $= \sum n_i D_i$
D_i : codim 1 sub. v. $\quad n_i$ order of 0 or ∞ .

Example: $\quad X = \mathbb{C} \xrightarrow[m:1]{\mu} \mathbb{C} = Y$, $t \to t^m = s$
$\qquad\qquad\quad \Psi$
$\qquad\qquad\quad P_0$

$$K_X = \operatorname{div}(\mu^* ds) = \operatorname{div}(m t^{m-1} dt) = (m-1) P_0 + K_Y$$

so $\quad K_X = \mu^* K_Y + (m-1) P_0$.

More Examples :

① Albanese fiber space

X smooth complete variety
$H^0(X, \Omega^1) = \{ \text{hol. 1-forms} \} \neq 0 \quad q\text{-dim}$

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "\text{alg. f. space}"'] & & A \\ & Y \arrow[ur, "\text{finite}"'] & \end{tikzcd}$$

$\Rightarrow$ Stein factorization

dim $A = q$, A abelian v.

② Iitaka fiber space

$H^0(X, mK) = \{ m\text{-ple canonical forms} \} \neq 0$

P_m-dim, basis $\omega_0, \cdots, \omega_{P_m - 1}$

$$X \dashrightarrow \mathbb{P}^{P_m - 1}$$
$$x \longmapsto (\omega_0(x) : \omega_1(x) : \cdots : \omega_{P_m-1}(x))$$

$$\begin{tikzcd} X \arrow[r, dashed] & \mathbb{P}^{P_m - 1} \\ & X' \arrow[ul, "\text{bi-rat'l}"] \arrow[u, "\text{morphism}"'] \end{tikzcd}$$

rat'l map

morphism : Φ_{mK} m-canonical map.

$\max(\dim \Phi_{mK}(X)) = \kappa(X)$ Kodaira dimension

maximal dim is attained at m

$$\begin{tikzcd} X' \arrow[rr] \arrow[dr, "\text{Iitaka fiber}"'] & \text{Stein fact.} & \mathbb{P}^{P_m - 1} \\ & Y \arrow[ur] & \end{tikzcd}$$

$$\begin{cases} \kappa(X) = \dim Y \\ \kappa(X_\eta) = 0 \end{cases}$$

Both examples are constructed using differential forms. Now another extreme :

③ Mori fiber space :

assume K_X is not nef :

$\deg K|_C = K.C < 0 \quad \exists C$ curve on X

$\Rightarrow \exists$ contraction associated to an extremal ray

$f : X \longrightarrow Y$. alg. fiber space (including birational morphism)

④ Moduli space & universal family

$$U \longrightarrow G(n, m)$$

universal Grassmannian

$\mathcal{C} \longrightarrow \mathcal{M}$ moduli of stable curves

$\mathcal{U} \longrightarrow \mathrm{Hilb}$ etc.

⑤ Twistor space for HyperKähler manifold

X : cpt Kähler mfd, $2n$-dim

$\mathbb{C} \simeq H^0(X, \Omega^2) \ni \omega$ hol. 2-form

$\wedge^n \omega =$ nowhere vanishing $2n$-form

g : Kähler metric

$(X, g, \omega) \rightsquigarrow \mathcal{X} \xrightarrow{f} \mathbb{P}^1$ family of HK mfds

For previous algebraic examples (alg. fiber space) we always have "positivity". For this analytic case, it is false:

$\mathcal{X} \simeq X \times S^2 \quad (C^\infty)$

f : smooth morphism of cpx manifolds

$f_* \omega_{\mathcal{X}/\mathbb{P}^1} := f_* \mathcal{O}(K_X - f^* K_{\mathbb{P}^1}) = \mathcal{O}(-n),\ n > 0.$

Abramovich - Karn

weak semi-stable model

cf. variety = resolution to smooth model

birational morphism

fiber space alteration

= birat'l morphism + finite covering.

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l, "\text{alteration}"'] \arrow[d, "f'"] \arrow[r, phantom, "\underset{\text{birat'l}}{\sim}"] & X \times_Y Y' \\ Y & Y' \arrow[l, "\text{alteration}"] \arrow[ul, phantom, "\substack{\text{birat'l to} \\ \text{fiber product}}"] \end{tikzcd}$$

f' weak s.s.

case : $\dim Y = 1$:

semi-stable : $\left\{ \begin{array}{l} X, Y \text{ smooth, all fibers are } \underline{\text{reduced}} \text{ normal crossing div.} \\ \operatorname{div}(f^* t) = D_1 + D_2 + \cdots \\ \cup D_i \ \text{NCD} . \end{array} \right.$ eg.  not.

General case :

X, Y smooth, $Y \supset E$, $X \supset D$ NCD $\qquad \begin{array}{c} x \in X \\ \downarrow f \\ y \in Y \end{array}$

$x \in X$, $y = f(x) \in Y$, local coor.

$x_1 \ldots x_n$, $y_1, \cdots, y_m$

$D = \operatorname{div}(x_1 \cdots x_s) = D_1 + \cdots + D_s$ at x

$E = \operatorname{div}(y_1 \cdots y_t) = E_1 + \cdots + E_t$ at y

$f^* E_i = D_{j_{i-1}+1} + \cdots + D_{j_i}$;

locally product of s.s. with $\dim Y = 1$.

This is called semi-stable model which is conjectured to exist, but not proved yet.

Rmk : the necessity to use alteration is already clear even in $\dim = 1$, eg. prev. example.

Weakly semi-stable model :

X : only Gorenstein quotient singularities

$$\begin{array}{l} X \ni x \\ \tilde{X} \xrightarrow{\pi} X \quad \text{(local covering)} \\ \text{smooth} \qquad X = \tilde{X}/G \qquad G : \text{finite gp} \end{array}$$

$\tilde{X}$: coor. $x_1, \cdots, x_n$

$\tilde{X} \supset (\mathbb{C}^\times)^n = \{ x_1 \cdots x_n \neq 0 \}$ mult gp str. $\supset G$

$g \in G$, $g = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix}$; α_i roots of unity

$\tilde{X}, Y$ smooth, $Y \supset E$, $\tilde{X} \supset \tilde{D}$, NCD, $\tilde{f} = f \circ \pi$

$x \in X$, $y \in \tilde{f}(x) \in Y$, local coor. $x_1 \cdots x_n$, $y_1 \cdots y_m$

$$\begin{tikzcd} \tilde{X} \arrow[d] \arrow[dd, bend right, "\tilde{f}"'] & \supset & \tilde{D} \arrow[d] \\ X \arrow[d] & \supset & D \arrow[d] \\ Y & \supset & E \end{tikzcd}$$

$\tilde{D} = \operatorname{div}(x_1 \cdots x_s) = D_1 + \cdots + D_s$ at x

$E = \operatorname{div}(y_1 \cdots y_t) = E_1 + \cdots + E_t$ at y

$\tilde{f}^* E_i = \tilde{D}_{j_{i-1}+1} + \cdots + \tilde{D}_{j_i}$

$E_1 \rightsquigarrow \tilde{D}_1 + \tilde{D}_2$, $E_2 \rightsquigarrow \tilde{D}_3 + \tilde{D}_4$ etc..

Now the condition reads : $f^* E_i = D_{k_{i-1}+1} + \cdots + D_{k_i}$ is a <u>reduced</u> divisor, quotient of NCD.

If $\dim Y = 1$:

s.s. reduction theorem (Mumford et. al)

in Book : Toroidal Embeddings I.

By Hironaka, X, Y smooth, fiber normal crossing but not reduced

$$\begin{array}{c} X \\ \downarrow f \\ Y \ni y \end{array} \qquad f^* y = \sum n_i D_i, \quad n_i \in \mathbb{N}.$$

l.c.m$(n_i) = m$

$(y')^m = y$

$$\begin{tikzcd} X \arrow[d] & \widetilde{X \times_Y Y'} \arrow[l] \arrow[d] \\ Y & Y' \arrow[l, "\pi"', "m:1"] \end{tikzcd}$$

~ — normalization

$X \times_Y Y'$ = weakly s.s. model

$y = \prod x_i^{n_i}$

$y = (y')^m$

Real Blow up:

$$\begin{tikzcd} X \arrow[d, "f"', "\text{w.s.s.}"] & X^{re} \arrow[l] \arrow[d, "f^{re}"] \\ Y & Y^{re} \arrow[l] \end{tikzcd}$$

f^{re} = topologically locally trivial (real analytic map)

eg. $(\mathbb{C}, 0)$ smooth pair $\leftarrow \mathbb{C}^{re}$

$G = e^{is}$

$re^{i\theta} \longleftarrow S^1 \times \mathbb{R}_{\geq 0} \ni (\theta, r)$

different from blow-up of real alg. v.

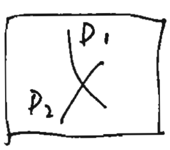
general case is product of this.

$(\mathbb{C}^2, +) = (\mathbb{C}, 0) \times (\mathbb{C}, 0)$

$(\mathbb{C}^2)^{re} = \mathbb{C}^{re} \times \mathbb{C}^{re}$ etc ...

$(X/G)^{re} = X^{re}/G$

$((X,D)/G)^{re} = (X,D)^{re}/G \longleftarrow$ free quotient!



X $\Rightarrow$ mfd with corner $(\mathbb{R}_{\geq 0})^2$

will use this approach to algebraic case to to adjunction theory ... $\square$

Prof. Yujiro Kawamata Lecture II.

Hodge Theory for algebraic Fiber Spaces 7/23

weakly semi-stable model $f : \mathfrak{X} \to S$ = smooth

$p \in \mathfrak{X}$, $q = f(p) \in S$

$\exists$ Galois covering $\tilde{U} \xrightarrow[\sigma]{G} U \ni p$

$\vdots$

$x_1, \cdots, x_n$ G acts diagonally

$V \ni q$, $y_1, \cdots, y_m$

$\sigma^* f^* y_i = x_{j_{i-1}+1} \, x_{j_{i-1}+2} \cdots x_{j_i}$

$\rightsquigarrow \left\{ \begin{array}{l} f \text{ flat equi-dim'l} \\ X \text{ quotient sing } (\Rightarrow CM) \\ f \text{ has reduced fibers} \end{array} \right.$

Real Blow-Up :

$$\begin{array}{ccc} \mathfrak{X}^{re} & \xrightarrow{\pi} & \mathfrak{X} \\ {\scriptstyle f^{re}}\downarrow & & \downarrow {\scriptstyle f} \\ S^{re} & \longrightarrow & S \end{array}$$

f^{re} topologically locally trivial family

MONODROMY :

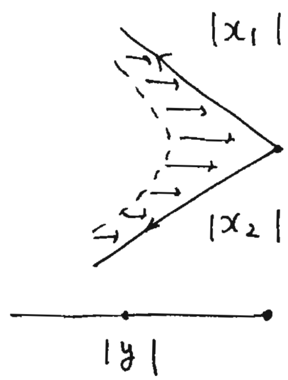
$\pi_1(S^{re}, *) \longrightarrow \mathrm{Aut}(H^p(X^{re}_*, \mathbb{Z}))$ $* \in S^{re}$

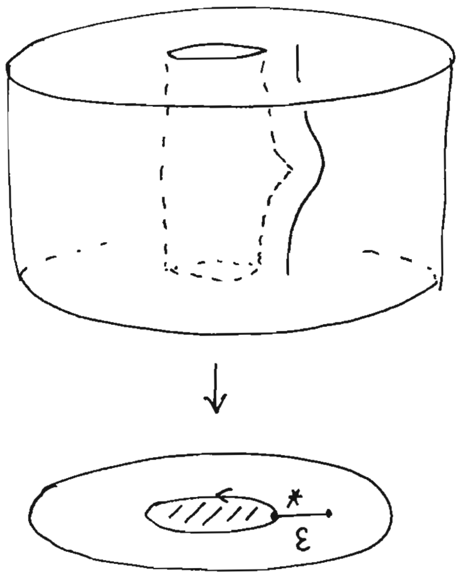
$X^{re}_s = f^{re\,-1}(s)$, $s \in S^{re}$

Simplest examples :

$y = x_1 x_2$

$|y| = |x_1| \cdot |x_2|$





usually need to take base point with dist $= \varepsilon$

now for real blow up, can take $\varepsilon = 0$. this makes things easier. Will prove:

- locally unipotent monodromy
- globally semi-simple.

$R^p f_*^{re} \mathbb{Z}_{\mathfrak{X}^{re}}$ local system over S^{re}

① Smooth fiber X of $f : \mathfrak{X} \to S$

$$\begin{array}{l} \mathbb{C}_X \xrightarrow[q.i.]{\sim} \Omega_X^{\cdot} \\ \cup \\ \mathbb{Z}_X \end{array}$$

stupid filtration:

$$\mathcal{O}_X \to \Omega_X^1 \to \cdots \to [\, \Omega_X^p \to \Omega_X^{p+1} \to \cdots \,]$$

$$\qquad\qquad\qquad\qquad\qquad \overset{..}{=} F^p$$

$$0 \to 0 \to 0 \cdots \to \Omega_X^p \to \Omega_X^{p+1} \to \cdots$$

as complexes

$$F^0 = \Omega_X^{\cdot} \supset F^1 \supset \cdots \supset F^n = \Omega_X^n[-n] \supset 0 \quad (\text{shifted by } n)$$

filtered complex $\Rightarrow$ spectral sequence

$$E_1^{p,q} = \mathbb{H}^{p+q}(X, Gr_F^p) \Rightarrow H^{p+q}(X, \Omega_X^{\cdot})$$

$$Gr_F^p \overset{..}{=} F^p/F^{p+1} = \Omega_X^p[-p], \qquad H^{p+q}(X, \Omega_X^{\cdot}) = H^{p+q}(X, \mathbb{C})$$

$$E_1^{p,q} = H^{p,q} := H^q(X, \Omega_X^p)$$

Th. degenerate at E_1

$\rightsquigarrow$ Hodge structure on $H^m(X,\mathbb{C}) = \bigoplus_{p+q=m} H^{p,q}$ (arbitrary int.)

Th. $\overline{H^{p,q}} = H^{q,p}$ (this does not follow from deg. at E_1).

② Variation of Hodge Structures

$$\begin{array}{ccc} f: \mathfrak{X} & \longrightarrow & S \\ \cup & & \cup \\ f_0: \mathfrak{X}_0 & \longrightarrow & S_0 \\ \parallel & & \\ f^{-1}(S_0) & & \end{array}$$

relative situation

smooth

$$f^{-1}\mathcal{O}_{S_0} \xrightarrow[q.i.]{\sim} \Omega^\bullet_{\mathfrak{X}_0/S_0} = \mathcal{O}_{\mathfrak{X}_0} \to \Omega^1_{\mathfrak{X}_0/S_0} \to \cdots$$

$$E_1^{p,q} = R^q f_{0*}\Omega^p_{\mathfrak{X}_0/S_0} \Rightarrow R^{p+q} f_{0*}\mathbb{C}_{\mathfrak{X}_0} \otimes \mathcal{O}_{S_0}$$

degenerate at E_1

$$\parallel$$
$$\mathcal{H}^{p+q} \quad \text{locally free}$$
$$\parallel$$
$$\mathcal{F}^0 \supset \mathcal{F}^1 \supset \cdots$$

holo. subbundles.

$H^m(X^{re}, \mathbb{Z})$: mixed Hodge structure

X smooth projective $\supset D$ NCD

$X_0 = X \setminus D$, $j: X_0 \subset X$

Deligne: $H^m(X_0, \mathbb{Z})$

cohomological mixed Hodge complex: sheaf of hol. forms with log poles

$$Rj_*\mathbb{C}_{X_0} \xrightarrow[q.i.]{\sim} j_*\Omega^\bullet_{X_0} \xleftarrow[q.i]{\sim} \Omega^\bullet_X(\log D)$$

Though LHS and RHS has no direct connection, but in the derived category, they are equal.

Canonical filtration

$\tau_{\le n}(K^\cdot) \subset K^\cdot$

$$\begin{array}{ccccccccc}
\cdots \to & K_{n-1} & \to & K_n & \xrightarrow{\alpha} & K_{n+1} & \to & \cdots \\
& \cup & & \cup & & \cup & & \\
\cdots \to & K_{n-1} & \to & \mathrm{Ker}\,\alpha & \to & 0 & \to & 0 & \cdots
\end{array}$$

$\tau_{\le n}(K^\cdot) \subset \tau_{\le n+1}(K^\cdot) \subset \cdots$

then $\quad Gr^{\tau}_{n}(K^\cdot) \cong H^n(K^\cdot)[-n]$

$$Rj_* \mathbb{C}_{X_0} \xrightarrow[q.i]{\sim} j_* \Omega^\cdot_{X_0} \xleftarrow[q.i]{\sim} \Omega^\cdot_X(\log D)$$

$W = $ cano. $\qquad$ cano $= W$

stupid $= F$

W : weight filtration $=$ cano filt $=$ filtration according to the order of log poles

(can be proved)

F : Hodge filtration

$$Gr^W_n(\Omega^\cdot_X(\log D)) \xrightarrow[\text{residue}]{\sim} \Omega^\cdot_{X[n]}[-n]$$

disjoint union of intersection of comp of D.

(may apply usual Hodge)

eg. $\quad X$

$X[0] = X$

$X[1] = $

$X[2] = $. $\quad$.

Theorem :
$$\begin{cases} E_1^{p,q} = H^q(X, \Omega^p(\log D)) \underset{\text{deg.}}{\Rightarrow} H^{p+q}(X_0, \mathbb{C}) \\ E_1^{p,q} = H^{p+q}(X, Gr^W_{-p}) \underset{\text{deg. at } E_2}{\Rightarrow} H^{p+q}(X_0, \mathbb{C}) \end{cases}$$

On $H^m(X^{re}, \mathbb{Z})$ mixed Hodge structure

via cohomological mixed Hodge complex

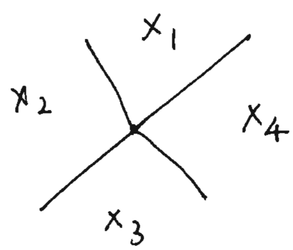
$\mathbb{Z}$ - level: $\quad 0 \to \mathbb{Z}_{X^{re}} \to \mathbb{Z}_{X^{re}[0]} \to \mathbb{Z}_{X^{re}[1]} \to \cdots$

Mayor-Vietoris

$X = f^{-1}(*) = X_1 \cup X_2 \cup \cdots$ irred. comp

generalized normal crossing variety

ex.



$xy = 0$

$zw = 0$

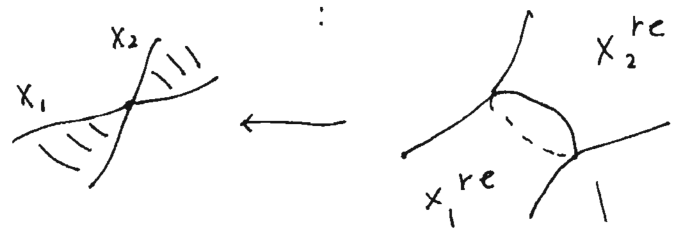
Standard maps

$$X[0] = \coprod X_j \xleftarrow{\text{rel. b-up}} X^{re}[0]$$

$$X[1] = \coprod X_{ij} \text{ of codim } 1 \xleftarrow[+ S^1 \text{ fiber}]{\text{real}} X^{re}[1] = \coprod (X_j^{re} \cap X_i^{re})$$

$$X[2] = \coprod X_{ijk} \text{ codim } 2 \xleftarrow[(S^1)^2\text{- fiber}]{\text{real} +} X^{re}[2]$$

$$\begin{array}{ccc} X & \xleftarrow{\pi} & X^{re} \\ \uparrow & & \uparrow \\ X[0] & \longleftarrow & X^{re}[0] \end{array}$$



get S^1-fibration

$$W_n = \left(\tau_{\le n} R\pi_* \mathbb{Q}_{X^{re}[0]}\right) \to \tau_{\le (n+1)} R\pi_* \mathbb{Q}_{X^{re}[1]}$$

$$\to \tau_{\le n+2}\left(R\pi_* \mathbb{Q}_{X^{re}[2]}\right) \to \cdots$$

$$Gr_n^W = R^n \pi_* \mathbb{Q}_{X^{re}[0]}[-n] \oplus R^{n+1} \mathbb{Q}_{X^{re}[1]}[-n-2] \oplus$$

$$R^{n+2} \pi_* \mathbb{Q}_{X^{re}[2]}[-n-4] \oplus \cdots$$

$\mathbb{C}$ - level:

$$R\pi_* \mathbb{C}_{X^{re}} \xrightarrow{\sim} R\pi_* M \xleftarrow{\sim} \Omega^\bullet_X(\log)$$

real str. $x = re^{i\theta}$

$\left[d\theta, \frac{dr}{r}, \log r \right]$ can also be defined alg'ly.

$\Omega^1_{\mathcal{X}}(\log)$ $\mathcal{X} = U = \tilde{U}/G$, take $\Omega^1_U(\log) = \Omega^1_{\tilde{U}}(\log)^G$

$\Omega^1_S(\log)$ - the usual

is locally free because the action is diag'l. and $\frac{dx}{x}$ is inv.

$\Omega^1_{\mathcal{X}/S}(\log) = \Omega^1_{\mathcal{X}}(\log)/f^*\Omega^1_S(\log)$ loc. free.

$\Omega^1_X(\log) = \Omega^1_{\mathcal{X}/S}(\log) \otimes \mathcal{O}_X$

get

$$\Omega^\bullet_X(\log) \to \Omega^\bullet_X(\log) \otimes \mathcal{O}_{X[0]} \to \Omega^\bullet_X(\log) \otimes \mathcal{O}_{X[1]} \to \cdots$$

Another Mayer-Vietoris ($\Rightarrow$ weight filt)

Th. ① $E_1^{p,q} = H^q(X, \Omega^p(\log)) \Rightarrow H^{p+q}(X^{re}, \mathbb{C})$

degenerate at E_1.

② $E_1^{p,q} = H^{p+q}(X, Gr^W_{-p}(R\pi_* \mathbb{Q}_{X^{re}})) \Rightarrow H^{p+q}(X^{re}, \mathbb{Q})$

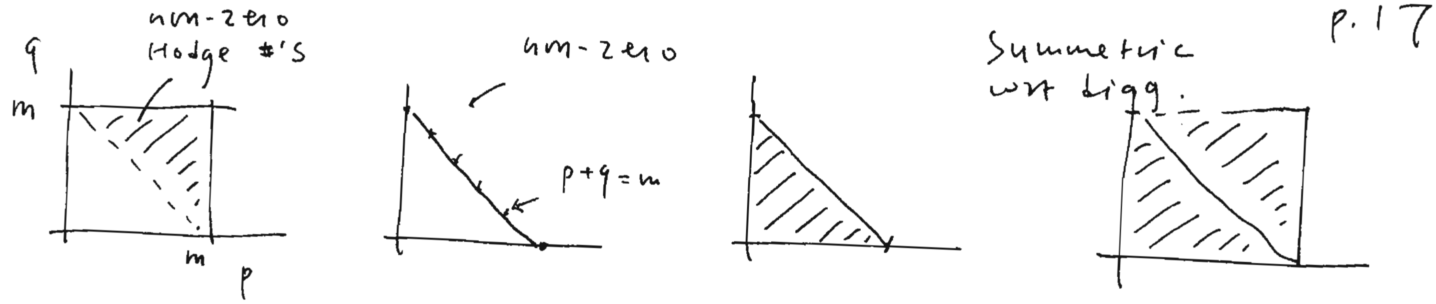
degenerate at E_2.

Hodge numbers:

$$Gr^W_n H^m(X_0, \mathbb{C}) = \bigoplus_{p+q=m+n} H^{p,q}, \quad \overline{H^{p,q}} = H^{q,p}$$

ie. $Gr^p_F(Gr^W_n H^m(Y, \mathbb{C})) = H^{p,q}$, $p+q = n+m$

in general.



$X_0 = X \setminus D$ compact X singular cpt X^{re}

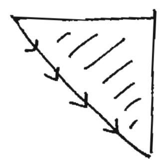
$\tau_{\le n} \to \tau_{\le n+1} \to \cdots$

Cor. $R^q f_* \Omega^p_{\mathcal{X}/S}(\log)$ is locally free over S

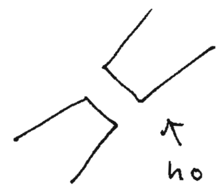
(this follows from de Rham + loc. trivial in part (1) of Th.)

Cor. Monodromy is unipotent (Even easier cor.)

$Gr^W_n (R\pi_* \mathbb{Z}_{X^{re}})$ ↺



bec.



Non-degenerate Hermitian Sym. Bilinear form h:

$f_* \omega_{\mathcal{X}/S} = f_* \Omega^n_{\mathcal{X}/S}(\log)$ $\quad n = \dim X$

$(R^q f_* \omega_{\mathcal{X}/S})$ $\quad \omega_{\mathcal{X}/S} = \mathcal{O}_{\mathcal{X}}(K_{\mathcal{X}/S}) = \mathcal{O}(K_{\mathcal{X}} - f^* K_S)$

E holo. v.b. on cpx. mfd S $(= S_0)$

$h : E \to \bar{E}^*$ (C^∞-isom.) $\quad h(u, \bar{v}) \in \mathbb{C}$, $\quad u \in E$, $\bar{v} \in \bar{E}$.

$E = R^n f_* \mathbb{C}_X \otimes \mathcal{O}_S \big|_{S_0}$

$h(u, v) = (u \wedge \bar{v})[X_s]$, $\quad u, v \in H^n(X_s, \mathbb{C})$

this is non-definite!

D connection $= h^{-1}\partial h + \bar\partial : E \to \Gamma^1_\infty(E)$

$\Theta = D \circ D$ curvature : $(1,1)$ form with value in $\mathrm{Hom}(E,E)$

$$\varphi = \frac{\sqrt{-1}}{2\pi} \operatorname{Trace} \Theta = \partial\bar\partial(-\log \det h)$$

For $0 \to F \xrightarrow{i} E \xrightarrow{p} G \to 0$ exact

$s\ (C^\infty)$

second fund. form $b = p D i \in \Gamma^1_{C^\infty}(\mathrm{Hom}(F, G))$

Lemma: h non-deg on E, $h|_F$ non-deg $\Rightarrow$

$$h_F(\Theta_F(u), v) = h_E(\Theta_E(u), v) - h_G(b(u), b(v)).$$

pf is by direct computation.

Theorem: (Griffiths)

$$f_* \omega_{X/S}\big|_{S^0} \subset R^n f_* \mathbb{C}_X \otimes \mathcal{O}_S\big|_{S^0}$$

$\Rightarrow \Theta_F$ is semi-positive (pos. semi-definite)

pf: ① infinitesimal period relation:

$$D(F^p) \subset \Gamma^1_{C^\infty}(F^{p-1})$$

$$F^0 = E \supset F^1 \supset F^2 \supset \cdots \supset F^n = F \supset 0$$

F^p $\quad dz_1 \wedge \cdots \wedge dz_p \wedge d\bar z_{p+1} \wedge \cdots$

$(p, q), \ p + q = n$

$$dz_1 \rightsquigarrow dz_1 + \varepsilon\, d\bar z_\alpha + \cdots$$

$$d\bar z_{p+1} \rightsquigarrow d\bar z_{p+1} + \varepsilon\, dz_\beta + \cdots$$

wedge + the fact "$\varepsilon^2 = 0$".

② Riemann-Hodge bilinear relation

$\dim X = 2n$

$$\mathrm{Ker}\left(H^{n-q}(X,\mathbb{C}) \xrightarrow[\text{Lefschetz}]{\wedge^{q}[L]} H^{n+q}(X,\mathbb{C}) \xrightarrow{[L]} H^{p+q+2}(X,\mathbb{C})\right)$$

$=: H_0^{n-q}(X,\mathbb{C})$ primitive

$$F^n(H^n(X,\mathbb{C})) \subset H_0^n(X,\mathbb{C})$$
$$\| $$
$$H^0(X,\Omega^n)$$

$$(-1)^{\frac{n(n-1)}{2}+q}(\sqrt{-1})^n h(H_0^{p,q}, H_0^{p,q}) \gg 0 \ .$$

important part : $(p,q) \to (p+1, q-1)$ change sign

$F^n \longleftrightarrow (F^{n-1} \cap H_0)/F^n$ sign-change.

Now Griffiths' pf follows easily.

$F_1 = F^n$

E flat $\Rightarrow \Theta_E = 0$

and h_G, h_F. diff sign. $\square$

Th: $f_* \omega_{X/S}$ is a numerically semi-positive vector bundle.

To be continued

For Lecture 3: Adjunction theory

Lecture 4: Fano Manifolds.

Lecture III. Adjunction Theory

First we need to finish Lect. II :

Def: X proj. v. E : locally free sheaf
$\mathbb{P}(E)$ proj. bundle over X : $\pi : \mathbb{P}(E) \to X$
$\mathcal{O}(1)$ tautological quotient line bundle of $\pi^* E$.
E is "numerically semi-positive"
$\Leftrightarrow$ $\mathcal{O}(1)$ nef. ie. $\forall C \subset \mathbb{P}(E)$, $(\mathcal{O}(1), C) \geq 0$.

Th. $f : X \to Y$ alg. fiber sp. X, Y smooth
$Y_0 \subset Y$, $Y - Y_0 = E$ normal crossing divisor
$f_0 = f|_{f^{-1}(Y_0)}$ smooth
$X_0 = f^{-1}(Y_0)$, $n = \dim X - \dim Y$.
Assume : local monodromies of $R^{n+q} f_* \mathbb{Z}_{X_0}$
around E are unipotent
$\Rightarrow$ $R^q f_* \omega_{X/Y}$ is num. semi-positive.

Rmk: This is birational invariant for X
but not for Y.

$$\begin{tikzcd} X' \arrow[dr, "\mu"] \arrow[dd, "f'"'] & \\ & X \arrow[dl, "f"] \\ Y & \end{tikzcd} \qquad R^q f'_* \omega_{X'/Y} \cong R^q f_* \omega_{X/Y}.$$

in fact X has Gorenstein quotient sing.
is oK. just like in smooth case. and
f weakly s.s. $\Rightarrow$ X Gorenstein canonical.

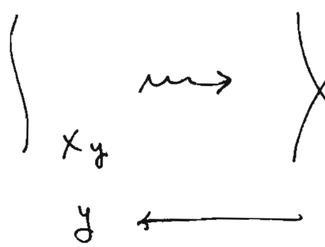
$\mathcal{F}_0 = R^g f_{0*}\omega_{X_0/Y_0}$ has a pos. definite metric with semi-positive curvature, $\mathcal{F}_0 = \mathcal{F}|_{Y_0}$

$\Rightarrow \mathcal{O}(1)|_{\pi^{-1}(Y_0)}$ has a metric with s.p. curvature.
(singular hermitian metric)

Def: Singular hermitian metric h: L line bundle on a variety Y, locally on Y, $h = e^{-\varphi} h_0$
h_0: C^∞-metric, φ: weight function $\in L^1$.

general fiber of $\mathcal{F}$ = hol. n-forms α

$$h(\alpha,\alpha) = * \int_{X_y} \alpha \wedge \bar{\alpha}$$



← log pole at intersections

$$\int \frac{dx}{x} \wedge \frac{d\bar{x}}{\bar{x}} = \infty .$$

h grows at the order of $|\log y|^m$,
$\varphi \sim \log \log |y|$.

$\Rightarrow$ curvature form $\Theta = \partial\bar{\partial}(-\log h)$

As a distribution $\Theta = \partial\bar{\partial}(-\log h) + \partial\bar{\partial}\varphi$

and 1st chern form: $c_1 = \frac{\sqrt{-1}}{2\pi}\Theta$

there is no boundary contribution to Θ:

$$\Theta = \underbrace{\Theta_{ab.c}}_{\text{absolute conti.}} + \underbrace{\Theta_{sing}}_{=0}$$

In general, non-unip monodromy $\Rightarrow \Theta_{sing} \neq 0$.

since $\Theta_{ab.c}$ is s.p. Θ is s.p. !

Remark: There is an algebraic proof due to Kollár via vanishing theorem. which is easier but more tricky. (after the existence of the analytic pf).

Adjunction Theory:

$\mathbb{Q}$-divisors: $D = \sum_{\text{finite}} d_j D_j$, $d_j \in \mathbb{Q}$, D_j prime div

nef $\Leftrightarrow$ $(D.C) \geq 0$ $\forall$ curve

A ample $\mathbb{Q}$-div $\Leftrightarrow$ $\exists$ $m \gg 0$ st. mA very ample

nef divisor is a limit of ample divisors:

$$\left.\begin{matrix} A \text{ ample} \\ D \text{ nef} \end{matrix}\right\} \Rightarrow A + D \text{ ample}$$

(eg. Nakai criterion)

$\varepsilon A + D$ ample $\varepsilon > 0$, $\lim_{\varepsilon \to 0} (\varepsilon A + D) = D$.

Sometimes it will be important to consider also $\mathbb{R}$-divisor $\sum d_j D_j$, $d_j \in \mathbb{R}$.

Remark: semi-ample
$\Rightarrow$ metric semi-positive $\Rightarrow$ nef
other directions are false

semi-ample $\overset{\text{def}}{\Longleftrightarrow}$ $\exists$ $m \gg 0$, $|mD|$ free.

Correction: In Kodaira's formula

$$K_S \sim_{\mathbb{Q}} f^*(K_C + \Theta)$$

K can. div is defined only up to linear equiv. $\sim$

$\omega = \mathcal{O}(K)$, $A \sim_{\mathbb{Q}} B \Leftrightarrow mA \sim mB$, $\exists$ $m > 0$.

In this way, sheaf theoretic expression is stronger than div. notation.

Theory of singularities:

Surface case:

Zariski decomposition (in his book)
(really the starting point of all MMT)

X : sm. proj. surface, D ef. div. $\sum d_j D_j$, $d_j \in \mathbb{N}$

$\exists$ $D = P + N$ as $\mathbb{Q}$-divisors, uniquely st.

P: nef $\mathbb{Q}$-div, N ef. $\mathbb{Q}$-div $= \sum n_j N_j$ st.

① $PN_j = 0 \quad \forall j$

② $[(N_j N_k)]_{j,k}$ negative definite.

the pf is a simple linear algebra. But its truth will implies many conjectures. eg. MMP. (in higher dim)

Connection with minimal models:

$$\begin{array}{lll} & X & \text{sm. proj. surf. } \kappa \geq 0 \\ & \downarrow & \\ f & \vdots & \text{contraction of } (-1) \text{ curves} \\ & \downarrow & \\ & X_{min} & \text{minimal model} \end{array}$$

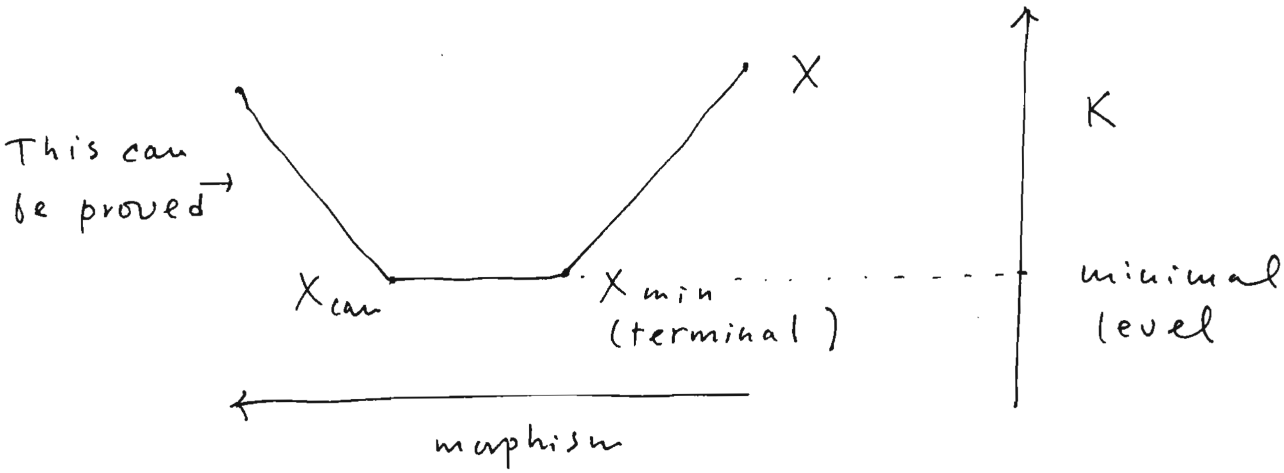
$K_X = f^* K_{X_{min}} + N$ is the Z-decomp.

$\kappa = 2$: $C \subset X_{min}$ $(K_{X_{min}} \cdot C) = 0 \iff C$ (-2) curve

$$\begin{array}{lll} & X_{min} & \text{(Artin's rational singularities)} \\ g & \downarrow & \text{contraction of } (-2) \text{ curves} \\ & X_{can} & \text{canonical model (Mumford)} \end{array}$$

$\rightsquigarrow$ can. ring is finitely, generated.

$K_{X_{can}}$ ample, $g^* K_{X_{can}} = K_{X_{min}}$



This picture is also true for higher dimensions!

$$\begin{array}{ccc} & X & \\ \alpha \swarrow & & \searrow \beta \\ X_1 & \dashrightarrow & X_2 \end{array}$$

bi-rat'l

may compare:

$$\alpha^* K_{X_1} \gtrsim \beta^* K_{X_2}$$

X min $\Longleftrightarrow$ K_X min

(broader sense: categorical sense in terms of morphisms)

X min $\Leftrightarrow$ minimal terminal (narrow sense)

Question: For a fixed birat'l ($K \geq 0$) class, there <u>exists</u> only a <u>finite</u> number (up to isom.) of minimal models?

For dim 3, general type finiteness is proved but for C-Y, it is still unknown.

Adjunction Formula :

X sm. variety , D sm. divisor

$$K_X + D \big|_D = K_D$$

log n-forms $\xrightarrow[\text{residue}]{}$ $(n-1)$ forms

log Pair : (X, B) X normal v. B ef. $\mathbb{Q}$-div

X sm. proj. surface , B n.c.d. (coef 1)

$(X \setminus B)$ open surface $\leftarrow$ Deligne's approach .

Kawamata's approach is really look at the pair :

if $m(K+B)$ effective $\exists\, m > 0$,

$$K_X + B = P + N = f^*(K_{X_{\ell t}} + B_{\ell t}) + N$$

Supp N is contracted , $X \xrightarrow{f} X_{\ell t}$ Grauert contraction

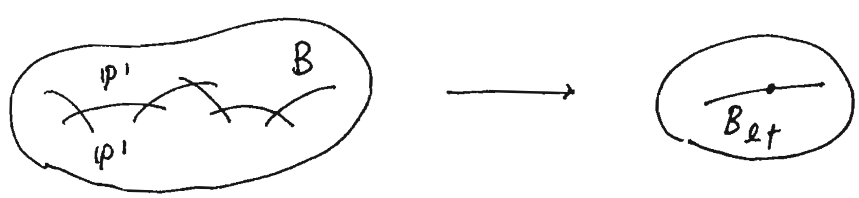
Artin (rational singularities) , $X_{\ell t}$ projective .

$B \longrightarrow B_{\ell t} = f_* B$ reduced .

$(X_{\ell t}, B_{\ell t})$ has singularities classified by

$\rightsquigarrow$ quotient sing. + standard divisor

ex.



cyclic quotient.

Def : (X, B) X normal , B ef. $\mathbb{Q}$-div

① $K_X + B$ $\mathbb{Q}$ Cartier. ie. $\exists\, m > 0$, $m(K_X + B)$ is Cartier div (pull back can be defined) .

② $\mu : Y \longrightarrow X$ resol. of sing.

$$\begin{array}{l} Y \\ \cup \\ E \quad \text{NCD} \quad (\text{log resol}) \\ \| \\ \sum E_j \end{array} \quad \begin{cases} Y - E \cong X - \mu(E) \\ \mu^{-1}(\operatorname{supp} B) \subset E \end{cases}$$

$$\frac{1}{m}\mu^*(m(K_X+B)) = \mu^*(K_X + B) = K_Y + \sum e_j E_j$$

log-canonical : $e_j \leq 1$

- klt (kawamata) : $e_j < 1$
- dlt (divisorial) : $e_j < 1$ $\forall E_j$ exceptional, $e_j \leq 1$
- plt (purely) : dlt and E_j with $e_j = 1$ are disjoint.

Rmk: In 2-dim case Mumford defines the intersection theory in all cases, hence condi ① is not needed. but for higher dim, M. Reid impose it. So that one may pull back etc.. and define int. theory.

Def. X normal

① K_X $\mathbb{Q}$-Cartier

② $\mu : Y \to X$ resol. $\mu^* K_X = K_Y + E$, $E = \sum e_j E_j$

X terminal $\iff$ $e_j < 0$ (sing in ter. min. model)

X cano $\iff$ $e_j \leq 0$ (sing in min model in broder sense)

$$\text{ter} \Rightarrow \text{can} \underset{\substack{\uparrow \\ B=0}}{\Rightarrow} \text{klt} \Rightarrow \text{plt} \Rightarrow \text{dlt} \Rightarrow \text{lc}$$

Thm: (X, B) dlt $\Rightarrow$ X has only rational sing. hence Cohen-Macaulay.

Rmk: lc may not be CM, eg. cone over abelian surf.

Th. X normal proj. (X, B) klt
L Cartier div. st. $L-(K_X+B)$ nef & big
$\Rightarrow$ $H^p(X, L)=0$ $\forall p>0$.

Also, klt $\Leftrightarrow$ L^2 analytically.

(X, B), $m(K+B) \longleftrightarrow L$ line bundle

$$\begin{array}{ccc} m(K+B) & \longleftrightarrow & L \\ \Psi & & \Psi \\ \varphi & & s \end{array}$$

(rational m-ple n-form)

$\varphi = \text{—} (dx_1 \wedge \dots \wedge dx_n)^{\otimes m}$ on X_{reg}

$\left| \int_{\text{locally}} \sqrt[m]{\varphi \wedge \bar{\varphi}} \right| < \infty$ easily from definition by calculating in the resolution.

Rmk: the finiteness is false for dlt and also the above vanishing thm is wrong.

(X, B), $\mu : Y \to X$ log resol. $B = \sum b_j B_j$, $0 < b_j < 1$

$\mu^*(K_X + B) = K_Y + E$, $E = \sum e_j E_j$.

consider the sheaf: <u>multiplier ideal sheaf</u>.

$$I = \mu_* \mathcal{O}(\ulcorner -E_+ \urcorner)$$

Ex. (X, B) klt $\Rightarrow \ulcorner -E_+ \urcorner \geq 0 \Rightarrow I = \mathcal{O}$

In general, under assumption on B, get $I \subset \mathcal{O}_X$

With I, would make things L^2:

$I \otimes \mathcal{O}(L)$ sheaf of L^2 sections

Th: X normal proj. (X, B) pair
L cartier, $L-(K+B)$ big & nef.
$\Rightarrow$ $H^p(X, I \otimes \mathcal{O}(L)) = 0$ $\forall p > 0$.

Def: (X, B) l.c. (non klt), call

E_j place of lc sing $\Leftrightarrow$ $e_j = 1$

$\mu(E_j)$ center of lc sing.

Prop:
$$\begin{array}{ccc} Y & \xrightarrow{\mu} & X \\ \cup & & \cup \\ E_j & \xrightarrow[\mu_0]{} & \mu(E_j) = W_j \end{array}$$
for E_j a place of l.c. sing. If W_j is minimal center

$\Rightarrow$ μ_0 alg. fiber space, W_j normal.

Prop: W_1, W_2 centers of lc. Then W an irred. comp of $W_1 \cap W_2$ $\Rightarrow$ W center of lc.

so has the notion of minimal center.

Rmk: scheme structure could be defined by $\mathcal{O}/I$.

so lc. $\Rightarrow$ $\operatorname{Spec}(\mathcal{O}/I)$ is reduced.

Th. (X, B°) klt, X projective, $B > B^\circ$,

(X, B) lc. $x \in X$, W min center for (X, B) through x. H ample, $\varepsilon > 0$

$$\Rightarrow K + B + \varepsilon H\big|_W \sim_{\mathbb{Q}} K_W + \underline{\overset{\exists}{B_W}} \quad \text{klt.}$$

This is the Main Result (very non-trivial).

which generalizes " $K + D|_D = K_D$ " via residue.

This appears in many intermediate steps in induction

eg. $\exists$ rational $\Rightarrow$ boundedness of mult. etc.

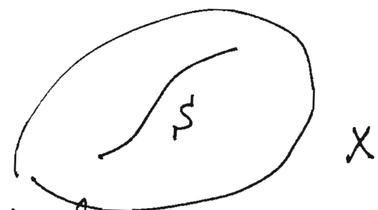
<u>To be continued.</u>

Fano Manifolds

Defined by Iskovskih : X proj. smooth, $-K_X$ ample
and by him : classification of Fano 3-folds

ex). 1) quartic 3-fold : $Bir(X) = Aut(X)$
2) $\mathbb{P}^3$: $Bir(\mathbb{P}^2)$ very large $\gg Aut(X)$

existence of a good member in $|-K|$,
$\exists S \in |-K|$, $S = K3$ surface
and use knowledge of S to
classify X.



Mukai : use $C \in |(-K)|_S|$: canonical curve
ladder : $C \subset S \subset X$ (Fujita's idea)

Kawamata's study of good member
– study of linear system

Def : (X, B) log Fano variety
(1) X projective
(2) (X, B) klt
(3) $-(K+B)$ ample

This is the "correct" category for study.

Recall : Base Point Free Theorem :
X proj, (X, B) klt, D Cartier div on X
assume ① D nef ② $D-(K+B)$ nef & big (ample)
$\Rightarrow \exists m_0 > 0$ st. $|mD|$ free $\forall m \geq m_0$.

(X, B), R: extremal ray

$\Rightarrow$ D satisfies condi (1), (2)

$(D.C) = 0 \iff [G] \in R$

$\Rightarrow$ $X \xrightarrow[\Phi_{|mD|}]{\varphi} Y$ contraction of R (alg. fiber sp.)

$\varphi(G) = pt \iff (D.G) = 0$

And ($K+B$ not nef $\Rightarrow \exists R$).

variety $\xrightarrow[\log MMP]{MMP}$ $\begin{cases} \text{minimal model:} \\ \quad \text{ie. } K+B \text{ nef, or} \\ \text{Mori fiber space:} \\ \quad \text{ie. } f: X \to Y \text{ contr. of } R \\ \quad \dim X > \dim Y \end{cases}$

for $\eta \in Y$, (X_η, B_η) is log Fano.

For minimal model:

$X \xrightarrow[m(K+B)]{} Y$: "biregular" Iitaka fiber space

this is the abundance conjecture:

$\exists\, m > 0$ st. $|m(K+B)|$ free.

For original BPF Thm. The problem now is:

effective bound m_0, or a "weaker - stronger":

Problem: (X, B) klt, D nef, $D-(K+B)$ ample

$? \Rightarrow |D| \neq \emptyset$?

Remark: It is equiv. to assume just big & nef

(exercise).

Fano index := r (largest) s.t.

$$-(K+B) \sim_{\mathbb{Q}} rH \ , \quad H \text{ Cartier ample} \ , \ r \in \mathbb{Q}$$

Truth of Prob $\Rightarrow |H| \neq \emptyset$, which is very strong !
(Kawamata wish a counterexample for the prob.)

Rmk: $r \leq n+1$ ($n = \dim X$) :

$P(t) = \chi(X, tH)$ poly of order n, $t \in \mathbb{Z}$

$H^p(X, tH) = 0 \quad \begin{cases} p > 0 \\ t > -r \end{cases}$

$tH - (K+B) = (t+r)H$

$H^0(X, tH) = 0$, $t < 0$, hence

$\chi(X, tH) = 0$ for $t = -1, -2, \cdots, -(n+1) \Rightarrow$

$-(n+1) > -r \Rightarrow$ contradiction, so $r \leq n+1$

and equality $\Rightarrow X \cong \mathbb{P}^n$, $B = 0$.

$\left\{ \begin{array}{l} \text{Non-Vanishing } |D| \neq \emptyset \leftarrow \text{Riemann-Roch formula} \\ \text{shape of general element } S \in |D| \leftarrow \text{adjunction theorem} \end{array} \right.$

Th: (Ambro; thesis) : (X, B) log-Fano

$-(K+B) \sim_{\mathbb{Q}} rH$, $r > n-3 \Rightarrow H^0(X, H) \neq 0$.

Th: (Kawamata) : $\dim X = 2$, (X, B) klt, D nef Cartier,

$D - (K+B)$ ample $\Rightarrow h^0(X, D) \neq 0$.

Th: (–) : (X, B), D as in Prob.

Prob is true if D ample $\Rightarrow$ Prob. true in general.

explaination: (reduction)

$\phi : X \xrightarrow{|mD|} Y$ alg. fiber space

D $m \gg 0$, E Cartier ample, $D = \phi^* E$ (BPF thm)

$H^0(D) \neq 0 \Leftrightarrow H^0(E) \neq 0$

$\exists B'$ on Y, (Y, B') klt

$E - (K_Y + B')$ ample

$\Big\rangle \to$ semi.posi thm.

Cor: $\nu(X, D) = \kappa(X, D) \leq 2 \Rightarrow$ Problem (too).

So it is very difficult to find counter examples

(can one find in toric varieties?)

Proof of Thm 1 (Ambro):

$$p(t) = \chi(tH) \qquad d = H^n > 0, \quad b = H^{n-1}B \geq 0$$

$$= \frac{t^n H^n}{n!} - \frac{t^{n-1} H^{n-1} K}{2(n-1)!} + \cdots + 1$$

$$= \frac{d}{n!} t^n + \frac{rd + b}{2(n-1)!} t^{n-1} + \cdots + 1$$

$$= \frac{d}{n!}(t+1)\cdots(t+n-3)\left(t^3 + At^2 + Bt + \frac{n(n-1)(n-2)}{d}\right)$$

$$A = \frac{n(n-r+s)}{2} - 3 + \frac{bn}{2d}$$

$$0 \leq (-1)^n p(-n+2) = -1 + \frac{d}{n(n-1)}\left((n-2)^2 - A(n-2) + B\right)$$

$$\| $$

$(-1)^n H^n(X, (-n+2)H)$ (call it P_{n-2})

then $P(1) = n - 1 + P_{n-2} + \dfrac{d(n-r+3)+b}{2} > 0$! $\square$

Rmk: this is first done by Alexeev for the case $r > n-4$.

(already very clever)

In Prob. $D-(K+B)$ ample

$\Rightarrow H^p(X, D) = 0 \ \forall p > 0$. so $h^0(X, D) = \chi(X, D)$

which is numerical! That's one reason why the prob. may be possible.

Th. $f: X \longrightarrow C$ = curve, alg. fiber space

(X, B) klt, D cartier on X, $D \sim_{\mathbb{Q}} K_{X/C} + B$

$\Rightarrow f_* \mathcal{O}(D)$ s.p. (log version, via covering tech.)

Proof of thm 2 (Kawamata):

may assume X smooth (min. resol.)

R.R.. $\chi(D) = \frac{1}{2} D(B+H) + \chi(\mathcal{O}_X)$

$\chi(\mathcal{O}_X) \geq 0 \Rightarrow \begin{cases} D \equiv 0 \rightsquigarrow \chi(\mathcal{O}) = 1 \\ D(B+H) > 0; \text{ OK}. \end{cases}$

$\chi(\mathcal{O}_X) = 1 - g < 0$, $X \xrightarrow{f} C$, $g(C) = g$

generically $\mathbb{P}^1$-bundle

$f_* \mathcal{O}(D - f^* K_C)$ s.p.

(condi. satisfied since:

$(D - f^* K_C) \sim_{\mathbb{Q}} K_{X/C} + B + H$, $H \sim_{\mathbb{Q}} B$ small.)

D f-nef $\Rightarrow$ D f-generated. ie.

$f^* f_* \mathcal{O}(D) \twoheadrightarrow \mathcal{O}(D)$

$\underset{\text{s.p.}}{f^* f_* \mathcal{O}(D - f^* K_C)} \twoheadrightarrow \underset{\text{s.p.}}{\mathcal{O}(D - f^* K_C)}$

$(D - f^*K_C)(B+H) \geq 0$

$D(B+H) = 2g-2, \quad \deg K_C = 2g-2$

$\Rightarrow$ contradiction. $\square$.

General member of linear system:

Thm (Ambro): (X,B) log-Fano, $r > n-3$,

$Y \in |H|$ general member (since know $\neq \emptyset$),

$\Rightarrow (X, B+Y)$ plt.

i.e. $\mu : Z \to X$:

$\mu^*(K_X + B + Y) = K_Z + \mu_*^{-1} B + \mu_*^{-1} Y + \sum e_j E_j$

$\forall e_j < 1$

$K_X + B + Y|_Y = K_Y + B_Y, \quad B_Y = B|_Y$

since Y Cartier div.

$RHS|_{\mu_*^{-1} Y} = K_{\mu_*^{-1} Y} + \underbrace{\boxed{}|_{\mu_*^{-1} Y}}_{\text{coeff} < 1}$

i.e. (Y, B_Y) klt.

$-(K_Y + B_Y) \sim_{\mathbb{Q}} (r-1) H|_Y$, hence $\exists$ ladder,

may proceed ...

Theorem (Kawamata): $\dim X = 4$

X canonical Gorenstein, $-K \sim D$ Cartier ample

$\Rightarrow$ ① $h^0(X, D) \neq 0$

② $Y \in |D|$ gen. $\Rightarrow (X, Y)$ plt ($\rightsquigarrow Y$ can. Gorenstein)

③ $h^0(D|_Y) \neq 0$

④ $Z \in |D|_Y| \Rightarrow (Y, Z)$ plt

⑤ X smooth $\Rightarrow Y$ smooth.

Proof by adjunction theorem:

suppose (X, Y) not plt,

$0 < c = \max\{ t \mid (X, B + tY) \text{ l.c.} \} \leq 1$

lc threshold.

$(X, B + cY)$ properly lc (lc & not klt)

W minimal center

$$K_X + B + cY + \varepsilon H \big|_W \sim_{\mathbb{Q}} K_W + B_W \quad \text{klt}$$

$$\wr_{\mathbb{Q}}$$

$$-rH + cH + \varepsilon H$$

$\dim W \geq 3$, $n \geq 4$, $r > 1$

$r - c - \varepsilon > 0 \quad \therefore (W, B_W)$ is log Fano

so $H^0(W, H_W) \neq 0$ by nonvanishing result proved before (Ambro, Kawamata).

get $*$: because $W \subset \mathrm{Bs}|H|$ (Bertini's thm)

$H^0(X, H) \to H^0(W, H) \to H^1(X, I_W \otimes \mathcal{O}(H)) = 0$

by perturbation, multiplier $= I_W$

(W is only center)

So $*$. $\square$.

Finally let's prove the adjunction theorem:

Th. $f: X \to Y$ alg. f. space, X, Y smooth

$\cup \qquad \cup$

$f_0: X_0 \to Y_0 \quad Y - Y_0 \ NC = E, \quad X_0 = f^{-1}(Y_0)$

$f^{-1}(E) \cup \mathrm{Supp}\, B$ NC. (X, B) sub klt

(coeff < 1) B may be non-ef.

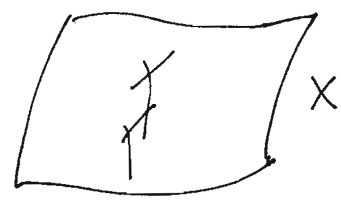
$\mathcal{O}_Y \to f_* \mathcal{O}_X(\lceil -BT \rceil)$ surjective over Y_0

$K + B \sim_{\mathbb{Q}} f^*(K_Y + L)$, L: $\mathbb{Q}$-Cartier

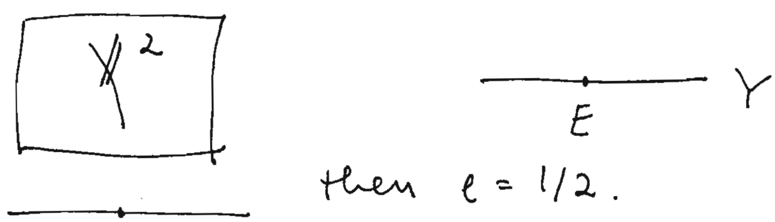
$B' = \sum_i e_j E_j$, e_j min st. $(X, B + f^*(E - B'))$ lc

$\Rightarrow L - B'$ nef. (call $M = L - B'$, and $\Delta = B'$ later)

ex. f semi-stable $\Rightarrow B' = 0$.
(& $B = 0$)



But for:



then $e = 1/2$.

So B' is something like how much need to do to the semi stable reduction case.

Proof: Semi-stable case: log version of s.p. thm.
general case: by covering

Th. Y proj. sm. E NCD $= \sum_{j=1}^{N} E_j$
$m_j \in \mathbb{N}$ $(j = 1 \dots N)$, then

$\exists\ \pi : X \to Y$ finite Galois, X smooth

$\pi^* E_j = m_j \overline{E_j}$, $\overline{E_j}$ reduced and $\sum \overline{E_j}$ NC.

ie. locally via covering is the semi-stable case with additional term B'. (correction term)

$B' \leftrightarrow$ eigenvalues of monodromy ! (rk 1 case)

for V.B. case still don't know the formulation.

Proof of adjunction thm:

$$\begin{array}{ccc} E & \subset & Y \\ {\scriptstyle f}\downarrow & & \\ V & & \downarrow{\scriptstyle \mu} \\ {\scriptstyle \sigma}\downarrow & & \\ W & \subset & X \end{array}$$

E : place of l.c. sing above W.

σ : resol. of W.

f : alg. fiber space.

$\mu^*(K+B) \sim_{\mathbb{Q}} K_Y + E + F$; eff $F < 1$

$\mu^*(K+B)|_E \sim_{\mathbb{Q}} K_E + F|_E$ — ample

$\sigma^*(K+B+\varepsilon H|_W) \sim_{\mathbb{Q}} K_V + (M + \varepsilon\sigma^* H - \varepsilon' Q) + \Delta + \varepsilon' Q$

this formula follows from positivity result.

Q : σ-exceptional eff, $0 < \varepsilon' \ll \varepsilon$

so $K + B + \varepsilon H|_W$ is klt. $\Box$.

Remark: If codim $W = 2$, then ε is not necessary.

reason: good moduli theory for $\mathcal{S}$ curves with $g = 0$ with marked pts. In this case $f: \begin{array}{c} E \\ \downarrow \\ V \end{array}$ is good and in fact M is semi-ample.

Finally, Fujita's conjecture (this is the source of all these l.c. center business).

Fujita Conjecture: X sm. proj. dim $= n$. H ample

$\Rightarrow$ $K_X + mH$ free : $m \geq n+1$

very ample : $m \geq n+2$.

still open!

Idea of proving this:

L ample, $L^n > n^n$ (eg. $L = (n+1)H$)

fix $x \in X$, R.R. $\exists$ $D \in |NL|$ | $\mathrm{mult}_x D > N \cdot n$

$\mathrm{mult} \underset{\underset{B}{\parallel}}{\dfrac{D}{\text{"}N(1+\alpha)\text{"}}} =: n$, (X, B) not klt at x.

l.c. threshold $c \leq 1$:

(X, cB) properly l.c at X.

W minimal center.

Suppose x <u>isolated</u> pt of W

$$\Rightarrow H^0(X, K+L) \twoheadrightarrow \underset{\underset{\mathbb{C}}{\parallel}}{H^0(X, K+L|_W)} \longrightarrow \underset{\underset{0}{\parallel}}{H^1(I_W \otimes \mathcal{O}(K+L))}$$

$\Rightarrow |K+L|$ free at x.

In this approach, it is necessary to extend Fujita's conj. to singular varieties:

Correct Version:

(X, B) klt, $x \in X$

$Y \xrightarrow{\mu} X$, $\mu^*(K+B) = K_Y + \sum e_j E_j$

minimal log discrepancy

$$\sigma := \min \left\{ 1 - e_j \;\middle|\; \mu(E_j) = \{x\} \text{ for all resolution } \mu \right\} > 0$$

ideal condition:

$\forall$ $W \ni x$, sub. v. thr x of $\dim = d$

$L^d W > \sigma^d \overset{?}{\Rightarrow}$ free at x. and "maybe"

$L^d W > (\sigma+1)^d \overset{?}{\Rightarrow}$ very ample at x.

Ask for proof in

- surface case ?
- Toric case ?

Partial result on surface:

Kawach - Masek : $L^2 > \operatorname{mult}(X, x)\, \sigma^2 \rightsquigarrow$ R.R.

The First

NCTS Summer School on Algebraic Geometry

July 19 — 30, 1999

Professor Eckart Viehweg
University of Essen

(Notes by Chin-Lung Wang)

Lecture I — 7/19, p.1

Safarevich Conjecture, Moduli, and Kodaira–Spencer Maps

Lecture II — 7/23, p.9

Construction of Moduli

Lecture III — 7/27, p.17

Positivity of Sheaves

Lecture IV — 7/30, p.26

Solutions to the Safarevich Problem

NCTS Summer School in Alg. Geom.

Prof. E. Viehweg

Lecture I, 7/14 at Academia Sinica

1954. Shafarevich Conjectures:

K number field, F curve over K, $g = g(F)$

1) $\#\{F/K$ with given set of places S of bad red.$\}$ is finite

2) $K = \mathbb{Q}$, $S = \phi$ then there no such curves.

$\underset{\Longrightarrow}{\text{Parshin}}$ Mordell. (Faltings, Yes).

F/K $\quad \mathcal{O}_K$: ring integers

$X \xrightarrow[f]{} \operatorname{Spec} \mathcal{O}_K$, f smooth over $\operatorname{Spec} \mathcal{O}_K - S$

1) $\operatorname{Spec} \mathcal{O}_K \longleftrightarrow$ Base curve B

$\operatorname{Spec} \mathcal{O}_{\mathbb{Q}} = \operatorname{Spec} \mathbb{Z} \longleftrightarrow \mathbb{A}^1 = B$.

Notations, $k = \bar{k}$, char $k = 0$ ($k = \mathbb{C}$)

$f : X \to B$, X, B non-singular proj $/k$

f : alg. fiber space + f flat

($\Leftrightarrow$ in this case $\dim f^{-1}(b)$ is constant)

f is a "family of algebraic varieties".

$S \subseteq B$, S : NCD

$B_0 = B - S$, $X_0 = f^{-1}(B_0) \xrightarrow[f_0]{} B_0$ smooth

let F = general fiber. $s = \# S$ if B curve.
F curve, $g = g(F)$, B curve, $S \subset B$ finite.

(I). $\#\{$Isom. classes of smooth non-iso-trivial families of curves $/ B_0$ of genus $g \} < \infty$

Def: $f : X \to B$ isotrivial $\iff$

$$X \times_B \overline{k(B)} \sim F \times_k \overline{k(B)}$$

binat'l

means all fibers all binat'l.

(make sense for higher dim'l also)

(II). If $2g(B) - 2 + s \leq 0$, then there are no non-isotrivial families

$(B, S) \neq (\mathbb{P}^1, \{0, \infty\}) \neq$ (Ellipt. ϕ)

Parshin (6?) $S = \phi$
Arakelov In general //

Problem: what about if $\dim F > 1$.

Assumption 1) F should be a minimal model

2) consider $f : X \to B$ (proj) with a polarization $\mathcal{L}$ ie. $\mathcal{L}$ inv. sheaf, f-ample.

(otherwise, $\exists$ families of K3 surfaces $X \to \mathbb{P}^1$ "twistor spaces". but this is non-algebraic.)

or just F = torus. but still total space no algebraic.

Faltings: (83) (I) is NOT true:

For (B, S) fixed, there exists pos. dim families of abelian varieties $/ B - S$.

Better : $\omega_{X/B} = \mathcal{O}(K_X - f^* K_B)$

$= \omega_X \otimes f^* \omega_B^{-1}$, $\omega_X = \Lambda^{\max} \Omega^1_X$.

$f: X \to B$ families of polarized minimal models of Kodaira dim $\kappa(F) \geq 0$.

i.e. ω_{X_0/B_0} f-nef.

let $\mathcal{L}$ be the polarization

$h(\nu) := \chi((\mathcal{L}|_F)^\nu)$ the hilbert polynomial

(I_a) Hope : Isom classes of non-isotrivial families $f_0 : X_0 \to B_0$ smooth polarized with given Hilb poly h should be parametrized by a scheme T of <u>finite type</u> / k

(I_b) Search for conditions implying that $\dim T = 0$.

(II) $2g(B) - 2 + s \leq 0 \Rightarrow T = \emptyset$.

Known :

(II) $\kappa(F) = 2$ (dim F), families of surfaces of general type

$(B, S) = (\text{Ell}, \emptyset)$ Migliomni

$(B, S) = (\mathbb{P}^1, \{0, \infty\})$ Kovacs

(II) OK if local Torelli theorem holds true

$\Rightarrow$ $\kappa(F) = 0$, $\dim F = 2$.

For $\dim F = 2$, $\kappa(F) = 1$, & Keiji Oguisio

true except $(B, S) = (\mathbb{P}^1, \{0, \infty\})$ & $\chi(\mathcal{O}_F) = 1$

2 multi fibers of multi m_1, m_2

$(m_1, m_2) = 1$, $m_1, m_2 \in \{2, 3, 4, 5\}$.

(I_a) F surface of general type , true by Bedulev, — .

(Ia) $\kappa(F)=0$ $\dim F=2$ follows from Hodge theory

(M. H. Saito, Zucker, Peters, Jost - Zuo, $\dim F \geq 2$)

$\kappa(F)=1$, open.

$\dim F > 2$, $\kappa(F)=\dim F$

ω_F ample,

MMP ($\dim F+1$) $\Rightarrow$ Ia true. Bedulov, —

Karu (recently) Existence of comp. of moduli

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§1 Moduli (Introduction)

Fix $h(t) \in \mathbb{Q}[t]$ hilb. poly.

$\mathcal{F}(k) = \{(X, \mathcal{L}) :$ X proj CM scheme + some prop.

$\omega_X^{[r]}$ invertible, nef, $\mathcal{L}$ ample and

$h(\nu) = \chi(\mathcal{L}^\nu)\} / \simeq$

$\omega_X^{[r]} := (\omega_X^r)^{\vee\vee}$.

$\mathcal{F}(T) = \left\{ (f : X \to T, \mathcal{L}) \;\middle|\; \begin{array}{l} f \text{ flat,} \\ \text{fibers are in } \mathcal{F}(k) \end{array} \right\}$

Definition (Mumford):

M coarse Moduli scheme $\overset{\text{def}}{\Longleftrightarrow}$

$\exists$ natural transformation $\phi : \mathcal{F} \longrightarrow \mathrm{Hom}(-, M)$

such that

(1). $\phi(\mathrm{spec}\, k) : \mathcal{F}(k) \xrightarrow{1-1} M(k)$

but in order to fix the structure sheaf, eg. elli case

(2) If $\varphi : \mathcal{F} \to \mathrm{Hom}(-, N)$ nat'l transf. $\mathbb{P}^1 \cup$ cusp.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[dr] & \mathrm{Hom}(-, N) \\ & \mathrm{Hom}(-, M) \arrow[u] \end{tikzcd}$$

$\Rightarrow \exists !\ M \to N$

Ex. Mumford,

$\mathcal{F} = M_g, \quad g \geq 2$

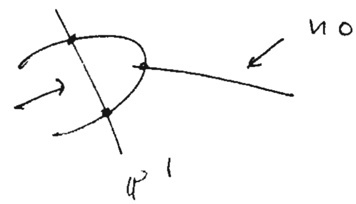
$M_g(k) = \{ C/k \mid C$ proj stable curve of genus $g \} / \cong$

i.e. C reduced 1-dim'l sing. are $\times$

st. ω_C ample.

$g = h^1(C, \mathcal{O}_C)$.

eg.



Theorem. (Mumford - Knudson)

M_g coarse moduli scheme for stable curves exists and M_g projective.

Moreover, $\exists$ invertible sheaf "λ_ν" such that

$$M_g(B) \ni f \quad \begin{matrix} X \\ \downarrow \\ B \xrightarrow{\varphi_f} M_g \end{matrix} \qquad \text{"} \varphi_f^* \lambda_\nu = \det (f_* \omega_{X/B}^{\nu}) \text{"}$$

unfortunately this is not true. the truth is

$\exists \ \lambda_\nu^{(p)} \quad p >> 0$ st. $\varphi_f^* \lambda_\nu^{(p)} = \det (f_* \omega_{X/B}^{\nu})^p$.

But for simplicity, we omit the p. (or think the $=$ in sense of $\mathbb{Q}$-divisor)

Addendum: $\lambda_\nu^{-\alpha} \otimes \lambda_{\nu\mu}^{\beta}$ ample on $M_g \quad \alpha, \beta > 0$

$\Rightarrow \lambda_\nu$ ample on M_g for $\nu \geq 2, \quad \nu, \mu >> 0$.

Interpretation of (II) $\longleftrightarrow$ M_g° locus of smooth curves is algebraically hyperbolic

i.e. $\nexists \ \mathbb{C}^* \subseteq M_g^\circ$

Elli $\subseteq M_g^\circ$.

2) [Kollár, Shepherd Barron, Alexeev]
There exists a definition of stable surfaces and a proj moduli space of stable surfaces, compactifying M_h° = moduli of surfaces of general type.

3) [Karu]:
MMP (dim + 1) ⇒ ∃ "stable canonically polarized n-fold"
Again Addendum true:
i.e. λ_ν ($\longleftrightarrow \det(f_* \omega_{X/B}^\nu)$) is ample on M_h, $\nu \geq 2$.
Very optimistic interpretation of (II).
find universal family after a cover

$$\begin{array}{ccc} Y_0 & \xrightarrow{\text{finite}} & M_h^\circ \\ \cap & & \cap \\ Y & \longrightarrow & M_h \end{array}$$

$D = Y - Y_0$ NCD
$\Omega_Y^1(\log D)$ = sheaf of diff forms with log poles along D.

locally $D = Z(x_1 \cdots x_s)$

$$\Omega_Y^1(\log D) = \left\langle \frac{dx_1}{x_1}, \dots \frac{dx_s}{x_s}, dx_{s+1}, \dots, dx_n \right\rangle \mathcal{O}_Y$$

$$\Omega_Y^1(\log D) \dashrightarrow \Omega_{\mathbb{P}^1}^1(\log S)$$

may think its the ampleness of $\Omega_Y^1(\log D)$ but this may not be true in general since we take a covering, and is outside somewhere ??

Def: $\mathcal{E}$ locally free on Y, $\mathcal{E}$ ample wrt. Y_0
$\Longleftrightarrow$ ∃ $\alpha > 0$, $\mathcal{H}$ ample inv. on Y
& ∃ $\bigoplus \mathcal{H} \hookrightarrow S^\alpha(\mathcal{E})$, ≅ over Y_0.

Remark: "Hope" true for M_g, $g \geq 2$

In general; open problems: M_h compactified moduli scheme (of can. polarized varieties), M_h° smooth part

$$\begin{array}{ccccc} Y - D & \xrightarrow[\varphi]{\text{finite}} & M_h^\circ & \overset{?}{\Rightarrow} & \det(\Omega_Y^1(\log D)) \\ \cap & & \cap & & = \omega_Y(D) \\ Y & \longrightarrow & M_h & & \text{ample w.r.t. } Y - D \end{array}$$

D: NCD

(I_a), $K(F) = \dim F$, Assume M_h exists

<u>Problem</u> (B) (boundedness) $f: X \to B$, B curve

f: families of minimal models on $f_0: X_0 \to B_0$.

$h(\nu) = \chi(\omega_F^\nu)$

<u>Show</u>: $\exists$ polynomial P depends only on h st.

$$\deg f_* \omega_{X/B}^\eta \leq P(g(B), \#S, \eta)$$

Now standard: $(B) \Rightarrow (I_a)$:

pf: remember λ_η ample on M_h.

$$\begin{array}{ccc} \mathbb{H} = \mathrm{Hom}\left((B, B_0), (M_h, M_h^\circ)\right) & \subseteq & \mathrm{Hom}(B, M_h) \\ \text{UI} & & \cup \\ \mathbb{H}_P\left((B, B_0), (M_h, M_h^\circ)\right) & \subseteq & \mathrm{Hom}_P(B, M_h) \\ \text{UI} & & \uparrow \\ & \text{scheme of finite type} & \varphi: B \to M_h \\ & & \varphi^* \lambda_\eta \leq P(\dots) \end{array}$$

Since the moduli is only coarse, consider

$$T = \{ \varphi: (B, B_0) \to (M_h, M_h^\circ), \text{ induced by } \text{"}f: X \to B\text{"} \in M_h(B) \} / \cong . \quad \square$$

§2. Diff forms and Kodaira-Spencer map.

Ex. X elliptic surface

$$\begin{tikzcd} & & \mathbb{P}^1 \arrow[d, phantom, "\parallel"] \\ f : X \arrow[r] & B \arrow[r, "\text{finite}"'] & M_1 \\ & & M_1^0 = \mathbb{A}^1 \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$f : X \to B$ non-isotrivial, $\exists\ B \longrightarrow M_1$, $B = \mathbb{P}^1$

$\overset{?}{\Rightarrow}$ f has at least 3 deg. fibers.

$\omega_{X/B} = f^* \Theta$, $\Theta \geq 0$ (Kodaira's formula)

$\deg(f_* \omega^2_{X/B}) > 0$.

consider: $0 \to f^* \Omega^1_B(\log S) \to \Omega^1_X(\log f^{-1}S) \to \Omega^1_{X/S}(\log f^{-1}(S)) \to 0$

exact sequence of vector bundles.

$$0 \to T_{X/B}\langle -f^{-1}(S)\rangle \to T^1_X\langle -f^{-1}S\rangle \to f^* T^1_B\langle -S\rangle \to 0$$

$$\begin{tikzcd} T^1_B\langle -S\rangle \arrow[r, "*"] \arrow[d, phantom, "\parallel"] & R^1 f_* T_{X/B}\langle -f^{-1}(S)\rangle \arrow[d, dash] \\ \mathcal{O}_B \arrow[ur, "\neq 0"'] & (f_* \omega^2_{X/B})^\vee \end{tikzcd}$$

← cheating a little here (need handle multi fibers)

← $\deg < 0$.

$B = \mathbb{P}^1$, $S = \{0, \infty\}$.

this proof is the "most complicated pf" in the ellipt case. but it holds true for general g, and also to higher dim. #.

<u>To be continued</u>.

$f: X \to B$ families of curves,
surfaces of general type
or. can. polarized manifolds.

Need diff forms & K-S Map:
positivity properties for $f_* \omega^\nu_{X/B}$ (not just $\nu = 1$)
$\rightsquigarrow$ moduli schemes.

§3. Construction of moduli

$h \in \mathbb{Q}[t]$, $\deg h = n = \dim(\text{---})$

$$M_h^\circ = \left\{ X \;\middle|\; \begin{array}{l} X \text{ normal reduced scheme with at} \\ \text{most RDP}, \omega_X \text{ ample}, \chi(\omega_X^\nu) = h(\nu) \end{array} \right\} \Big/ \cong$$

Need compatification of moduli Problem.

Ex. $n=1$, $M_h^\circ = M_g^\circ$

$$M_g^\circ = \left\{ X \;\middle|\; \begin{array}{l} X \text{ stable curve}, \omega_X \text{ ample} \\ \text{only transversal int. as sing.} \end{array} \right\} \Big/ \cong$$

$n=2$: $\exists$ definition of stable surfaces [Kollár, S-B].
More general definition will be given later.

M_h bounded $\iff \exists H$ & $\mathfrak{X} \xrightarrow[f]{} H$ fiber space
H scheme of finite type
st. each $X \in M_h(k)$ occurs as
a fiber of f.

Ex. $M_h^\circ(k)$ bounded.

Def: X $\mathbb{Q}$-Gorenstein, $\omega_X^{[r]}$ invertible some $r \gg 0$, X semi-log can. sing.

$\Leftrightarrow$ i) X satisfies Serre's condition S_2

ii) X NCD in Codim 1

iii) $\forall$ $f: Y \to X$, Y normal $\mathbb{Q}$-Gorenstein

$$\omega_Y^{[r]} = f^* \omega_X^{[r]} \otimes \mathcal{O}(\Sigma a_i E_i) \quad a_i \geq -r.$$

Def: Stable n-folds:

connected proj n-dim'l X

* semi-log-can. sing.
* smoothable
* $\omega_X^{[r]}$ ample, (invertible)

$M_h(k) = \{ X : X$ stable n-fold, Hilb poly $= h \}$

$M_h(T) = \{ f: \mathfrak{X} \to T$, that fiber $\in M_h(k)$ and $\omega_{\mathfrak{X}/T}^{[r]}$ invertible $\} / \simeq$

Properties:

A). Boundedness: $\exists$ μ st. $\forall$ $X \in M_h(k)$ $\omega_X^{[\mu]}$ very ample (& invertible)

$n = 1$ exercise

$n = 2$ Alexeev

$n \geq 2$: Thm (Karu): Assume MMP $(n+1)$

$\Rightarrow M_h$ is bounded (in dim n).

Idea of pf: M_h° bounded $\rightsquigarrow$ $\mathfrak{X}_0 \xrightarrow{f_0} H_0$

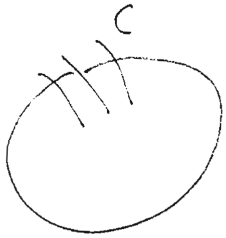
compactify $\mathfrak{X} \xrightarrow{f} H$, Apply weakly s.s. alteration

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "f'"] & H' \arrow[d, "\text{finite}"] \\ \mathcal{X} \arrow[r, "f"] & H \end{tikzcd}$$

Need to know

$$\mathcal{Y} = \mathrm{Proj}\Big(\bigoplus_{\nu} f'_* \, \omega^{[r]^{\nu}}_{\mathcal{X}'/H'}\Big)$$

need to be finitely generated.



$$\bigoplus f'_* \, \omega^{[r]^{\nu}}_{\mathcal{X}/C} \,/C$$

need Siu-Kawamata's thm on invariance of plurigenus. □

Cor. There exists a Hilbert scheme H and universal family $f: \mathcal{X} \to H \in M_h(H)$ together with

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[dr] & \mathbb{P}(f_* \omega^{[r]}_{\mathcal{X}/H}) \simeq \mathbb{P}^{h(r)-1} \times H \arrow[d] \\ & H \end{tikzcd}$$

Idea (old): $r \gg 0$

$$|\omega_X^{[r]}| : X \hookrightarrow \mathbb{P}^{h(r)}, \quad \forall X \in M_h(k)$$

⇒ subvarieties of bounded degree ($\leftrightarrow h$) $\subseteq \mathbb{P}^{\bullet}$ are parametrized by $\mathrm{Hilb}(\cdot, h)$ & $\mathcal{Y} \to \mathrm{Hilb}$.

→ need locally closedness.

B). H quasi-projective:

$\mathcal{X} \xrightarrow{f} H$, $f_* \omega^{[r]}_{\mathcal{X}/H} \cong \bigoplus^{h(r)} N$, N inv.

$$N^{-n} \otimes \underbrace{S^{\mu} f_* \omega^{[r]}_{\mathcal{X}/H}}_{\text{just copies of } \mathcal{O}} \twoheadrightarrow f_* \omega^{[\nu r]}_{\mathcal{X}/H} \otimes N^{-\mu}$$

⇒ $X \to \mathrm{Grass}(\cdots)$, which is projective.

& ample sheaf is of the form

$$\lambda_\eta := \det (f_* \omega^{\eta}_{\mathcal{X}/H})$$

$$\lambda^{\alpha}_{\nu\mu} \otimes \lambda^{-\beta}_{\nu} \quad \alpha, \beta \in \mathbb{N} \quad \text{will do.} \quad \square .$$

c). "stable reduction"

true by definition for $n \geq 2$.
exercise for $n = 1$.

D). $G = SL(h(\nu), k)$ acts on H

$G \times H \xrightarrow{\sigma} H$ action is proper with finite stabilizers.

proper $:=$ $G \times H \xrightarrow{(\sigma, pr_2)} H \times H$ is proper

this actually forces birat'l maps of general fiber in family extends to the central fiber.
(or maybe isom. on general fiber $\Rightarrow$ birat'l)

THEOREM : A, B, D $\Rightarrow$ $\exists$ coarse quasi-projective moduli scheme M_h, and
C $\Rightarrow$ M_h projective.

Ex. MMP $(n+1)$ $\Rightarrow$ M_h as above
without $\Rightarrow$ $M^{\circ}_h \longleftrightarrow X$ with normal RDP
similarly X with can. sing. (Kawamata).

Proof: Assume H/G exists

$\underset{?}{\Longrightarrow}$ $H/G = M_h$

Def: H q. proj. G red. linear alg. gp

$\pi : H \to Z = H/G$ geom. quotient

$\Leftrightarrow$ 1) π compatible with G-action

2) $\mathcal{O}_Z = (\pi_* \mathcal{O}_H)^G$

3) $W_i \subset H$ G-inv closed $\Rightarrow \pi(W_i)$ closed, $i=1,2$

& $W_1 \cap W_2 = \phi \Rightarrow \pi(W_1) \cap \pi(W_2) = \phi$

4) $\pi^{-1}(w)$ one G-orbit.

Cor. If H/G geom. quotient exists $\Rightarrow H/G \cong M_h$.

pf: gluing: $g : \mathcal{Y} \to T$

$g_* \omega_{\mathcal{Y}/T}^{[r]} |_H \xrightarrow{\sim} \oplus \mathcal{O}_U$ on an open set.

:

(2nd) universal property follows from 1) and 2).

(easy exercise).

So the important thing is to construct H/G:

Seshadri's elimination of finite isotropies (~to)

G: reductive linear alg. gp H quasi-proj

$\sigma : G \times H \to H$ proper gp action with f stable

$\Rightarrow \exists$
$$\begin{array}{ccc} V & \xrightarrow{\pi} & Z \\ \downarrow p & & \\ H & & \end{array}$$
with 1) σ lifts to $\Sigma : G \times V \to V$

2) $\pi : V \to Z$ geom. quotient & principal G-bundle in Zariski top, $\pi^{-1}(u) \cong G \times U$, U open.

3) V/H Galois cover, gp Γ, and Γ commutes with G.

Pf is a simple construction :

$$\begin{array}{c} G \times Y \\ \downarrow \\ V_X \\ \downarrow p_X \\ H \end{array}$$

not proper

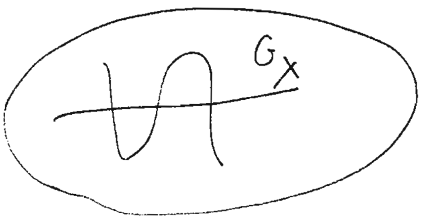
1) true by definition

2) true in nbd of $p_X^{-1}(G_X)$

Now use glueing :

Consider

$$\begin{tikzcd} & V_X \times_H V_y \arrow[dl] \arrow[d] \arrow[dr] & \\ V_X \arrow[r] & H & V_y \arrow[l] \end{tikzcd}$$



take normalization $V_{XY} = \widetilde{V_X \times_H V_y}$.

1), 2) in nbd of G_X and G_y.

After a finite number of steps : get

$V' \xrightarrow{p'} H$ s.t. 1), 2). surj. not proper

irred.

V = normalization of H in Galois hull of

$K = \langle k(V_i) \rangle \supset k(H)$, $\Rightarrow$ 1), 2), 3).

this is approximately the proof. $\square$

Cor. $\tilde{H}/G = Z/\Gamma$ if the later exists.

2 problems. $\sim$ and Γ.

Remark: Z q. proj $\Rightarrow$ Z/Γ exists and q. proj. but one Z is not q.p.

Recall:

$$\begin{tikzcd} \mathfrak{X} \times_H V \arrow[dr] & & \\ & V \arrow[r] \arrow[d] & Z \\ \mathfrak{X} \arrow[r] & H & \end{tikzcd}$$

locally $E \times U \to U$

$$\uparrow$$

$\mathfrak{X} \times_H V \,|_{E \times U} = \mathfrak{X}_U$

$\Rightarrow$ glue to $Y \xrightarrow{g} Z$

$\exists\ g: Y \to Z$ st. $\forall z_0 \in Z$

$\{z \in Z \mid g^{-1}(z_0) \simeq g^{-1}(z)\}$ is finite $(*)$

- Thm: $g: Y \to Z$ family with property $(*)$ & $g \in M_h(Z)$ then Z is quasi-proj

This thm is true, but very hard to prove. But if with Z proper, then the pf is much easier, & ample sheaves $\det(g_* \omega_{Y/Z}^{[r]})$ $\nu \gg 0$.

Cor. Z/P exists $= \tilde{H}/G$.

Prob: $\tilde{H}/G$ Normalization of unknown object M_h.

Two ways out:

A). Show that M_h exists as a <u>proper</u> alg. space

$\Rightarrow \tilde{M}_h \xrightarrow{p} M_h$ with λ_ν on M_h,

$p^*\lambda_\nu$ ample $\Rightarrow M_h$ proj. scheme.

this method only works for proper moduli prob.

B). Construct H/G by GIT.

use Methods of proof of Thm $\circ$ (stability).

c). For our problem, $\tilde{M}_h$ is enough

Remarks: for M_h°, B) is the only known method.

For proof of Thm:

Def: a) $\mathcal{E}$ locally free on Z, $\mathcal{E}$ is semi-positive: if Z proper, $\forall\ \tau: C \to Z$, C curve,

$\forall\ \tau^*\mathcal{E} \to \mathcal{L}$ inv $\Rightarrow \deg_C(\mathcal{L}) \geq 0$.

For Z arbitrary, $\mathcal{E}$ semi-positive $\Leftrightarrow$

$\forall\ \tau : Z' \hookrightarrow Z$, Z' q. proj

$\forall\ \mathcal{H}'$ inv. ample on Z, $\forall\ \alpha > 0$

$$S^{\alpha}(\tau^* \mathcal{E}) \otimes \mathcal{H}' \text{ ample}$$

Remark: Z q. proj, $Z' = Z$ enough.

We Need:

A) $g_* \omega^{[r]}_{Y/Z}$ semi positive for all $r \gg 0$ & divisible.

easy (Z proper); hard (Z not proper)

B) A) + (*) $\Rightarrow$ Theorem. $\square$

<u>To be continued</u>.

Positivity of Sheaves

- $f: X \to B$: curve $\qquad \kappa(F) = \dim F$
- $g: Y \to Z \in M_h(Z)$

$f_* \omega_{X/B}^{\nu}$. for $\nu = 1$ already in Kawamata's lecture.

$\mathcal{E}$ loc. free on Y

Y proper, $\mathcal{E}$ num. s.p. (nef) $\overset{def}{\iff}$ $\forall \tau: C \to Y$

$\forall \tau^* \mathcal{E} \to \mathcal{L} \to 0$, $\deg \mathcal{L} \geq 0$.

Y arbitrary, $\mathcal{E}$ s.p. (w.p.) $\overset{def}{\iff}$ $\forall \tau: Y' \to Y$ q. projective

$\forall \mathcal{H}'$ ample inv on Y', $\forall a > 0$

$S^a(\tau^* \mathcal{E}) \otimes \mathcal{H}'$ ample.

Ex. $f: X \to B$: B arbitrary.

$f_* \omega_{X/B}$ n.s.p. if monodromy are unipotent
($B - B_0$ NCD, B smooth)

without monodromy condition, can only say

$\exists \mathcal{E} \subset S^?(f_* \omega_{X/B})$, $\mathcal{E}$ num. s.p.

But for B curve, this is $\iff$ $f_* \omega_{X/B}$ n.s.p.

Main Reference : Mori : Bowdoin '85 (general)

For B curve : Esnault, —, Compositio '76

also : Bedulev, — . (preprint)

Starting Point:

Theorem (Fujita): $f: X \to B^1$ family of manifolds

$\Rightarrow f_* \omega_{X/B}$ n.s.p, X, B nonsingular.

One possible proof using alg. method:

pf: Kollár's vanishing:

X projective mfd, $\mathcal{L}$ semi-ample on X

D ef. st. $H^0(X, \mathcal{L}^\nu(-D)) \neq 0$ for some $\nu > 0$

$\Rightarrow H^i(X, \mathcal{L} \otimes \omega_X(D)) \twoheadrightarrow H^i(D, \mathcal{L} \otimes \omega_D) \ \forall i$.

Take $\mathcal{L} = f^* \mathcal{O}_B(pt)$, $D = F =$ fiber, get

$$\begin{array}{ccc} H^0(X, \mathcal{L} \otimes \omega_X(F)) & \twoheadrightarrow & H^0(F, \mathcal{L} \otimes \omega_F) \\ \| & & \| \\ H^0(B, (f_* \mathcal{L} \otimes \omega_X) \otimes \mathcal{O}(p)) & \twoheadrightarrow & f_*(\mathcal{L} \otimes \omega_X) \otimes k(p) \end{array}$$

ie. $f_* \omega_{X/B} \otimes \omega_B \otimes \mathcal{O}_B(2p)$ is globally generated.

Now

$$\begin{tikzcd} X^r = X \times_B \cdots \times_B X \arrow[r, "f^r"] & B \\ X^{(r)} \arrow[u, "\delta"] \arrow[ur, "f^{(r)}"'] & \end{tikzcd}$$

resd. $-\delta$

so $f_*^{(r)} \omega_{X^{(r)}/B} \otimes \omega_B(2p)$ is also globally gen.

$\downarrow$ over some open set

$$f_*^\nu \omega_{X^\nu/B} \otimes \omega_B(2p) = \left(\otimes^\nu (f_* \omega_{X/B})\right) \otimes \omega_B(2p)$$

$$\longrightarrow S^\nu(f_* \omega_{X/B}) \otimes \omega_B(2p)$$

Now B curve $\Rightarrow f_* \omega_{X/B}$ n.s.p.

Same proof works except the last step.
and also sing. occures in the resolution map.

Theorem (Kawamata): $\dim B \geq 1$

$f : X \to B$ family of manifolds, X, B nonsingular
$B - B_0$ NCD, $\Rightarrow f_* \omega_{X/B}$ n.s.p.
for $f : X \to B \in M_n(B)$.

Recall Multiplier ideals:

X mfd, $D \geq 0$ ef div, $N \in \mathbb{N}$, $\tau : X' \to X$ st.
$\tau^* D$: NCD.
$$\omega_X\left\{-\frac{D}{N}\right\} = \tau_* \omega_{X'}\left(-\left[\frac{\tau^* D}{N}\right]\right)$$
$$\left(= \omega_X\left(\lceil -\frac{D}{N} \rceil\right) \text{ in Kawamata's notation}\right)$$

1). independent of τ
2). "$\mathcal{L}^N = \mathcal{O}(-D)$" $\Rightarrow \mathcal{L} \otimes \omega_X\left\{\frac{-D}{N}\right\}$ has similar properties as ω_X (eg. vanishing)

Reason: a) $R^i \tau_* \omega_{X'}\left(-\left[\frac{D'}{N}\right]\right) = 0$

b) $\exists$ cyclic covering $\gamma : Z' \to X'$ st.
$$\gamma_* \omega_{Z'} \rightleftarrows \mathcal{L} \otimes \omega_{X'}\left(-\left[\frac{D'}{N}\right]\right) \quad \text{a factor. (split)}$$

Cor. $\mathcal{L}$ inv. on X, $\mathcal{L}^N = \mathcal{O}_X(D)$
$$\Rightarrow f_*\left(\omega_{X/B}\left\{-\frac{D}{N}\right\} \otimes \mathcal{L}\right) = (*) \quad \text{n.s.p.}$$
for B curve.

Reason: $(*) \rightleftarrows f_* \omega_{Z'/B}$.

Properties:

$$e(\mathcal{L}) := \mathrm{Min}\left\{ e \;\middle|\; \omega_{X/B}\left\{-\frac{D}{e}\right\} = \omega_{X/B},\ \forall D \geq 0 \right\}$$

where $\mathcal{L} = \mathcal{O}_X(D)$

i.e. the minimal number to kill all multiply ideals.

1) $e(\mathcal{L}) < \infty$

2) $\mathcal{L}$ very ample : $e(\mathcal{L}) \leq c_1(\mathcal{L})^{\dim X} + 1$

3) Behave nicely under product : $X \times \cdots \times X \xrightarrow{pr_i} X$

$$e\left(\otimes\, pr_i^* \mathcal{L}\right) = e(\mathcal{L})$$

Cor. $f_* \omega^{\nu}_{X/B}$ n.s.p. $\forall \nu > 0$

Pf: $f_* \omega^{\nu}_{X/B} \otimes \mathcal{O}_B(\eta)$ n.s.p. for some $0 < \eta = \nu\rho - 1$ with ρ min.

$\Rightarrow f_* \omega^{\nu}_{X/B} \otimes \mathcal{O}_B(\nu\rho)$ ample

$\Rightarrow \omega^{\nu\mu(\nu-1)}_{X/B}(\mu(\nu-1)\cdot\underline{\nu\cdot\rho})$ "gen. by global sections along general fiber".

$\mathcal{O}(D) :=$

in fact not quite correct unless take away some base locus in fiber.

eg. WF semi-ample. (let's only do this case)

take $\mathcal{L} = \omega^{\nu-1}_{X/B}((\nu-1)\rho)$, $N = \nu\mu$

$\Rightarrow f_*\left(\omega^{\nu-1}_{X/B}((\nu-1)\rho)\otimes\omega_{X/B}\{-\frac{D}{N}\}\right)$ n.s.p.

$\downarrow \cong$ over some open set

$f_* \omega^{\nu}_{X/B}((\nu-1)\rho) \Rightarrow$ n.s.p.

So must $\nu(\rho-1)-1 < (\nu-1)\rho$ ie. $\rho < \nu+1$

ie. the twist is only bounded by ν !

Same true if

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow \\ G:\ B' & \longrightarrow & B \end{array} \quad \text{finite cover}$$

$f'_* \omega^{\nu}_{X'/B'} \subset G^* f_* \omega^{\nu}_{X/B}$

$\simeq$ over some open set

Some may take cover to make " $\rho < \nu + 1$ " number < 1. Then done. $\square$

Main Prob. for general B: D near singular fiber would be very bad!

Main ingredients for above pf:

- cyclic cover.
- product.

Next cor. (weak stability) true for arbitrary morphism and arb. variety, but again we only deal with semi-ample fiber case: and B curve.

Assume that $f_* \omega^{\nu}_{X/B} \neq 0$ $(\nu \geq 2)$

Equivalently:

a) $f_* \omega^{\nu}_{X/B}$ ample

b) $\det(f_* \omega^{\nu}_{X/B})$ ample

c) $\exists\, \eta, \alpha > 0$: $\mathcal{O}(\eta p) \hookrightarrow \bigotimes^{\alpha} (f_* \omega^{\nu}_{X/B})$

a) $\Rightarrow$ b) $\Rightarrow$ c), $\alpha = \mathrm{rk}(f_* \omega^{\nu}_{X/B})$

For c) $\Rightarrow$ a). replace B by some finite cover

$\rightsquigarrow \eta \gg 0$.

$$\begin{tikzcd} X^\alpha = X \times_B \cdots \times_B X \arrow[r, "F^\alpha"] & B \\ X^{(\alpha)} \arrow[u, "\delta"] \arrow[ur, dashed, "f^{(\alpha)}"'] & \end{tikzcd}$$

$\mathcal{M} = \delta^* \bigotimes^\alpha pr_i^* \, \omega_{X/B}$, 2 morphisms :

$$f_*^\alpha \delta_* \, \omega_{X^{(\alpha)}/B} \otimes \mathcal{M}^{\nu-1} \longrightarrow f_*^\alpha \, \omega^\nu_{X^\alpha/B} = \bigotimes^\alpha f_* \omega^\nu_{X/B}$$

$$\bigotimes^\alpha f_* \omega^\nu_{X/B} \longrightarrow f_*^\alpha \left(\delta_* \mathcal{O}_{X^{(\alpha)}} \right) \otimes \bigotimes pr_i^* \, \omega^\nu_{X/B} = f_*^{(\alpha)} \mathcal{M}^\nu$$

$\mathcal{M}^\nu$ has section with zero divisor $\Gamma + \eta F$

$(\mathcal{M}^\nu(\overset{\alpha}{F}))^N$ has lots of sections

since $f_* \mathcal{M}^\nu$ is n.s.p.

Remember we assume ω_F semi-ample

let D general section, $D|_{F^\alpha}$ nonsingular

Apply Fujita's positivity thm :

$$f_*\left(\omega_{X/B}^{\nu-1} \otimes \omega_{X/B}\left\{ -\frac{D}{N} \right\} \otimes \mathcal{O}((\nu-1)\rho F^\alpha) \right) \leftarrow \text{cancel}$$

consider

$$\mathcal{M}^{\nu(N+1)(\nu-1)} = \mathcal{M}^{\nu N}(NF^\alpha) \otimes \mathcal{O}(\Gamma + (\eta - N)F^\alpha)^{\nu-1}$$

$$= \mathcal{O}\Big(\underbrace{(\nu-1) D + (\nu-1)\Gamma + (\nu-1)(\eta-N) F}_{\Delta} \Big)$$

$\mathcal{L} = \mathcal{M}^{\nu-1}$

cor. $\Rightarrow$ $f_*^{(\alpha)} \omega_{X^{(\alpha)}/B}\left\{ -\frac{\Delta}{\nu(N+1)} \right\} \otimes \mathcal{M}^{\nu-1} = (*)$

$$\Big\downarrow \quad \simeq \text{ on open set}$$

$$f_*^{(\alpha)} \, \omega^\nu_{X^{(\alpha)}/B} \subset \overset{\alpha}{\bigotimes} \left(f_* \omega^\nu_{X/B} \right).$$

On general fiber:

$$\omega_{F^\alpha}\left\{-\frac{D(\nu-1)+P(\nu-1)}{\nu(N+1)}\right\} \simeq \omega_{F^\alpha}$$

for $N \gg 0$

$(\eta - N) > 0$, $\left[\frac{(\nu-1)(\eta-N)}{\nu(N+1)}\right] > 1$ (may chose)

this is the place to play. see below

hence $(*) \hookrightarrow f_*^{(\alpha)} \omega^{\nu}_{X^{(\alpha)}/B}(-F^\alpha)$

$\simeq$ over the general fiber

and we are done. $\square$

More Precise (will be needed later for Shaf. conj (B) and (II)):

<u>Cor.</u> $r(\nu) = \alpha = \text{rank}(f_* \omega^{\nu}_{X/B})$, $e(\omega^{\nu}_F) = e(\nu)$

$\Rightarrow S^{r(\nu)\, e(\nu)}(f_* \omega^{\nu}_{X/B}) \otimes \det(f_* \omega^{\nu}_{X/B})^{-1}$ n.s.p.

(quite precise numbers!)

Mainly because:

$$e(M^{\nu}_{F\, r(\nu)}) = e(\omega^{\nu}_F). \quad \text{EXERCISE.}$$

Rem. & Cor:

Thm: Assume $f: X \to B$ is semi-stable, Then

f isotrivial $\Leftrightarrow \deg[\det(f_* \omega^{\nu}_{X/B})] = 0 \;\; \forall \nu \geq 0$

If $\kappa(F) \geq 0$, either F general type (Kollár / Viehweg)

or F has minimal model F' and

$\omega_{F'}$ semi-ample (Kawamata).

Cor. $f : X \to B$, F as in thm.

$$\begin{tikzcd} \sigma : Y \arrow[rr] \arrow[dr, "g"'] & & X \arrow[dl, "f"] \\ & B & \end{tikzcd}$$ dominant

g isotrivial $\Rightarrow$ f isotrivial.

$f_* \omega_{X/B}^\vee < g_* \omega_{Y/B}^\vee$.

Recently, Mok-Hwang have similar result of this cor for families of Fano varieties.

To finish the discussion of semi-positivity, we need

§ Strong Positivity:

Recall:

$$\begin{tikzcd} & V \arrow[r, "\pi"] \arrow[d] & Z & \mathcal{Y} \arrow[l, "g"'] \in \mathcal{M}_h(Z) \\ \mathcal{X} \arrow[r, "f"] & H & & \end{tikzcd}$$

"universal family"

$\mathcal{E}_\nu := g_* \omega_{\mathcal{Y}/Z}^\vee$ semi-positive (n.s.p if Z proper)

The proof is very hard (if Z not proper) which does not follow from above discussion.

(Most book in moduli spent 1/2 to do this!)

Proof: If Z proper $\rightsquigarrow$ reduce to curve (since have numerical criterion)

$$\begin{array}{l} X \\ \downarrow f \in \mathcal{M}_h(B) \\ B \subseteq Z \end{array}$$

"as above"

$f_* \omega_{X/B}^\vee = g_* \omega_{\mathcal{Y}/Z}^\vee |_B$ n.s.p !

(a little cheating: should be $\omega^{[\nu]}$!).

Theorem: $\lambda_\nu := \det(g_* \omega^\nu_{Y/Z})$

$\exists\ \alpha, \beta, \mu \gg 0$ st. $\lambda^\alpha_{\nu\mu} \otimes \lambda^\beta_\mu$ ample Z.

If Z proper, $\exists\ \mu \gg 0$ st. $\lambda_{\nu\mu}$ ample.

a) Similar to GIT

b) observed by Kollár via Numerical criteria (which is simpler)

Will explain 1st pf a) here:

let $\xi = \xi_\nu = g_* \omega^\nu_{Y/Z}$

$\mathbb{P} = \mathbb{P}(\oplus^\nu \xi^\vee) \xrightarrow{\pi} Z \Rightarrow \pi^* \oplus^\nu \xi^\vee \twoheadrightarrow \mathcal{O}(1)$

① $\mathcal{O}(1) \otimes \det(\pi^* \xi)$ (n.) s.p.

$$\begin{tikzcd} \mathcal{O}(1) \otimes \det(\pi^* \xi) \\ \bigoplus^r \pi^*(\underbrace{\xi^\vee \otimes \det(\xi)}_{\Lambda^{r-1}\xi}) \arrow[u, two heads] \end{tikzcd}$$

② $\overset{r}{\oplus}\, \mathcal{O}(-1) \underset{s}{\hookrightarrow} \pi^* \xi$, let D = degenerate divisor = zero$(\det s)$

$\underline{\mathcal{O}(-r)(D) = \pi^* \det \xi}$

on $V = \mathbb{P} - D$, $\underline{\oplus^\nu \mathcal{O}(-1)|_V \xrightarrow{\sim} \pi^* \xi|_V}$ $(*)$

Rmk:

$$\begin{tikzcd} (*) \rightsquigarrow & V \arrow[r, "\pi|_V"] \arrow[d, "\rho"] & Z \\ & H & \end{tikzcd}$$

this is somehow reverse the previous construction.

① + ② = ③ : $\pi^* \det(\xi)^{r-1} \otimes \mathcal{O}(D)$ is (n.) s.p.

Next time will see ③ + Rmk $\Rightarrow$ Thm. <u>To be conti</u>

Recall.

$$\begin{tikzcd} & & \mathcal{Y} \arrow[dl, "g"'] & \in \mathcal{M}_h(Z) \\ V \arrow[r] \arrow[d, "p"', "\text{finite}"] & Z & & \\ H \arrow[r, hook] & \mathbb{P}^M & & \end{tikzcd}$$

plucker embedding

Having H, may reconstruct V via shishadri's method

Having Z may also reconstruct V:

$$V \subseteq \mathbb{P} = \mathbb{P}(\oplus^r \mathcal{E}_\nu^\vee), \quad D = \mathbb{P} - V$$

$\mathcal{E}_\nu = g_* \omega^\nu_{\mathcal{Y}/Z}$ semi.pos. $\lambda_\nu = \det \mathcal{E}_\nu$

③ $\mathcal{O}_{\mathbb{P}}(D) \otimes \pi^* \lambda_\nu^{\nu-1}$ semi. pos.

④ $\mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{O}_{\mathbb{P}}(D) = \mathcal{O}_{\mathbb{P}}(r) \otimes \pi^* \lambda_\nu$

$$\left.\begin{array}{ccc} \mathcal{O}_Z & \longrightarrow & S^r(\oplus^r \mathcal{E}^\nu) \otimes \lambda_\nu \\ \| & & \\ \mathcal{O}_Z & \rightleftarrows & (\otimes^r \mathcal{E}^\nu) \otimes \lambda_\nu \end{array}\right\} \begin{array}{l} \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(D) \\ \text{natural splitting} \end{array}$$

Cor. If $\mathcal{L}$ inv. on Z & $\pi^* \mathcal{L} \otimes \mathcal{O}_{\mathbb{P}}(\Delta)$ ample

$\Delta \geq |D|$, $\Rightarrow \mathcal{L}$ ample.

Idea of pf (simple exercise, in fact):

$$\begin{array}{ccc} H^0(\mathbb{P}, \pi^*(\mathcal{L}^\eta)(\eta\Delta)) & \twoheadrightarrow & H^0(\mathbb{P}_p, \pi^*\mathcal{L}^\eta \otimes \mathcal{O}(\eta\Delta)) \\ \uparrow\downarrow & & \uparrow\downarrow \\ H^0(Z, \mathcal{L}^\eta) & \twoheadrightarrow & H^0(p, \cdots) = k \end{array} \quad \Rightarrow \text{b.p.f.}$$

similarly for sep. 2 pts. tangents etc. //

Blow up $\mathbb{P}' \xrightarrow{\sigma} \mathbb{P}^M$

ample $\to$ $\pi'^* \lambda_\nu^{-\alpha} \otimes \lambda_{\nu\mu}^{\beta}(\Delta')$

by ③ since s.p + ample = ample. get

$$\pi'^* \lambda_\nu^{\alpha'} \otimes \lambda_{\nu\mu}^{\beta} (\Delta' + \underbrace{(\alpha' - \alpha)}_{> |D'|} D') \, ; \quad \alpha' \gg 0 \quad \text{ample}$$

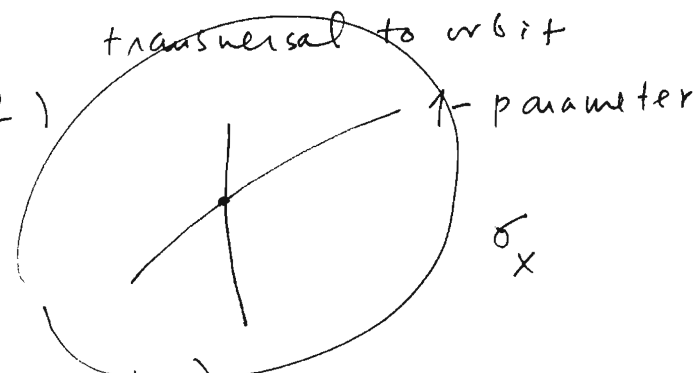
$\Rightarrow \lambda_\nu^{\alpha'} \otimes \lambda_{\nu\mu}^{\beta}$ ample on Z. done.

((sometimes) $\Rightarrow$ λ_ν is ample on Z already)

cor. $\widetilde{M}_h$ exists with ample sheaf induced by λ_ν.
$\widetilde{M}_h$ – normalization of M_h

For people experienced in GIT, the above is "Hilbert - Mumford criterion":

H.M. $\leadsto$ $\Delta' \geq \mathbb{P}' - V$
(but with diff ample sheaf)



Remark:
Similar argument shows that $H = H^s(\lambda_\nu)$ (stable points)
hence $\Rightarrow$ M_h exists.

§ Shafarevich Problem :

Up to now,

A). $\tilde{M}_h$ exists, λ_ν ample

B). $f : X \to B$ non-isotrivial, F surface (curve) of general type, or can. polarized mfd.

$\Rightarrow \deg f_* \omega^\nu_{X/B} > 0 \quad (\nu \geq 2,\ f_* \omega^\nu_{X/B} \neq 0)$

(for $n > 2$ use MMP to do boundedness)

c) $S^{r(\nu) e(\nu)} (f_* \omega^\nu_{X/B}) \otimes N^{-1}$ ample for $\deg N < \deg (f_* \omega^\nu_{X/B})$. $\quad r(\nu) = \mathrm{rk}\ f_* \omega^\nu_{X/B}$.

Today, remains to show

Theorem (Bedulev, —) ; F general type, assume $\Phi_{|\omega^\nu_F|} : F \longrightarrow \mathbb{P}$ has at most 1-dim'l fibers ($\nu \gg 0$) & non-isotrivial

$s = \# S$, f smooth over $B_0 = B - S$.

a) (Migliorini, Kovacs, Qi Zhang) $\longleftrightarrow$ (II)

$$2g(B) - 2 + s = \deg \omega_B(s) > 0$$

b) f semi-stable :

$$n(2g(B) - 2 + s) \cdot \nu \cdot e(\nu) \cdot r(\nu) \geq \deg (f_* \omega^\nu_{X/B})$$

c) f not semi-stable $\longrightarrow$.

Proof: a) If < 0, add points to S

$\deg \omega_B(S) = 0 \rightsquigarrow f$ smooth $/ \; \underset{k^*}{E}$

so may assume f semi-stable (via covering)

hence a) is a special case of b).

b). Assume bound does not hold:

use property c). $\mathcal{A} = \omega_B(S)^n$,

$$S^{er}(f_* \omega^\nu_{X/B}) \otimes \mathcal{A}^{-er\nu} \text{ ample}$$

$\mathcal{A} = \omega_B(S)^{n-m}$, $m \geq 0$ is even better.

$\Rightarrow \omega_{X/B} \otimes f^* \mathcal{A}$ is 1-ample wrt. $X_0 = f^{-1}(B_0)$

Def: $\mathcal{L}$ inv $/X$, proj. $X_0 \subseteq X$, $\Gamma = X - X_0$,

i) $\mathcal{L}$ is semi-ample wrt $X_0 \iff$ for some $\eta > 0$

$$\iota_\eta : H^0(X, \mathcal{L}^\eta) \otimes \mathcal{O}_X \longrightarrow \mathcal{L}^\eta \text{ surj on } X_0.$$

ii) $\mathcal{L}$ 1-ample wrt $X_0 \iff$ for some η in i)

the induced map $\phi_\eta : X_0 \twoheadrightarrow V \subseteq \mathbb{P}(H^0(X, \mathcal{L}^\eta))$

(induced by ι_η) is proper, birational, &

$$\dim \phi_\eta^{-1}(v) \leq 1, \quad v \in V$$

this is old notation, new notation is $\Omega^1_X(\log)$

$$\Sigma_1, \quad 0 \to f^* \underset{\substack{\| \\ \Omega^1_B(\log S)}}{\omega_B(S)} \to \Omega^1_X\langle\Gamma\rangle \to \Omega^1_{X/B}\langle\Gamma\rangle \to 0$$

$$n=2: \quad 0 \to f^*\omega_B(S) \otimes \Omega^1_{X/B}\langle\Gamma\rangle \to \Omega^2_X \to \underset{\substack{\| \\ \omega_{X/B}}}{\Omega^2_{X/B}\langle\Gamma\rangle} \to 0$$

motivation:

$$\begin{tikzcd}
\otimes \omega_{X/B} : f_* \omega^2_{X/B} \arrow[r] \arrow[dr] & R^1 f_* \left(\omega_{X/B} \otimes \Omega^1_{X/B} \langle \Gamma \rangle \right) \otimes \omega_B(S) \arrow[d, "\text{Kodaira - Spencer theory}"] \\
& \left(R^2 f_* \omega_{X/B} \right) \otimes \omega_B(S)^2
\end{tikzcd}$$

want to show $\neq 0$. but even if use Torelli thm + KS theory, still can not conclude it.

Actual way to prove this: via vanishing thm:

$\mathcal{L}$ 1-ample, Assume V in ii) allows projective morphism $V \xrightarrow{\gamma} W$, W affine, then

$\Rightarrow$ $\exists$ blowing up $\tau : X' \to X$ centers in Γ such that $\Gamma' = \tau^* \Gamma$ NCD.

$\exists$ $0 \leq \Sigma \leq m\Gamma'$ with

$H^q(X', \Omega^p_{X'} \langle \Gamma' \rangle \otimes \tau^* \mathcal{L}^{-1} \otimes \mathcal{O}_{X'}(\Sigma)) = 0$, $p+q < \dim X$.

Rmk: If $\mathcal{L}$ ample, this is Nakano-Akizuki-Kodaira.

(See eg. LN H. Esnault, —, DMV-Lecture notes).

$\Sigma^\bullet_m := \Lambda^m \Sigma_1$, tautological sequence

$$0 \to f'^* \omega_B(S) \otimes \Omega^{m-1}_{X'/B} \to \Omega^m_{X'} \langle \Gamma \rangle \to \Omega^m_{X'/B} \langle \Gamma' \rangle \to 0$$

$\Sigma^\bullet_m \otimes \tau^* \mathcal{L}^{-1}(\Sigma)$: $H^{n-m}(\text{Middle term}) = 0$ $\Rightarrow$

$$H^{n-m}(\Omega^m_{X'/B} \langle \Gamma' \rangle \otimes \tau^* \mathcal{L}^{-1}(\Sigma)) \hookrightarrow H^{n-m+1}(\Omega^{m-1}_{X'/B} \langle \Gamma' \rangle \otimes \tau^* \mathcal{L}^{-1} \otimes f^* \omega_B(S)).$$

choose $\mathcal{L}_m = \omega_{X/B} \otimes f^*\omega_B(S)^{n-m}$

call $\mathcal{L}_{m-1}^{-1}(\Sigma) = \tau^*\mathcal{L}_m^{-1}(\Sigma) \otimes f^*\omega_B(S)$.

$m=0$: $\quad H^n(\tau^*\mathcal{L}_0^{-1}(\Sigma)) = 0 \qquad$ by iteration of inclusion

$$\cup\!\shortmid$$

$m=n$: $\quad H^0\big(\Omega^n_{X/B}\langle P'\rangle \otimes \tau^*\omega_{X/B}^{-1} \otimes \mathcal{O}(\Sigma)\big) \neq 0$

$$\tau^*\omega_{X/B}^{-1} \;\vee\; \Omega^n_{X/B}\langle P\rangle^{-1}$$ since $f: X \to B$ is semi-stable

$$\underbrace{\Omega^n_{X/B}\langle P'\rangle \otimes \tau^*\omega_{X/B}^{-1}}_{\hookrightarrow\, \mathcal{O}(E) \quad E \geqslant 0}$$

a contradiction. $\square$.

About Vanishing Theorem: Idea:

X proj. $\mathcal{L}$ inv. $\mathcal{L}^N = \mathcal{O}(D)$, D NCD

$\mathcal{L}^{(1)} := \mathcal{L}(-[\frac{D}{N}])$, Then

i) $\mathcal{L}^{(1)-1}$ has a flat connection

$\nabla : \mathcal{L}^{(1)-1} \longrightarrow \mathcal{L}^{(1)-1} \otimes \Omega^1_X\langle D\rangle$ (comes from cyclic cover)

ii) $H^p(\Omega^q_X\langle D\rangle \otimes \mathcal{L}^{(1)-1}) \Rightarrow H^{p+q}(\Omega^{\bullet}_X\langle D\rangle \otimes \mathcal{L}^{(1)-1})$

degenerate at E_1 term.

iii) Residues along components of $D \notin \mathbb{Z}$

($\Leftarrow D = \sum a_i D_i$, $N \nmid a_i$) & $\mathbb{V} = \ker(\nabla|_{X_0 = X - |D|})$

$\Rightarrow H^{p+q}(\Omega^{\bullet}_X\langle D\rangle \otimes \mathcal{L}^{(1)-1}) = 0$.

iv) So, $H^p(\Omega^q_X\langle D\rangle \otimes \mathcal{L}^{(1)-1}) = 0$ for $p+q < \mathrm{c.d.}(X-D)$.

c.d. $(X-D) = \dim X$ if $X-D$ affine

or if $\exists$ $X-D \xrightarrow[\Phi]{} W$, $\Phi^{-1}(w) \leq 1$

birat'l.

Idea of the pf:

If. nec.; blow up st. $\mathcal{L}'^{\eta}(-\Sigma_1)$ globally generated with $\Sigma_1 > 0$.

May assume that $|\Sigma| = P' = (\tau^* P)_{red.}$ and

η + multiplicities, as in Kawamata's talk,

can be achieved by perturbation a little bit.

Induced: $X' \twoheadrightarrow Z_1 \supset V$ over V fiber dim ≤ 1

$\Rightarrow$ $D = H + \Sigma_1$ satisfies the assumption, $X' - D \to V - V \cap H$.

$\to$ get vanishing for

$$\Omega_X^q \langle H + \Gamma' \rangle \otimes \mathcal{L}^{-1} \otimes \mathcal{O}\left(\left[\frac{\Sigma_1}{\eta}\right]\right)$$

$$\Omega_{X'}^q \langle \Gamma' \rangle \to \Omega_{X'}^q \langle H + \Gamma' \rangle \longrightarrow \Omega_H^{q-1} \langle \Gamma'|_H \rangle \longrightarrow 0$$

+ induction on dimensions. //.

Final Remarks:

let $T^1_{X/B}\langle -\Gamma \rangle := \left(\Omega^1_{X/B}\langle \Gamma \rangle\right)^\vee$.

$$0 \to T^1_{X/B}\langle \Gamma \rangle \to T^1_X\langle \Gamma \rangle \to f^* T^1_B \langle S \rangle \to 0$$

$$T^1_B\langle S \rangle \to R^1 f_* T^1_{X/B}\langle \Gamma \rangle$$

not yet use 1-dim'l fiber.

for moduli schemes $\underset{\text{finite}}{B \to M_h}$, $\underset{\text{univ. family}}{X = \mathfrak{X}}$

should get $T^1_B\langle S\rangle \xrightarrow{\sim} R^1 f_* T^1_{X/B}\langle\Gamma\rangle$

dualize: $R^{n-1} f_*\left(\Omega^1_{X/B}\langle\Gamma\rangle \otimes \omega_{X/B}\right) \xrightarrow{\sim} \Omega^1_B\langle S\rangle$

For $n=1$ (family of curves):

$$\underset{\text{ample}}{f_* \omega^2_{X/B}} \xrightarrow{\sim} \Omega^1_B\langle S\rangle$$

(at least proven for curves)

$f_*\omega^2_{X/B}$ nef & $\underset{\text{ample}}{A} \subset \underset{\cong \text{ over } B-S}{S^2(f_*\omega^2_{X/B})}$

So may say $f_*\omega^2_{X/B}$ ample wrt $(B-S)$.

$\Rightarrow \Omega^1_B\langle S\rangle$ ample wrt $(B-S)$

$\Rightarrow B-S \not\supset E$ (elliptic curve) or $k^\times$.

May this be general situation for $n \geq 2$?

Problem: Positivities for $\Omega^1_{M_h}\langle M_h - M_h^\circ\rangle$ for larger class of moduli.

Remark: $\Omega^1_{M_h}\langle M_h - M_h^\circ\rangle$? semi-positive &

$(Y, Y_0) \subseteq (M_h, M_h^\circ) \underset{?}{\rightsquigarrow} \Omega_Y^{\dim Y}\langle Y - Y_0\rangle$ ample wrt Y_0.

END.

The First

NCTS Summer School on Algebraic Geometry

July 19 — 30, 1999

Professor H’el`ene Esnault

University of Essen

(Notes by Chin-Lung Wang)

Lecture I — 7/19, p.1

Motivation for Chern—Simons/Cheeger—Simons Theory

Lecture II — 7/23, p.9

Weil Homomorphisms

Lecture III — 7/26, p.16

Cohomology of Cheeger—Simons Differential Characters

Lecture IV — 7/29, p.25

Riemann—Roch for Rank One Irregular Connections on Curves

NCTS Summer School in Alg.-Geom.

Prof. H. Esnault.

Lecture I. 7/19 at Academia Sinica

Chern-Simon / Cheeger-Simon Theory

0. Motivation:

		E : vector bundle	$c_n(E)$
X mfd	(topological)	$H^{2n}(X, \mathbb{Z})$	Betti
	C^∞	$H^{2n}_{DR}(X)$	de Rham
	analytic	$H^{2n}_{\mathcal{D}}(X, n)$	Deligne (-Beilinson)
	algebraic	$CH^n(X)$	Chow groups

Supplementary structure on E:

$\to$ richer char. class of E

connection $\nabla : E \xrightarrow[k\text{-linear}]{} \Omega^1 \otimes E$

Ω^1 1-forms (C^∞, analytic, alg)

st. Leibnitz rule.

curvature : $\nabla^2 : E \to \Omega^2 \otimes E$: $\mathcal{O}$-linear

In particular, ∇ flat $\longleftrightarrow$ $\nabla^2 = 0$

topologically / analytically (E, ∇) flat $\longleftrightarrow$ local system

$\mathbb{E} = \{ e \in E, \nabla e = 0 \}$

$\longleftrightarrow H^1(X, GL_r(\mathbb{C}))$ pointed set

$r = \mathrm{rk}\, E$:

$r=1$: example of richer invariant

$H^1(X, GL_1(\mathbb{C})) = gp = \mathrm{Hom}(H_1(X,\mathbb{Z}), \mathbb{C}^\times)$
$\quad\; \| \qquad\qquad\qquad = \mathrm{Hom}(\pi_1(X), \mathbb{C}^\times)$
$\;\;\mathbb{C}^\times \qquad\qquad\quad$ = characters of $\pi_1(X, x)$

= set of chern-Simon invariants

classical chern-Simon theory:

$$\begin{tikzcd}
r=1: & H^1(X, \mathbb{C}^\times) \arrow[r] \arrow[d, "\text{boundary via exp.}"'] & H^{2n-j}(X, \mathbb{C}/\mathbb{Z}(n)) \arrow[d] \\
& H^2(X, \mathbb{Z}(1)) & H^{2n}(X, \mathbb{Z}(n))
\end{tikzcd}$$

$H^1(X,\mathbb{C}^\times) \overset{\exp}{\cong} \mathbb{C}/\mathbb{Z}(2\pi i)$ — call it 1.

$\mathbb{Z}(n)$: $(2\pi i)^n$

$H^2(X,\mathbb{Z}(1))$: top chern class

$H^{2n}(X,\mathbb{Z}(n))$: torsion classes.

Want good behavior in families, compatible with inverse images (so. coh. theory).
defect of compatible with direct images is the
— Riemann-Roch Theorem —.

setup: $f: X \longrightarrow S$ morphism, proper
$\qquad\quad \uparrow$
$\qquad\quad E$ v.b.

in order to take direct image, need to know still in the same category.

f: proper, Grauert $\Rightarrow R^i f$ coherent

S, X smooth (Hilbert) $\Rightarrow \exists$ resol. via v.b.'s.

Grothendieck Riemann-Roch :

$$ch(Rf_* E) = F(f, ch E)$$

$$\cap$$

$$\bigoplus CH^n(S) \otimes \mathbb{Q} \xrightarrow[/\mathbb{C}]{} \bigoplus H^{2n}_{\mathcal{D}}(X, n) \otimes \mathbb{Q} \to \bigoplus H^{2n}(X, \mathbb{Z}) \otimes \mathbb{Q}$$

More precisely :

$F(f, ch E)$ = 2 pieces of information

$f_* (Td(f) \odot ch(E))$

direct image (trace)

Todd class $\in CH(X) \otimes \mathbb{Q}$

For (E, ∇), like classes which behave "correctly" in alg. morphism.

1st analogy (Deligne)

X smooth curve $\xrightarrow{f} S = \operatorname{Spec} \mathbb{F}_q$

$\rho : \pi_1(X) \longrightarrow GL(1, \mathbb{Q}_\ell)$, rk 1 local system

$Rf_*(\rho) = H(\bar{X}, \rho) \in K_0(\mathbb{Q}_\ell) = \{ \text{v.s.} / \mathbb{Q}_\ell \} /$ exact seq.

base ext. to $\bar{\mathbb{F}}_q$

$\det H(X, \rho)$

1-dim'l v.s / $\mathbb{Q}_\ell$.

becomes a $\pi_1(\mathbb{F}_q)$ module

$$\pi_1(\mathbb{F}_q) = Gal(\bar{\mathbb{F}}_q / \mathbb{F}_q) \cong \hat{\mathbb{Z}} \ni \text{Frob}.$$

give $\rho_{Im} : \pi_1(\mathbb{F}_q) \longrightarrow GL(1, \mathbb{Q}_\ell)$

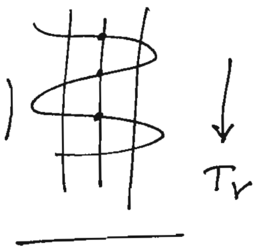
$(\det H(\bar{X}, \rho))$

Theorem: (Deligne)

(motivation) $\quad \rho'_{\mathrm{Im}} = F(f, \rho)$ (of shape of Grothendieck R–R.)

$= -\operatorname{Tr} \rho \int$ divisor of a meromorphic section of ω = sheaf of 1-forms on X.

$$= -f_*\big(\underbrace{c_1(\Omega^1_{X/\mathbb{F}_q})}_{\substack{CH^1(X) \\ \| \\ \mathrm{Pic}(X)}} \cup \underbrace{\rho}_{\in \mathrm{Hom}(\pi_1(X),\, \mathrm{Aut}\, \mathbb{Q}_\ell)}\big)$$



geometry if one starts with ∇.

$$H^1(\bar{X}, \rho) \sim \sum (-1)^i \underbrace{R^i f_*(\Omega^{\cdot}_{X/S} \otimes E)}_{\text{relative de Rham coh.}}$$

$\otimes E$ ↑ defined by connection

↑ comes a $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ action $\quad \xrightarrow[\nabla \text{ flat}]{}$ coh. sheaf with a connection: Gauß-Manin conn.

Wants:

$$ch\left(\textstyle\sum (-1)^i \text{"GM"}, \text{ Gauß-Manin connection}\right) = F(f, ch(E, \nabla)).$$

Another motivation from topology:

$\mathcal{D}$-modules: X smooth $/k$

ex. (E, ∇) flat connection, $E \xrightarrow{\nabla} \Omega^1_X \otimes E$

$\nabla \leftrightarrow A$ = matrix of 1-forms $= \sum A_i \, dx_i$

(alg.) choice of coor. x_i

T locally given by ∂x_i, $\langle \partial x_i, dx_j \rangle = \delta_{ij}$

∂x_i acts on $e_j \in E$ by $\partial x_i (e_j) = A$

action of $\mathcal{D}$: $\partial_{x_i} \partial_{x_j} = \partial x_j \, \partial x_i$.

∇ flat $\longrightarrow$ action of $\mathcal{D}$ on E.

$\exists$ more complicated examples:

① $U \underset{j}{\hookrightarrow} X$ open smooth

(E, ∇) flat on $U \to j_* (E, \nabla) = \mathcal{D}$-mod on X.

② $Z \underset{i}{\hookrightarrow} X$ closed imbedding

(E, ∇) conn. on $Z \longrightarrow i_* (E, \nabla)$.

$f : X \to S$ proper morphism

M: $\mathcal{D}$-module on $X \longrightarrow Rf_* M$ defined as $\mathcal{D}$-modules on X

$\to$ R.R. Question

$ch(Rf_* M) = F(f, ch M)$.

Problem: One needs a good theory of char classes of $\mathcal{D}$-modules. Don't have this yet!

If $\mathcal{D}$-module connection (E, ∇) do have this / also for $j_*(E, \nabla)$.

in particular, on $U(\mathbb{C})$; $(E, \nabla)|_U$

$$\left\{\begin{array}{lcl} \text{top} & \overset{1-1}{\longleftrightarrow} & \text{local system} \\ C^\infty & & \text{Chern-Simon} \\ \text{an} & & \text{Cheeger } H^{2n-1} \end{array}\right.$$

2nd Motivation (C^∞):

Theorem (Bismut-Lott, Bismut)

$f: X \to S$ proj / $\mathbb{C}$ smooth
$\uparrow$
$\mathcal{V}$ local system

$ch(\mathcal{V}) \in H^{2n-j}(X, \mathbb{C}/\mathbb{Z}(i))$
$ch(Rf\,\mathcal{V}) = F(f, ch\,\mathcal{V})$.

Want thm $f: X \to S$ proj smooth, (E, ∇) flat conn which should have the fcn of Deligne in # theory and

$\swarrow$ G.R.R $\qquad\qquad \searrow$ Bismut.

Classes: * $\exists$ good classes in group of "alg. differential characters"

$\swarrow$ alg. cycles $\qquad\qquad \searrow$ top

$\oplus\, CH(X)$ $\qquad\qquad$ Chern-Simon, Cheegers

* explicit understanding of those classes at generic point of X (w. S. Bloch).

RR: general RR thm:

— for regular connection ("half" flat)
$r=1$, piece of classes detected at generic point of var.
$r \geq 2$: loose some information
— for irregular connections
for rk 1 irr. conn. $\to$ formula

$$ch(Rf\nabla) = F(f, ch\nabla)$$

plus extra information
(should give swan conductor in # theory)
Higher rk : $\to$ new invariant $\to$ conjecture.

Recall of Chern classes (after Beilinson / Kazhdan, unpublished)

Chern – Weil homomorphism:
$G = GL(r, \mathbb{C})$, $\mathfrak{g} = M(r \times r, \mathbb{C})$
$\omega = \oplus \omega_n : \oplus S^n(\mathfrak{g}^*)^G \longrightarrow \oplus H^{2n}_{DR}(BG) = H^{2n}(BG, \mathbb{C})$

G acts on $\mathfrak{g}$ via
adjoint rep'n

$$p \longmapsto \omega_n(p)$$

is an isomorphism.
Atiyah class of a vector bundle.
torsor, exact obstruction to the existence
of a connection on a V.b. E. $\in H^1(X, \Omega^1 \otimes \mathrm{End}\, E)$

α locally $\exists$ connection by declaring a given local basis as a flat basis.

$u_i \quad (e_i) \; ; \; u_j \quad (e_j)$

$\nabla_i - \nabla_j : E \longrightarrow \Omega^1 \otimes E \qquad \in H^1(X, \Omega^1 \otimes \mathrm{End}\, E)$

(β) 2nd way to think of it :

$X \times X \supset \Delta^{(1)}$ — 1st inf'l nbd of Δ

obtain $\mathcal{P}(E)$ = sheaf of principal part of E

get $0 \to \Omega^1 \otimes E \longrightarrow \mathcal{P}(E) \to E \longrightarrow 0$

$\in \mathrm{Ext}^1(E, \Omega^1 \otimes E) = H^1(X, \Omega^1 \otimes \mathrm{End}\, E)$.

(γ) let $P \xrightarrow{\pi} X$ be principal G-bundle asso. to E — $GL(r)$

has a G-action

$\Rightarrow \pi_*(\Omega^1_P)$ have G-action

$\Rightarrow$ take inv. $\pi_*(\Omega^1_P)^G$ hence get

Atiyah extension :

$$0 \to \Omega^1_X \to \pi_*(\Omega^1_P)^G \to \mathrm{End}\, E \longrightarrow 0$$

we will use this to define chern classes.

to be <u>continued</u> .

Weil homomorphism à la B-K :

Atiyah extension of a v.b. E/X

$p: P \to X$ principal G-bundle, $G = GL(r)$

$\mathfrak{g} = \mathrm{Lie}\, G = M_{r\times r}(\mathbb{C})$

$$0 \to \Omega^1_X \to \Omega^1_{X,E} \to \mathrm{End}\, E \to 0$$

$$\Omega^1_{X,E} = (p_* \Omega^1_P)^G \quad : \text{ G-inv form in principal bundle.}$$

BK $\to$ differential graded algebra (DGA) $\supset \Omega^\bullet_X$

$$d: \mathcal{O}_X \cong \Omega^0_{X,E} \longrightarrow \Omega^1_{X,E}$$

$$\Omega^n_{X/E} = \bigoplus_{a+b=n} \Omega^{a,b}_{X,E} \supset \Omega^{n,0}_{X,E} \supset \Omega^n_X$$

$$\Omega^{a,b}_{X,E} := \Lambda^{a-b}\, \Omega^1_{X,E} \otimes S^b\, \mathrm{End}\, E$$

$$\Omega^{a,b}_{X,E} \xrightarrow{d'} \Omega^{a+1,b}_{X,E} \quad \leftarrow \text{comes from } d \text{ on } P$$

$$\Omega^{a,b}_{X,E} \xrightarrow{d''} \Omega^{a,b+1}_{X,E}$$

$$\omega_1 \wedge \dots \wedge \omega_{a-b} \otimes \varphi_1 \cdots \varphi_b \mapsto \sum (-1)^j \omega_1 \wedge \dots \hat{\omega}_j \wedge \dots \omega_{a-b} \cdot (\mathrm{Im}\, \omega_j)\, \varphi_1 \cdots \varphi_b$$

$$(d')^2 = (d'')^2 = 0, \quad d'd'' = d''d', \quad d = d' + d''$$

$$(\Omega^\bullet_X, d) \longrightarrow (\Omega^\bullet_{X,E}, d) \text{ sub complex}$$

is a filtered quasi-isomorphism (purely alg'ly):

$$(\Omega^{\geq p}_X, d) \longrightarrow (F^p \Omega^\bullet_{X,E}) = \bigoplus_{a \geq p} \Omega^{a,b}_{X,E} \text{ as an algebra}$$

Hodge filtration

$$\begin{array}{lccc}
\Omega^{\geq n}: & \Omega_X^n & \longrightarrow & \Omega_X^{n+1} \\
 & \downarrow & & \downarrow \\
F^n: & \Omega^{n,0} & & \Omega^{n+1,0} \oplus \Omega^{n,1} \\
 & \downarrow & & \downarrow \\
 & \Omega^{n,1} & & \Omega^{n+1,1} \oplus \Omega^{n,2} \\
 & \downarrow & & \\
 & \Omega^{n,2} & &
\end{array}$$

$$\Omega_X^1 = \operatorname{Ker} \Omega^1_{X/E} \longrightarrow \operatorname{End} E$$

$$\overset{\shortparallel}{\Omega^{1,0}_{X,E}} \qquad \Omega^{1,1}_{X,E} = \Lambda^0 \Omega^1_{X/E} \otimes S^1 \operatorname{End} E \cong \operatorname{End} E.$$

Weil homomorphism:
Apply for the def. of $\Omega^{a,b}$ to
the classifying simplicial scheme (space) of
principal G-bundles together with its
universal G-bundle. $EG \longrightarrow BG$.

top: top. space M is a BG up to some dim N
if ① $\exists$ G prin. bundle $EG \to BG$, EG contractible

$$\begin{array}{cccc}
 & P = X \times_{BG} EG & \longrightarrow & EG \\
\exists & \downarrow & & \downarrow \\
 & X & \longrightarrow & BG
\end{array}$$

② $\forall$ $P \to X$, X mfd, $\dim X \leq N$, the above diagram exists.

To do this algebraically, we need to replace alg. V. by simplicial schemes.

Algebraically, BG exists (with ① and ② without bounds on dim X) in the category of simplicial (schemes) varieties.

ie. $X_\cdot$ simplicial variety is a contravariant functor (Hodge III)

$$(\Delta^\cdot) \longrightarrow \{\text{schemes}\} \ ; \ n \mapsto X_n$$

satisfies obvious relations on ∂_i maps: $X_{n+1} \longrightarrow X_n$.
(n+2 such maps)

sheaf $\mathcal{F}$: a complex of sheaves :

$$n \longmapsto \mathcal{F}_n \ : \text{cpx of sheaves on } X_n \ \text{st.}$$

$$\mathcal{F}_{n+1} \longleftarrow \partial_i^* \mathcal{F}_n$$

topological realization of $X_\cdot$: (mfd)

$$\coprod X_n \times \Delta^n \ : \quad \begin{tikzcd} X_{n+1} \times \Delta^{n+1} \arrow[r, bend left, "\partial_i"] & X_n \times \Delta^n \arrow[l, bend left, "\delta_i"] \end{tikzcd}$$

eg. $\mathbb{C}, \pi$ simplicial sheaf ($X/\mathbb{C}$ for simplicity)

$$H(X_\cdot, \text{simplicial sheaf } \mathbb{C}, \pi) \cong H(\text{top reali}, \mathbb{C}, \pi)$$

BG :

$$\begin{tikzcd} G^{\ell+1} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & {} \\ G^{\ell+1}/G \cong G^\ell \end{tikzcd}$$

forgetful maps

ℓ ↑ right transl.

$$\begin{tikzcd} G \times G \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] \arrow[d] & G \arrow[d] \\ G \arrow[r, shift left] \arrow[r, shift right] & \cdot = \operatorname{Spec} k \end{tikzcd}$$

$$G \times G/G \ \underset{h}{\overset{g}{\rightrightarrows}} \ G/G$$

$(g, h)\delta = (g\delta, h\delta)$

right transl. (Hodge III)

BK : Atiyah class can be extended to simplical cat.

$$(*) \quad 0 \to \Omega^1_{BG} \to \Omega^1_{BG, E_{un}} \to \operatorname{End} E_{un} \to 0$$

$$\Omega^1_{X,E} \qquad \operatorname{End} E_{un} = \mathfrak{g}^*$$

Koszul complex:

$(*)$ - resolution of $\Lambda^n \Omega^1_X = \Omega^n_X$ follows

$$0 \to \Omega^n_X \to \Big[\Lambda^n \Omega^1_{X,E} \to \Lambda^{n-1} \Omega^1_{X,E} \otimes \mathfrak{g}^* \to \Lambda^{n-2} \Omega^1_{X,E} \otimes S^2 \mathfrak{g}^* \to \cdots \to S^n \mathfrak{g}^* \to 0 \Big]$$

in general; in particular on BG

$\Rightarrow$ connecting morphism. $X = BG$

$$\begin{array}{ccc} H^0(X, S^n \mathfrak{g}^*) & \longrightarrow & H^n(X, \Omega^n_X) \\ X = BG \ \Vert & & \\ (S^n \mathfrak{g}^*_{\text{speck}})^G & & \\ \text{S1} & & \\ \text{inv. poly.} / k & & \end{array}$$

$$X_1 \overset{\delta_2}{\underset{\delta_1}{\rightrightarrows}} X_0 \ ; \quad G \times G \overset{g}{\underset{h}{\rightrightarrows}} G$$

$$\mathcal{F}_1 \longleftarrow \delta_1^* \mathcal{F}_0 \qquad \text{what's } H^0 ?$$

Prop: $BG \ (G = GL(-))$

$$\begin{array}{ccc} H^{2n}_{DR}(X = BG) & & H^n(X, \Omega^n_X) \\ \uparrow \wr & \nearrow & \\ \mathbb{H}^n(X, \Omega^{\geq n}_X) & & \end{array}$$

$H^i(BG, \Omega^j_{BG}) = 0$ for $i \neq j$.

cor: conn. hom. $(S^n \mathfrak{g}^*_k)^G \xrightarrow[w_n]{\sim} H^n(BG, \Omega^n_{BG}) \simeq H^{2n}_{DR}(BG)$

In fact, for $k = \mathbb{C}$, $G(\mathbb{C})$:

Def'n : $\left(S^n \times \mathfrak{g}^* \right)^G_{\mathbb{Z}} = \mathrm{Ker}\left(S^n \mathfrak{g}^{\times}_{\mathbb{C}} \right)^G \to H^{2n}(BG(\mathbb{C}), \mathbb{C})$

$\to H^{2n}(BG(\mathbb{C}), \mathbb{C}/\mathbb{Z}(n))$

$H^{2n}_{DR}(BG) \simeq H^{2n}(BG(\mathbb{C}), \mathbb{C})$

Rmk: Up to now, mostly are true for alg. gp G up to 2-torsion (Stiefel-Whitney class)

$n \rightsquigarrow \mathrm{Tr}\, M^n$ (–, Newton class). done. $\square$

Chern-Simons (classical Theory):

firstly, chern character via Chern-Weil theory.

question (CS): $P \xrightarrow[p]{} X$ principal G-budle

find ? functorial + good property for products in the following:

$p^* E \simeq \oplus^r \mathcal{O}_\infty$ (canonical trivialization)

(E, ∇), Γ_n -inv polynomial

$d? = \Gamma_n(\nabla, \nabla, \cdots, \nabla) \in H^0(P, \Omega^{2n}_\infty, cl)$ closed

$\exists\, ? \in H^0(P, \Omega^{2n-1}_\infty)$.

Answer, let $F(A) = dA - A^2 = \nabla^2$

$T\Gamma_n = n \int_0^1 \Gamma_n(A, \underbrace{F(tA), \cdots, F(tA)}_{n-1})\, dt$!

transgression

ex. $n=1$, $P_1(M) = \mathrm{Tr}\, M$

$$TP_1 = \int_0^1 P_1(A)\,dt = \int_0^1 (\mathrm{Tr}\, A)\,dt = \mathrm{Tr}\, A$$

$$\mathrm{Tr}(dA) = d\,\mathrm{Tr}\, A, \quad P_1(\nabla^2) = d\,TP_1, \quad \mathrm{Tr}(dA - A^2).$$

$P_2 = \mathrm{Tr}\, M^2$:

$$TP_2 = 2\int_0^1 \mathrm{Tr}(A, F(tA))\,dt$$

$$= 2\int_0^1 \left[t\,\mathrm{Tr}\, A\,dA - t^2(\mathrm{Tr}\, A^3)\right] dt$$

$$= 2\left(\tfrac{1}{2}\mathrm{Tr}\, A\,dA\right) - \tfrac{2}{3}(\mathrm{Tr} A^3) = \mathrm{Tr}\, A\,dA - \tfrac{2}{3}\mathrm{Tr}\, A^3.$$

Easy to check this is the answer.

Summary: P_n – inv. poly.

$$(E, \nabla) \rightsquigarrow H^0(P, \Omega_\infty^{2n-1})$$

$$d(TP_n) = P_n(\nabla^2).$$

If P_n has zero periods $\left(\in (S^n \mathfrak{g}_{\mathbb{C}}^*)_{\mathbb{Z}}^n\right)$

$\to$ class of $P_n(\nabla^2) \in H^{2n}(X, \mathbb{C}/\mathbb{Z}(n))$

$\to$ $\exists$ chain $u \in \mathcal{C}^{2n-1}(X, \mathbb{C}/\mathbb{Z}(n))$ st.

$TP_n - p^* u =$ coboundary.

Answer: $\nabla^2 = 0 \Rightarrow P_n \in H^0(P, \Omega^{2n-1}_{\infty, cl})$

$$\downarrow$$

$$[P_n] \in H^{2n-1}_{DR}(P)$$

$\to [P_n] \in H^{2n-1}(P, \mathbb{C}/\mathbb{Z}(n))$ well-defined on X.

Def: class $H^{2n-1}(X, \mathbb{C}/\mathbb{Z}(n)) \ni c_n(E, \nabla) = c_n(\nabla)$.

does not live algebraically.

Cheeger - Simons:
(really the beginning of new notions)
differential characters, cohomology theory.

X : C^∞ mfd :

$$\begin{tikzcd}
\mathbb{H}^{2n}(X, \mathbb{Z}(n) \to \Omega^0_\infty \to \Omega^1_\infty \to \cdots \to \Omega^{2n-1}_\infty) \arrow[d] & \\
& \mathrm{Ker}(H^0(X, \Omega^{2n}_{\infty, cl})) \arrow[r] & \dfrac{H^{2n}_{DR}(X)}{\mathrm{Im}(H^{2n}(X, \mathbb{Z}(n)))} \arrow[r] & 0 \\
H^{2n-1}(X, \mathbb{C}/\mathbb{Z}(n)) \arrow[uu] & CW.\ Ch(\nabla^2) \arrow[u] \\
CS(E, \nabla; \nabla^2 = 0) \arrow[u] \\
0 \arrow[uu]
\end{tikzcd}$$

one $\nabla^2 = 0$. then comes from CS.

Ancestor of Deligne - Beilinson Coh. ~78.

To be continued.

coh. of Cheeger-Simons (diff. characters). C^∞

$$\begin{tikzcd} & H^{2n}(X, \mathbb{Z}(n) \to \mathcal{O}_\infty \xrightarrow{d} \Omega^1_\infty \to \cdots \to \Omega^{2n-1}_\infty) \arrow[dr, two heads] & \\ H^{2n-1}(X, \mathbb{C}/\mathbb{Z}(n)) \arrow[ur] & & \mathrm{Ker}\,(H^0(X, \Omega^{2n}_\infty, d) \to H^{2n}(X, \mathbb{C}/\mathbb{Z}(n))) \to 0 \\ 0 \arrow[u] & & \\ \text{C.S.} \arrow[uu] & c_n((E,\nabla)) \arrow[uuu] \arrow[r] & c_n(\nabla^2, \cdots, \nabla^2) \\ & & \text{C.W.} \end{tikzcd}$$

in case $c_n(\nabla^2, \cdots, \nabla^2) = 0$.

Argument roughly:

$\exists$ classifying space of (E, ∇) ($\dim X \leq \cdots$, $\mathrm{rk}\, E \leq \cdots$)

$\to$ univ. case, also classifying space for bundles M

$\Rightarrow$ $H^{2p-1}(M, \mathbb{Z}) = 0$.

$\to$ diff character = diff form

$\to$ Chern-Weil form

$\to$ pull back. done. $\square$

Weil Homomorphism:

$$S^n(\mathfrak{g}^*)^G \longrightarrow H^{2n}_{DR}(BG) \xrightarrow{[E]} H^{2n}_{DR}(X)$$

but

$$\begin{array}{ccc} S^n(\mathfrak{g}^*)^G & \longrightarrow S^n(\mathfrak{g}^*_{E_{nn}}) = & \Omega^{n,n}_{BG, E_{n,n}} \\ \text{SII} & & \Big\downarrow d = d' + d'' \\ \oplus\,\mathbb{Z} & & \\ c_n, c_1 c_{n-1}, \cdots & & \Omega^{n+1,n} = \Omega^1 \otimes S^n(\mathfrak{g}^*_E). \end{array}$$

Here Beilinson: $\Omega^{a,b} := \Lambda^{a-b}\Omega^1_{X,E} \otimes S^b \mathfrak{g}^*$.

obtains
$$\begin{tikzcd}(S^n\mathfrak{g}^*)^G \arrow[r] \arrow[rr, bend right, "w"'] & \Omega^{n,n}_{(X,E),\mathcal{U}} \arrow[r] & c(F^n\Omega^\cdot_{X,E})[2n]\end{tikzcd}$$

w induces: $(S^n\mathfrak{g}^*)^G \xrightarrow[{[E]}]{} H^{2n}_{DR}(X)$

Beilinson: Weil coh. (dep. on E) analytic theory

$U_E(n) :=$ cone $\left(\mathbb{Z}(n) \oplus (S^n\mathfrak{g}^*)^G[-2n] \xrightarrow{2\oplus w} \Omega_{X,E}\right)[-1]$

$\rightsquigarrow \mathbb{H}^{2n}(U_E(n)) \longrightarrow \ker\left\{ H^0(X,(S^n\mathfrak{g}^*)^G) \oplus H^{2n}(\mathbb{Z}(n)) \to H^{2n}(\Omega^\cdot_{X,E})\right\}$

$H^{2n-1}(\mathbb{C}/\mathbb{Z}(n))$ $\quad\uparrow\quad 0$

Ψ
$c_n(E)$
$c_{n-1}(E)\, c_1(E) \ldots$

in particular, on (BG) ..

given ∇ on E, $\longleftrightarrow$ split $\Omega^\cdot_X \rightleftarrows \Omega^\cdot_{X,E}$

$\longleftrightarrow$ $\Omega^1_X \rightleftarrows \Omega^1_{X,E}$

$$\begin{tikzcd}
\mathcal{O}_X \arrow[r] \arrow[d, no head, "\wr"] & \Omega^1_{X,E} \arrow[r] \arrow[d, shift left, "\nabla"] & \Lambda^2\Omega^1_{X,E} \oplus \mathfrak{g}^*_E \arrow[r] \arrow[d, shift left, "\nabla"] \arrow[d, bend left, "\nabla^2"] & \cdots \\
\mathcal{O}_X \arrow[r] & \Omega^1_X \arrow[r] \arrow[u, shift left] & \Omega^2_X \arrow[r] \arrow[u, shift left] & \cdots
\end{tikzcd}$$

$\nabla^2 = 0 \iff$ filtered q. isom.

$\exists\, \nabla \iff$ split q. is. $\Omega^\cdot_X \longrightarrow \Omega^\cdot_{X,E}$

$\exists\, \nabla,\ \nabla^2 = 0 \iff$ split filtered q. is.

$$\begin{tikzcd}
\exists \nabla,\ G(E) \arrow[d, "\nabla"] & & UE(n) \arrow[d, "\nabla"'] \\
\mathbb{H}^{2n}(\mathrm{cone}(\mathbb{Z}(n) \oplus F^{2n} \longrightarrow \Omega^{\cdot}_{X,E})[-1]) & &
\end{tikzcd}$$

$\qquad\qquad$ || Cheeger - Simons diff. char.

Chern - Simons diff forms :

$(2n-1)$ - diff. form on principal bundle, ∇

$\nabla^2 = 0 \longrightarrow H^{2n-1}(\mathbb{C}/\mathbb{Z}(n))$.

Algebraization — alg. diff. char :

$X/\operatorname{spec} k$, char $k = 0$:

$$AD^n(X) := \mathbb{H}^n_{Zar}(X, \mathcal{K}_n \to \Omega^n_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{2n-1}_X)$$

alg. diff forms

Eventually : $\mathcal{K}_n$ Zariski sheaf of Milnor K-theory

F : field : $\quad K^M_n(F) = \dfrac{F^* \otimes_{\mathbb{Z}} \otimes \cdots \otimes F^*}{(\ldots x \otimes \cdots \otimes (1-x) \otimes \cdots)}$

/ or local ring $\qquad x \in K^\times - \{1\}$.

$\mathcal{K}_n = \operatorname{Im} K^M_n(\mathcal{O}) \longrightarrow K^M_n(k(X))$

additive sp.

For X smooth, $CH^n(X) = \mathbb{H}^n(X, \mathcal{K}_n)$

$K_1 = \mathcal{O}^\times_X \xrightarrow[d\log]{} \Omega^1$

$$\begin{tikzcd}
K^M(\mathcal{O}) \arrow[r] \arrow[d] & \Omega^n \\
\mathcal{K}_n(\mathcal{O}) \arrow[r] & K^M_n(k(X)) \arrow[u, bend right, "d\log"']
\end{tikzcd}$$

(cf. Gabber's result. all char. are torsion)

$$\begin{array}{ccccccc} 0 \to AD^n_{flat}(X) & \longrightarrow & AD^n(X) & \longrightarrow & \mathrm{Ker}\left\{ H^0(\Omega^{2n}_{cl}) \to \dfrac{H^n(X,\Omega^{2n})}{\mathrm{Im}\, CH^n(X)} \right\} \to 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 \to H^{2n-1}(\mathbb{C}/\mathbb{Z}(n)) & \longrightarrow & CS_{an}\,\mathrm{Diff}^n(X) & \longrightarrow & * \end{array}$$

The strongest wh. theory:

$$\begin{array}{ccc} AD^n_{flat}(X) \subset AD^n(X) & \xrightarrow{c_n(E,\nabla) \mapsto c_n(E)} & CH^n(X) \\ \downarrow /\mathbb{C} & & \swarrow \quad \downarrow \quad \searrow /\mathbb{C} \\ H^{2n-1}(\mathbb{C}/\mathbb{Z}(n)) & & \left(H_{et} \qquad H(\mathbb{Z}) \atop H_{\mathcal{D}} \quad \ldots \right) \end{array}$$

construction: (B-K, Weil homomorphism)

generalized splitting principal.

Example: $\mathrm{rk}\, E = 2$:

$$P = \mathbb{P}(E) \qquad 0 \to \Omega^1_{P/X}(1) \to \pi^* E \to \mathcal{O}(1) \to *$$

$$\pi \downarrow \qquad \nabla \rightsquigarrow \quad \Omega^1_{\mathbb{P}} \xrightarrow{\tau} \pi^* \Omega^1_X$$

$$X \qquad\qquad \uparrow \text{alg.}$$

define $\tau \circ \nabla$ on $\pi^* E$ — respects $*$.

① $(\mathcal{O}(1), \tau\cdot\nabla) \in H^1(\mathbb{P}, \mathcal{O}^\times \xrightarrow[\tau d\log]{} \pi^*\Omega^1_X)$

② Product: $H^2(\mathbb{P}, K_2 \xrightarrow[\tau d\log]{} \dfrac{\Omega^2_{\mathbb{P}}}{\Omega^2_{\mathbb{P}/X}}$

$\xrightarrow[\tau d\log]{} \pi^*\Omega^3_X = \dfrac{\Omega^3_{\mathbb{P}}}{\langle \Omega^1_{\mathbb{P}/X} \rangle})$.

in general case may have horizontal direction using this kind of ideas:

For $X/\mathbb{C}$, flat bundle (E,∇), $\nabla^2 = 0$

$$\begin{array}{ccc} H^{2n-1}(X, \mathbb{C}/\mathbb{Z}(1)) & & H^{2n}(X,\mathbb{Z}) \\ \mathbb{C}^\times & & \uparrow \\ \| & & K_0(\mathbb{C}) \\ K_1(\mathbb{C}) & & \| \\ & & H^{2n}(X,\mathbb{Z}) \otimes K_0(\mathbb{C}). \\ H^{2n-1}(X,\mathbb{Z}) \otimes K_1(\mathbb{C}). & & \end{array}$$

$\to H^{2n-2}(X,\mathbb{Z}) \otimes K_2(\mathbb{C})$ mod torsion

$H^{2n-p}(X,\mathbb{Z}) \otimes K_p^M(\mathbb{C})$ $\quad p \le n$

$H^n(X,\mathbb{Z}) \otimes K_n^M(\mathbb{C})$. $\longrightarrow$ conjectures.

while AD complicated.

Simpler part:

$$AD^n(X) \longrightarrow AD^n(\operatorname{Spec} \overset{\eta}{\overset{\|}{k(X)}}) = \begin{cases} \dfrac{\Omega^{2n-1}_\eta}{d\,\Omega^{2n-2}_\eta} & n \geq 2 \\[2ex] \dfrac{\Omega^1_\eta}{d \log k(X)^\times} & n = 1 \end{cases}$$

S. Bloch: Close at generic pt $\longrightarrow$

given by TP eq'n à la Chern-Simons

(E,∇) alg, trivialized by Zariski covering of X

$\nabla \leftrightarrow$ matrix of 1-forms A, compute that

$$TP_n(A) - TP_n(\underbrace{g}_{g\text{-inv on } U} A g^{-1} + dg\, g^{-1}) \quad \text{closed on } U$$

locally exact. $n \geq 2$.

locally $d\log$ exact $n = 1$.

$\Rightarrow \quad TP_n(A) \in H^0(X, \Omega^{2n-1}/d\,\Omega^{2n-2}) \qquad n \geq 2$

$\qquad \in H^0(X, \Omega^1/d\log \mathcal{O}^\times) \qquad n=1$

Bloch Ogus : $\quad C \begin{cases} \Omega_\eta / d\,\Omega_\eta^{2n-2} & n \geq 2 \\ \Omega_\eta / d\log k(X) & n = 1. \end{cases}$

---

class at generic pt $\to$ just like

$$CH^n(X) \longrightarrow \text{Griffiths}^n(X)$$

relation $\sim$ by $\mathbb{P}^1$ deformation ; $\sim$ by 1 parameter deformation

THEOREM : (Bloch, —) :

$f : X \to S$ projective morphism / Spec k, char $k = 0$
X, S smooth, $d = \dim f = \dim X - \dim S$,
smooth / Spec $k(S)$, Y = rel. NCD $= f^{-1}(\Sigma) + Z$
with Σ NCD on S.
Given $D : E \to \Omega^1_X(\log Y) \otimes E$ with conditions :

hence

$$\begin{array}{ccccc} & & \Omega^2_{k(X)} & \longrightarrow & \Omega^2_{k(X)/S} \\ \downarrow & & \uparrow & & \\ \Omega^2_S \otimes k(X) & \longrightarrow & * & \longrightarrow & \Omega^1_S \otimes \Omega^1_{k(S)/S} \end{array}$$

assume $\nabla^2 : E \to f^*\Omega^2_S(\log \Sigma) \otimes E$
$\qquad\qquad \cap\ \Omega^2_X(\log Y)$
(condition for the existence of a Gauß-Manin conn.)

classes w_n = class at generic pt associated to $P_n(M) = \mathrm{Tr}\, M^n$. Then

I. $w_n\left(\Sigma(-1)^i Rf_*(\Omega^{\cdot}_{X/S}(\log Y)\otimes E, \nabla_{X/S}), GM \text{ conn.}\right)$

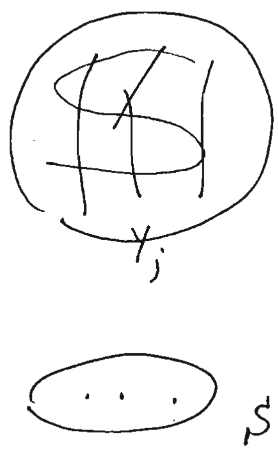
$= (-1)^d f_*\left[c_d(\Omega^1_{X/S}(\log Y), \mathrm{res}_Z)\cdot w_n(E,\nabla)\right]$, $n \geq 2$.

II. $n=1$: true if $\otimes \mathbb{Q}$, but also true $/\mathbb{Z}$ if replace (E,∇) by $(E-(\mathrm{rk}\,E)\mathcal{O}, \nabla_E(\mathrm{rk}\,E)d)$.

comments:

① LHS: coh. sheaf $R^if_*\Omega_{X/S}\log Y\otimes E$ not loc. free. (TPth) extends to coh. sheaves with connections.

② RHS: the product "$\cdot$":



$(E,\nabla)|_Z$

cycles like multi-sections with coeff.

$\cdot\ w_n \in \dfrac{H^0(X,\Omega^{2n-1}(\log Y))}{d\,H^0(X,\Omega^{2n-2}(\log Y))}$ $n \geq 2$

$\cdot\ \dfrac{H^0(X,\Omega^1(\log Y))}{d\log H^0(X,\mathcal{O}^\times)}$

take repr of $c_d(\Omega^1_{X/S}(\log Y))$ does not contain Y_j.

then $(E,\nabla)|_S$ + trace not defined.

Instead of this "section S",

$CH^d(X) = H^d(X,K_d) \leftarrow \mathbb{H}^d(X, K_d \to \oplus K_d|_{Y_i} \to \oplus_{i<j} K_d|_{Y_i\cap Y_j} \to \cdots)$

res $\Omega^1_X(\log D) \longrightarrow \oplus \mathcal{O}_{Z_i}$

if write $CH^d(X, Z) = \mathbb{H}^d(X, K_d \to \oplus K_d|_{Y_i} \to \cdots)$

then get pairing

$$f_*\left(CH^d(X,Z) \cdot \{\text{chern-Simons coh.}\}\right)$$

(3) In Previous lect. R.R.:

$$f: X \to S \qquad\qquad ch(Rf_*E),\ ob(S)$$

$$\uparrow \qquad\qquad\qquad \|$$

$$E \quad ch(E),\ ob(X) \qquad F(f, ch\,E)$$

the thm is of this shape.

- LHS depends on $X, U \xrightarrow[f]{} S$, $(E, \nabla)|_U$
- RHS is OK: CS classes are recongnized on $\operatorname{Spec} k(X)$, in particular on U.

information of $c_d(\Omega^1_{X/S}(\log Y), res_2) \in CH^d(X, Z)$ depends only on (X, U)

$\rightsquigarrow$ RHS eq'n is of Grothendieck type.

ON THE PROOF:

(1) Reduction to case $X = \mathbb{P}^1 \times S \xrightarrow{\gamma} S$ and S = fcn field ($S \sim \operatorname{Spec} k(S)$) $=: K$

$X \supset Y$ horizontal divisor, modulo torsion

$Y \in \mathbb{P}(K)$, ΣY_i, y_i rat'l point of the base.

(2) Reduction to the case: to allow more general coherent sheaves with this type of connections.

need deal with torsion $\to$ no torsion in E.

ie. $E \simeq \oplus \mathcal{O}(n_i)$. hence may assume

$E \simeq \oplus \mathcal{O}$ trivial bundle (alg. trivial bundle)

(Bolibrud: $S = \mathbb{C}$, $\mathbb{P}^1$ – pts (ICM '94)

$\rho : \pi_1(U(\mathbb{C})) \longrightarrow GL(r, \mathbb{C})$,

if ρ is not irreducible, then $\forall (E, \nabla)$

$$\nabla : E \longrightarrow \Omega^1_{\mathbb{P}^1}(\log \Sigma) \otimes E$$

st. $E^{\nabla}|_U \cong \rho$.

$E \neq \mathcal{O}(m) \otimes$ trivial bundle.)

③ Boils down to the following kind of eq'ns

$\Leftarrow$ higher trace formula in $\widehat{\ }$ Milnor's K-theory:

dlog –

$N \geq \delta \geq r \geq 1$: let $a_i \neq 0$

consider $\omega = \sum_i d\log(z - a_i) - (\delta + 1)\dfrac{dz}{z} = \dfrac{F(z)\,dz}{z\prod(z - a_i)}$

$\mathrm{res}_{a_i}\omega = \mathrm{res}_\infty \omega = 1$, F has roots β_i.

key lemma: $\sum d\log(\beta_1 - a_1) \wedge \cdots \wedge d\log(\beta_i - a_i)$

$$= \sum_1^r (-1)^{j-1} d\log F(a_j) \wedge d\log(a_j - a_1) \wedge \cdots \wedge \widehat{d\log(a_j - a_j)} \wedge \cdots \wedge d\log(a_j - a_r)$$

this eq'n $\Leftrightarrow$ R.R.

To be continued.

RR Thm for rk 1 irregular connections on curves

Deligne (~73)

$U \hookrightarrow X$ smooth alg v. $/\mathbb{C}$
open

local system on U :

$V_\rho \longleftrightarrow \rho : \pi_1(U) \longrightarrow GL(N, \mathbb{C})$

How to extend to all X ?

Analytic way: (Riemann - Hilbert correspondences)

$V_{\rho c} \xleftrightarrow{1-1} (E, \nabla)_{ana.}$ ∇ flat connection

$V_\rho \longrightarrow (E = \mathcal{O}_{an} \otimes_{\mathbb{C}} V_\rho , d \otimes \wedge)$

solution of system of differential equations.

$E^\nabla \longleftarrow (E, \nabla)$. Require strong topology.

$(E, \nabla)_{ana}$ underlies an algebraic structure :

$$((E_{alg}, \nabla_{alg})) \otimes_{\mathcal{O}_{X_{Zar}}} \mathcal{O}_{X_{an}} = (E, \nabla)_{ana}.$$

but (E_{alg}, ∇_{alg}) not unique ! many choices.

Yes, if one requires ∇_{alg} at ∞ $(= X - U)$

has only logarithmic poles, then

(E_{alg}, ∇_{alg}) is unique !

Related to connection with regular singularities at ∞.

i.e. $\exists$ extension of (E_{alg}, ∇_{alg}):

$\overline{E}_{alg}, \overline{\nabla}_{alg} : \overline{E}_{alg} \longrightarrow \Omega^1(\log\infty) \otimes \overline{E}_{alg}$

alg. vector bundle on X.

"top" $/_U$ $\longleftrightarrow$ "reg. sing." connections are controlled by topology.

For irregular singularities, invariants of connections are only partly controlled by topology.

Rank 1 case:

$$X \longrightarrow \operatorname{Spec} K = S$$

(L, ∇) connection, $U/_K \subset X \supset D = X - U$

K = function field $/_k$ char $k = 0$.

take $\overline{L} \xrightarrow{\overline{\nabla}} \Omega^1_X(*D) \otimes \overline{L}$ an extension.

$D = \sum D_i$ defined over K.

$\nabla_{X/S} : \overline{L} \longrightarrow \Omega^1_{X/S}(*D) \otimes \overline{L}$

rank 1 sheaf

n_i = smallest $n_i \in \mathbb{N}$ st.

$\nabla_{X/S} : \overline{L} \to \Omega^1_{X/S}(\sum n_i D_i) \otimes \overline{L}$, but not

$\mathfrak{D} = \Omega^1_{X/S}((n_i - 1) D_i + \sum_{j \neq i} n_j D_j) \otimes \overline{L}$

call $\omega = \Omega^1_{X/S}$. have

$\nabla_{X/S} : \overline{L} \longrightarrow \omega(\sum n_i D_i) \otimes \overline{L}$

Recall:

$$(L,\nabla)|_{\mathcal{U}} \in \mathbb{H}^1(U, \mathcal{O}^\times \xrightarrow{d\log} \Omega^1_U)$$
$$(\bar{L},\bar{\nabla}) \in \mathbb{H}^1(X, \mathcal{O}^\times \longrightarrow \Omega^1_X(*D))$$
$$(\bar{L},\bar{\nabla})_{X/S} \in \mathbb{H}^1(X, \mathcal{O}^\times \longrightarrow \underset{\omega(\mathcal{D})}{\underset{\shortparallel}{\Omega^1_{X/S}(\textstyle\sum n_i D_i)}})$$

- want to make $*D$ more precise, call $f^{-1}(\omega(\mathcal{D}))$.

$$0 \longrightarrow f^*\Omega^1_S(*D) \to \Omega^1_X(*D) \twoheadrightarrow \Omega^1_{X/S}(*D)$$

relative connection.

Assume: $\nabla^2 : \bar{L} \to f^*\Omega^2_S(*D) \otimes \bar{L}$ (vertically)

eg. ∇ flat.

$\Rightarrow$ In fact, $\bar{\nabla} : \bar{L} \to \Omega^1_X(\log D)(\mathcal{D}-D) \otimes \bar{L}$

computations $\Rightarrow$

$$(\bar{L},\bar{\nabla}) \in \mathbb{H}^1(X, \mathcal{O}^\times \to \Omega^1_X(\log D)(\mathcal{D}-D))$$

R.R: $\mathcal{U} \hookrightarrow X \longrightarrow \operatorname{Spec} K = S$

(L,∇) on $\mathcal{U}$, ∇^2 ...

Katz: take $(\bar{L},\bar{\nabla})$ as before with extra condition that

$$\left(\bar{L} \xrightarrow{\bar{\nabla}_{X/S}} \omega(\mathcal{D} \otimes \bar{L})\right) \underset{\text{q. isom.}}{\longrightarrow} \left(L|_U \xrightarrow{\nabla_{X/S}} \omega_{U/S} \otimes L|_U\right)$$

a ... if ∇ with pole everywhere along D.

Gauß-Manin connections = ?

$$\begin{array}{ccccc} & & \bar L & \xrightarrow{\sim} & \bar L \\ & & \downarrow \nabla & & \downarrow \nabla_{X/S} \\ f^*\Omega^1_S \otimes \bar L(\mathcal{D}-D) & \longrightarrow & \Omega^1_X(\log D)(\mathcal{D}-D)\otimes \bar L & \longrightarrow & \omega(\mathcal{D})\otimes \bar L \\ \downarrow & & \downarrow \nabla & & \\ f^*\Omega^1_S\otimes\omega(\mathcal{D}-D)\otimes\bar L & \xrightarrow{\sim} & \dfrac{\Omega^2_X(\log D)(\mathcal{D}-D)\otimes\bar L}{f^*\Omega^2_S(*D)\otimes\bar L} & & \end{array}$$

think Leibnitz rule

$$GM^i: R^if_*(\bar L \to \omega(\mathcal{D})\otimes\bar L) \longrightarrow \Omega^1_S\otimes R^if\left(\bar L(\mathcal{D}-D)\to\omega(\mathcal{D})\bar L(\mathcal{D}-D)\right)$$

$$H^i(\nabla_{X/S}) = R^if(L|_U \to \omega_{U/S}\otimes L) \qquad (L|_U \to \omega_{U/S}\otimes L|_U)$$

GM connection

want to compute:

$$\det\left(\sum (-1)^i H^i(\nabla_{X/S})\right)$$
$$= \left(\Lambda^{\mathrm{rk} H^0} H^0, \Lambda^{\mathrm{rk} H^0} GM^0\right)\otimes\left(\Lambda^{\mathrm{rk} H^1} H^1, \Lambda^{\mathrm{rk} H^1} GM^1\right)\cdots$$

rk 1 connection on $\operatorname{Spec} K = S$

$$\in H^1(\operatorname{Spec} K, \mathcal{O}^\times \longrightarrow \Omega^1_S) = \Omega^1_K / d\log K^\times.$$

Know already:

thm: ① Regular case (no sing.) $f: X \leftrightarrows S$ (L, ∇)

$$(\det)^{-1} = f_*(c_1(\omega)\cdot c_1(L,\nabla))$$

$c_1(\omega)$ in $\mathrm{Pic}(X)$, $c_1(L,\nabla)$ in $H^1(X, \mathcal{O}^\times \to \Omega^1)$

product into $H^2(X, \mathcal{K}_2 \to \Omega^2_X)$.

then to $\mathbb{H}^2(X, K_2 \to f^*\Omega^1_S \otimes \omega)$

$$\begin{tikzcd} \mathbb{H}^2(X, K_2 \to f^*\Omega^1_S \otimes \omega) \arrow[d, "Tr"] \\ \Omega^1_S / d\log \mathcal{O}_S^\times \end{tikzcd}$$

trace

② regular (i.e. logarithmic) singularities :

$$(\det)^{-1} = f_*\left(c_1(\omega(D)) \cdot c_1(\bar{L}, \bar{\nabla})\right)$$

$\mathrm{Pic}(X, D) \times \mathbb{H}^1(X, \mathcal{O}^\times \to \Omega^1_X(\log D))$

$\uparrow$

// generalized jacobians of Serre.

$\{(M, s) \mid s : \mathcal{O}_D \xrightarrow{\sim} M_D\}$

$\mathbb{H}^1(X, \mathcal{O}^\times \twoheadrightarrow \mathcal{O}_D^\times)$

$c_1(\omega(D))$; res : $\mathcal{O}_D \xrightarrow{\sim} \omega_D|_S \cong \mathcal{O}_S$

the product we have

$\mathcal{O}_X^\times \to \mathcal{O}_D^\times$

$$\begin{tikzcd} \mathcal{O}_X^\times \times (\mathcal{O}_X^\times \to \Omega^1_X) \arrow[r] \arrow[d] & (K_2 \to \Omega^2_X) \arrow[d, "Tr"] \\ \mathcal{O}_X^\times \to \Omega^1_X(D) \arrow[ur] & K_2 \to f\,\Omega^1_S \otimes \omega_{X/S} \end{tikzcd}$$

③ Irregular case :

$\mathrm{Pic}(X, \mathcal{D}) \times \mathbb{H}^1(X, \mathcal{O}^\times \to \Omega^1(\log D)(\mathcal{D} - D))$

not D. in order to pair.

$c_1(\bar{L}, \bar{\nabla})$

? need char. class here.

like $c_1(\omega)$, $(c_1(D), \mathrm{res})$, but now ...

Need new idea.

$$\begin{tikzcd} \bar{L} \arrow[r, "\nabla_{X/S}"] \arrow[dr] & \omega(\mathcal{D}) \otimes \bar{L} \arrow[d] \\ & \frac{\omega(\mathcal{D})}{\omega} \otimes \bar{L} = \omega_{\mathcal{D}} \otimes \bar{L}|_{\mathcal{D}} \end{tikzcd}$$

Leibnitz formula

$\to \mathcal{O}_{\mathcal{D}}$ linear map.

$pp \quad \nabla_{X/S} : \bar{L}|_{\mathcal{D}} \xrightarrow{\sim} \omega_{\mathcal{D}} \otimes \bar{L}_{\mathcal{D}} \quad ; \quad \mathcal{O}_{\mathcal{D}} \to \omega_{\mathcal{D}}$.

the class is :

$$u(\omega(\mathcal{D}), pp\,\nabla_{X/S})\ !$$ Now can write down

THEOREM : (S. Bloch, —) :

$$(\det)^{-1} = f_* \left(u(\omega(\mathcal{D}), pp\nabla_{X/S}) \cdot u(\bar{L}, \bar{\nabla}) \right).$$

More precisely, $\exists$ cup product + trace $j : U \hookrightarrow X$.

$$\begin{array}{c} \mathrm{Pic}(C, \mathcal{D}) \times H^1(X, j_* \mathcal{O}^\times \to \Omega^1(\log D)(\mathcal{D} - D)) \\ \downarrow \\ \Omega^1_K / d \end{array}$$

Some Comments :

① Reg. Sing. Case : then $D = \mathcal{D}$.

$u(\omega(D), pp\ \nabla_{X/S}) \in \mathrm{Pic}(X, D)$ – today

$u(\omega(D), \mathrm{res}_D) \in \mathrm{Pic}(X, D)$ – tuesday

global geometry (Atiyah class)

$\Rightarrow$ 2 RHS are the same.

② Another formulation :

$N = 2g - 2 + \sum m_i = \deg \omega(\mathcal{D})$

on $\mathrm{Pic}^N(X, \mathcal{D})$ $\exists$ special K point :

$a(\omega(\mathcal{D}), pp\,\bar{\nabla}_{X/S})$ some choice of $\bar{\nabla}$

Formulation:

$$X - |D| \xrightarrow{\alpha} \mathrm{Pic}^1(X, \mathcal{D})$$

$$\begin{array}{c} \psi \\ x \longmapsto \mathcal{O}_X(x),\ \mathcal{O} \hookrightarrow \mathcal{O}_X(x) \end{array}$$

Similarly for $\mathrm{Pic}^N(X, \mathcal{D})$ ...

$$\begin{array}{c} \mathrm{Pic}^N(X, \mathcal{D}) \\ \downarrow \\ S \end{array}$$

Gives rk 1 connections

choice of x_0

$\mathrm{Pic}^0(X, \mathcal{D}) \simeq \mathrm{Pic}^N(X, \mathcal{D})$

comm. alg. gp

$(M, s) \longrightarrow (M(Nx_0), s)$

Def'n: ① invariant relative rank 1 connection, i.e.

$$\begin{tikzcd} G \times G \arrow[r, "\mu"] \arrow[d, "pr_1"'] \arrow[dr, "pr_2"] & G \\ G & G \end{tikzcd}$$

$\mu^*(L, \nabla) = p_1^*(L, \nabla) \otimes p_2^*(L, \nabla)$

② absolute inv. rk 1 connection

$\rightarrow$ Same def + triviality along 0-section.

Prop: Via α^*:

(1,1) correspondence between

inv. rk 1 conn
rel:
abs:
(rk 1)

$\longleftrightarrow$

$L \to \omega(\mathcal{D}) \otimes L$

$L \to \Omega^1(\log D)(\mathcal{D} - D) \otimes L$

Vect $\longleftrightarrow$ Vect.

In particular, via α^*, (L, ∇) thought of as a rk 1 vertical conn. on $\mathrm{Pic}^N(C, \mathcal{D})$. Then RHS

$(\bar{L}, \bar{\nabla})\,|$ special point.

One word about the proof:

Invariant connection

→ New R.R. thm:

$$\begin{array}{ccc} Pic^N(X, D) & & (M, \Delta) \\ \downarrow g & & \downarrow \\ Pic^N(X) & & M \end{array}$$

$B = g^{-1}(\omega(D)) \simeq$ torsor under $\mathcal{O}_D^\times$ ($\simeq \prod G_a \times \prod G_m$)

(L, ∇) on B.

$d + \beta$, β = "exact form"

Kontsevich: Comparing in some cases:

DR coh ⟷ $d + df\,\beta$. f exact β

Higgs coh ⟷ df ↦ dim are the same.

"K's thm" come out to the diff. form.

though the pf has nothing to do with Kontsevich.

③ Further Comment: (Higher rank case.)

Assume (E, ∇) irregular on $U \subset X \to S$, $r = \operatorname{rk} E > 1$ curve.

$(\bar{E}, \bar{\nabla})$: $pp\,\nabla_{X/S}$: $\bar{E}_D \to \omega_D \otimes \bar{E}_D$
(polar part)

$\det pp\,\nabla_{X/S} : (\det E)|_D \xrightarrow{\sim} \omega_D^r \otimes (\det \bar{E})|_D$

S'll see, ∃ ex. for which

$(\det)^{-1} \neq f_*(u\,\omega(D), pp \det \nabla) \cdot u \det(\bar{E}, \bar{\nabla})$!

(this is what we have for $\bar{\nabla}$: log sing.) END

Need again new ideas. (in progress).

The First

NCTS Summer School on Algebraic Geometry

July 19 — 30, 1999

Professor Ching-Li Chai
University of Pennsylvania

(Notes by Chin-Lung Wang)

Langlands Program and Non-Abelian Class Field Theory

Lecture I — 7/20, p.1

Lecture II — 7/26, p.6

Lecture III — 7/27, p.14

Lecture IV — 7/30, p.20

NCTS Summer School in Alg. Geom.
Prof. Ching-Li Chai
Lecture I. 7/20 at Wu-Ling Farm.

General Langlands' Program ~
non abelian C.F.T. (class Field Theory)

Review: classical C.F.T.

F - global field or local field

Galois side: Analytic side:

$$\left\{ \begin{array}{c} \mathrm{Gal}(F^{sep}/F) \to \mathbb{C}^\times \\ \text{characters of} \\ \text{finite order} \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \mathbb{A}_F^\times/F^\times \longrightarrow \mathbb{C}^\times \\ F^\times \longrightarrow \mathbb{C} \end{array} \right\}$$

characters of finite order

$\mathrm{res}_F : \mathbb{A}_F^\times/F^\times \longrightarrow \mathrm{Gal}(F^{sep}/F)$ reciprocity norm map

This is Artin's formulation.

Another formulation: L- functions

$$\begin{array}{c} \text{abelian Hecke} \\ \text{L-functions} \\ \text{(Dirichlet)} \end{array} \overset{1-1}{\longleftrightarrow} \begin{array}{c} \text{Hecke's} \\ \text{L-function} \end{array}$$

Outline of the shape of non-abelian generalization

Analytic Side

automorphic representation (generalization of modular form + consider all levels simultaneously)

Auto. repr of $G(\mathbb{A}_F)$

irred. adm. repr of $G(F)$

G = conn. reductive. alg. gp $/F$.

Galois Side

are parametrized by <u>parameters</u>

$$L_F \longrightarrow {}^L G$$

$\bigcup$ $\quad$ "L-group"

$\mathrm{Gal}(F^{sep}/F)$

" extension of $\mathrm{Gal}(F^{sep}/F)$ by a cpx reductive Lie group "

Remark: one parameter may correspond to several repr of LHS (L-indistinguishable / stable conj class etc)

Serious Prob: when F is a global field, don't really know what L_F is!

But the situation in the local case is a lot better

eg. Local Langlands' conj for GL_n has been proved (M. Harris, R. Taylor, Henniart)

When F is a local field,

$$L_F = W_F \times SL_2(\mathbb{C})$$

W_F the Weil group:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I & \xrightarrow{\text{inertial gp}} & \mathrm{Gal}(F^{sep}/F) & \longrightarrow & \mathrm{Gal}(k^{sep}/k) & \longrightarrow & 1 \\ & & \| & & \cup & & \underset{\cong}{\hat{\mathbb{Z}}} & & \\ 1 & \longrightarrow & I & \longrightarrow & W_F & \longrightarrow & \mathbb{Z} \subset \hat{\mathbb{Z}} & \longrightarrow & 1 \end{array}$$

gen. by frob. $= \mathbb{Z}$

k residue field

${}^L G = \hat{G} \rtimes W_F$

via simple co-characters :

dual root system determine dual groups

exchange characters $\longleftrightarrow$ co characters.

$\hat{G}$ has root system dual to that of G.

${}^L G$ is defined up to inner automorphism.

eg. $G = GL_n$, $\hat{G} = GL_n$

G = type C_n , $\hat{G}$ = type B_n etc.

From the number theoretic point of view :

when $G = GL_n$, $L_F \to {}^L G$ is very close to

what we want the ("Galois repr'n")

$\pi = \bigotimes_{v \in \Sigma_F} \pi_v$

φ_v : parameter of π_v

It looks like we get a family of local Galois repr'n.

This reminds the notion of compatible system of ℓ-adic repr'n.

A substantial part of number theory are talking about this. $\to$

eg. $X/_F \rightsquigarrow H^*(X \otimes_F \bar{F}, \bar{\mathbb{Q}}_\ell)$

ℓ-adic representations.

Prophecy / expectation :

If $\pi = \otimes_v \pi_v$ is an <u>algebraic</u> auto. repr'n,

then π "should" correspond to a "motive".

For simplicity, let $G = GL_n$:

Algebraic: 2 aspects

- archimedean parameters $\pi_\infty = \otimes_i \pi_{\infty_i}$ should be algebraic:

  Aside: $F \cong \mathbb{R}$, $1 \to \bar{F}^\times \to W_F \to \mathrm{Gal}(\bar{F}/F) \to 1$

  $F \cong \mathbb{C}$, $W_F := F^\times$ the only non-trivial ext.

  i.e. $W_F = \bar{F}^\times \amalg j\bar{F}^\times$ with $jzj^{-1} = \bar{z}$.

  when we talk about homo $W_F \longrightarrow \cdots$ this is the cpx Hodge structure in alg. geom.

  i.e. Hodge str $/\mathbb{R}$ with value in $\mathbb{C}$.

  Now algebraic in the sense that the restriction of φ_∞ to $\mathbb{C}^\times$ are algebraic ("$z^{m_1}\bar{z}^{m_2}$")

  (Weil: Grothen characters)

- $\mathbb{Q}(\pi_f)$ = field of moduli of π_f is algebraic

  f finite Adelic component

  i.e. the fixer is an alg. number field.

Given such a repr. try to construct "motive" in the sense of L-functions. In good case, may use Shimura varieties, eg. for GL_2, Deligne has done this.

Remark: In function field case, the Hodge part is totally different. That's the problem.

$X_{/F}$ alg. v. over number field F.
X smooth. proj. say.

$\rightsquigarrow H^*(X \otimes_F \bar{F}, \bar{\mathbb{Q}}_\ell)$

[From the point of L-functions:
Identify {auto. L-fun} $\underset{?}{\longleftrightarrow}$ {motivic L-func}

Auto L-functions:

$\pi = \otimes_v \pi_v$, need $r : {}^LG \longrightarrow GL(V)$

get $L(s, \pi, r)$

should be meromorphic, has functional eq'n

$$L(s, \pi, r) = \varepsilon(s, \pi, r) \cdot L(1-s, \pi, \tilde{r})$$

If one admits

$$\begin{tikzcd} L_F \arrow[r, "\varphi"] \arrow[dr] & {}^LG \arrow[d, "\alpha"] \\ & {}^LH \end{tikzcd}$$

(finite collection)

Langlands' functoriality

α: {packets for G} $\longrightarrow$ {packets for H}.]

Expect:

- $L(s, H^*(\bar{X}, \bar{\mathbb{Q}}_\ell))$ should be automorphic L-function of GL_n, $n = \dim(H^*(\bar{X}, \bar{\mathbb{Q}}_\ell))$.

  But what we want is the smallest such gp.

Question: what is the "smallest" gp H st. this

$L(s, H^*(\bar{X}, \bar{\mathbb{Q}}_\ell))$ is $L(s, \otimes_v \pi_v, r)$?
auto. repr. of H

Take an archimedean place ∞_1 of F:

natural Hodge str. $\rightsquigarrow H^*(X_{\infty_1}(\mathbb{C}), \mathbb{Q})$

$\rightsquigarrow$ "$\hat{G}$" $MT(\cdots)$ conn. reductive gp $/\mathbb{Q}$.

via Tannakian category.

Optimistically.

To be continued

F $\left\langle \begin{array}{l} \text{local} \\ \text{global} \end{array} \right.$ fields G : reductive conn alg. gp / F

Analytic Side:

- irreducible admissible repr of $G(F)$, local
- irred. automorphic repr "of $G(\mathbb{A}_F)$". global

Galois Side:

<u>admissible</u> homomorphism:

$$L_F \longrightarrow {}^L G$$

- local : W_F' or $W_F \times SL_2(\mathbb{C})$ $(SU_2(\mathbb{R}))$
- global : ?? (unknown yet)

1. Weil groups:

Explicitly:

F : local, non archimedean:

$$\begin{array}{ccccccccc} 1 \to & I & \to & Gal(F^{sep}/F) & \to & Gal(K^{sep}/K) & \to 1 \\ & \parallel & & \cup\!\text{I} & & \cup & \\ 1 \to & I & \longrightarrow & W_F & \longrightarrow & \{Frob^{\mathbb{Z}}\} & \to 1 \end{array}$$

discrete top.

topology: from I.

use W_F to get more characters.

$F \cong \mathbb{C}$, $W_F \cong F^\times \cong \mathbb{C}^\times$

$F \cong \mathbb{R}$: $1 \to \bar{F}^\times \to W_{\mathbb{R}} \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1$

non-split

$j \mapsto \sigma$, $jzj^{-1} = \bar{z}$

$\mathbb{R}^\times \xrightarrow{\sim} W_{\mathbb{R}} / W_{\mathbb{R}}^c$: reciprocity law map

$-1 \longmapsto j\, W_{\mathbb{R}}^c$

$x \longmapsto \sqrt{x} \cdot W_{\mathbb{R}}^c$

typical example in class field theory

In general : $F^\times$

$$\mathbb{A}_F^\times / F \xrightarrow{\sim} W_F^{ab} \quad \text{reciprocity law map.}$$

This is what we want for Weil gp.

$$W_F \longrightarrow \mathrm{Gal}(F^{sep}/F) \quad \text{with dense image}$$

(i.e. capture all top. information and get more rep'n. for local F, we completely know them.)

Functoriality :

$F \longrightarrow {}^\sigma F$ (by Frob.) get

$W_F \longrightarrow W_{\sigma F}$

global - local compatibility : let F global

$$\begin{array}{c} W_F \\ \uparrow \\ W_{F_v} \end{array}$$

and just like C.F.T.

for E/F finite separable ext. get

$$\begin{array}{ccccc} W_E^{ab} & \longrightarrow & \overset{E^\times}{\mathbb{A}_E^\times / E^\times} & \longrightarrow & \mathrm{Gal}(E^{ab}/E) \\ \uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow \\ W_F^{ab} & \longrightarrow & \overset{F^\times}{\mathbb{A}_F^\times / F^\times} & \longrightarrow & \mathrm{Gal}(F^{ab}/F) \end{array}$$

one is inclusion
one is transfer

In Short : Weil gp are suitable version of "Galois group".

2. Langlands dual groups :

G/F

(1). Root data for G/F^{sep} :

T : max. torus

B : Borel subgroup (over F^{sep})

(minimal parabolic subgp. eg.  's).

get $X^*(T)$, Δ^* : simple roots, and dual :

$(X^*(T), \Delta^*, X_*(T), \Delta_*)$

To be canonical, take $\varprojlim$ (of these)

Now, when G/F, has action of $\Gamma_F = \mathrm{Gal}(F^{sep}/F)$.

on $(X^*, \Delta^*, X_*, \Delta_*)$.

Now, take the dual root data $(X_*, \Delta_*, X^*, \Delta^*)$

which is by def an "abstract root system"

general theory $\to$ can produce reductive gp over any alg. closed field. say $\mathbb{C}$.

call it $\hat{G}$. $/\mathbb{C}$.

Next thing : to produce (Weil version of it)

$$\begin{tikzcd} 1 \arrow[r] & \hat{G} \arrow[r] & {}^L G \arrow[r] & W_F \arrow[r] \arrow[l, bend left, "\text{split}"] & 1 \end{tikzcd}$$

then will get an action of W_F on $\hat{G}$. hence get

$\hat{G} \rtimes W_F$ (choosing a splitting)

(There are also finite Galois versions.)

(trivial) example:

1) GL_n : $(\mathbb{Z}^n, e_i - e_j, \mathbb{Z}^{n*}, x_i - x_j)$

$\widehat{GL_n} = GL_n$

2) SL_n : (Fact: dual of simply connected semi-simple gp get adjoint gp)

$\widehat{SL_n} = PSL_n$

3) Forgetting the center:

$A_n \longleftrightarrow A_n$; $B_n \longleftrightarrow C_n$; $D_n \longleftrightarrow D_n$

(we do not bother exc. gp here)

4) $PSp_{2n} \longleftarrow Spin_{2n+1}$

5) GSp_{2n} : $\widehat{GSp_{2n}}^{der}$ is simply connected!

6) E/F finite extension,

$Res_{E/F}(GL_2/E)$ (Weil restriction)

L-dual group $= GL_2/\mathbb{C} \rtimes W_F$

action of $W_F \twoheadrightarrow Gal(F^{Gal}_{null}/F)$ on the root system

$\simeq$ action $Gal(F^{sep}/F)$ on $Hom(E, F^{sep})$.

4. $\varphi : L_F \longrightarrow {}^LG$ is admissible (homomorphism)

if (roughly):

- semi-simple
- (locally) relevant

Wait,

3. L_F : F local.

If $F \cong \mathbb{R}$ or $\mathbb{C}$, take $L_F \cong W_F$

If F is local non-archemedean, then one has to enlarge W_F (Deligne):

Two versions: (Deligne-Weil gp)

- $W_F' = W_F \ltimes \mathbb{G}_a$

commutation relation $\underset{\mathbb{G}_a}{w\, x}\, w^{-1} = |w|\, x$ via the reciprocity law map.

- Jacobson-Morosov : $W_F' \rightsquigarrow W_F \times SL_2(\mathbb{C})$.

for the equivalence, we want homomorphism which is holomorphic (alg) in $SL_2(\mathbb{C})$ factor.

Motivation:

Grothendieck's semi-stable reduction theorem:

Not an alg. notion

$\hookrightarrow \rho_\ell : Gal(F^{sep}/F) \to GL(V/\overline{\mathbb{Q}_\ell})$ $\ell \neq char(K)$ residue field

$\rho_\ell|_U$ is unipotent, $U \subset I$ finite index

$N = \log \rho_\ell|_U \in Hom(V(1), V)$ Tate twist

(Arithmetic analogue of top'l quasi-unipotency thm)

But, one can "take away" the monodromy to make this into an algebraic notion.

$$1 \to I \to W_F \to Gal(K^{sep}/K) \to 1$$

$$\Phi \mapsto Frob.$$

$$\underbrace{\rho_\ell}_{\text{Galois repr.}}(\Phi^n \underset{\substack{\cap \\ I}}{\sigma}) = \underbrace{\rho'(\Phi^n\sigma)}_{\text{repr of } W_F'} \exp(t_\ell(\sigma) N) \qquad t_\ell : I \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \xrightarrow{\sim} \mathbb{Q}_\ell(1)$$

the point is, $\rho'(W_F)$ is finite, hence algebraic

eg. can do conjugation. etc.

4. (continue).

- semi-simple : easy via 2nd version

  $W_F \times SL_2(\mathbb{C}) \to GL_n$. semi-simple in the usual sense (direct sum of irreducibles)

  In general, $\varphi : W_F \times SL_2(\mathbb{C}) \to {}^L G$ is semi-simple if :

$$\begin{tikzcd}
W_F \times SL_2(\mathbb{C}) \arrow[rr] \arrow[dr] & & {}^L G \arrow[dl] \\
& GL(V) &
\end{tikzcd}$$

- relevant :

$$\underset{\substack{\circlearrowleft \\ \Gamma_F}}{\left\{\begin{array}{l} \text{conjugacy class} \\ \text{of parabolic} \\ \text{of } G/F^{sep} \end{array}\right\}} \xleftrightarrow[1-1]{} \left\{\begin{array}{l} \text{conj. classes} \\ \text{of parabolics} \\ \text{of } \hat{G} \end{array}\right\}$$

Real problem: If a conj class is fixed by Γ_F, it may or may not come from a parabolic of G rational over F.

$\varphi : L_F \to {}^L G$ is relevant if,

If $\varphi(L_F) \subset$ Levi of a F-ratl conj class of a parabolic

then this parabolic is relevant.

Ex. B: quaternion division alg.

$\rightsquigarrow B^\times$

Say, $W_F \xrightarrow[\varphi]{} {}^LG = GL_2 \times W_F$, then $\varphi = \mu \oplus \nu$ is not allowed for μ, ν 2 characters.

5. Analytic Side:

F: local Archemedean,

admissible = H-C modules (Harish-Chandra)

G: Lie group, usually consider just continuous representations V, (say 50 years ago)

\ but "same repr" may have many diff top. ver's!

"Algebraic / essential" part:

$K \subseteq G$ max. cpt.

take the space of K-finite vectors (picking out a subspace which is dense)

$$V_{K\text{-finite}} \begin{array}{l} \circlearrowleft U(\mathfrak{g} = \mathrm{Lie}(G)) \\ \circlearrowleft K \end{array} \Big\rangle \text{ compatible}$$

Def: An admissible repn of G is a $(U(\mathfrak{g}), K)$ module (or equivalently $(\mathfrak{g}, K)$-mod) s.t. each irred. repn of K occurs with $< \infty$ multiplicities.

$\to$ Irred. admissible $(\mathfrak{g}, K)$-module.

F: local non-archemedean: $G(F)$.

Def: An admissible repr of $G(F)$ is one which is smooth (stabilizer are open) and $\forall$ cpt open K. (as in previous).

Automorphic Representations :

Idea : (Consider only sub quotient of very explicit representations)

$$\left\{ f : G(F) \backslash G(\mathbb{A}_F) \longrightarrow \mathbb{C} \;\middle|\; \begin{array}{l} \text{smooth / Schwartz} \\ \text{which grows slowly} \\ \text{for every place (or just} \\ \text{for one place)} \end{array} \right\}$$

Number field case :

- $Z(U(\mathfrak{g})) =$ finite
- growth condition at ∞

function field case :

- equiv. to condi. at one place.

these functions = " automorphic forms " $\circlearrowleft G(\mathbb{A}_F)$.

Now we want to think about subquotient of this.

<u>to be continued</u>

$$\left\{ \text{automorphic forms on } G(F)\backslash G(\mathbb{A}_F) \right\} \quad *$$

$$\cup\!|$$

$$\left\{ \text{cuspidal auto. forms on } '' \right\} \quad **$$

$$f : G(F)\backslash G(\mathbb{A}_F) \longrightarrow \mathbb{C}$$

$$\int_{N(F)\backslash N(\mathbb{A}_F)} f(nx)\, dn = 0$$

$\forall$ N: unipotent radical of a F-rat'l parabolic of G

(with this. then $L^2 \equiv$ bounded $\equiv$ decay very fast)

irred. auto. repr means those which appears in $*$
cuspidal repr .. in $**$.

Fact: π automorphic repr $\rightsquigarrow$ $\pi = \bigotimes'_{v \in \Sigma_v} \pi_v$

restrict product w.r.t. $(\mathcal{H}_v, K_v)$

$\mathcal{H}_v$: loc. const. measure on $G(F_v)$, v non-arch (inv. by K_v)

K_v: G/F_v is un-ram. quasi-split and split over an un-ram. ext'n of F_v

$\rightsquigarrow K_v =$ hyperspecial max'l cpt .

for almost all v, π_v will be unramified

$\Rightarrow V_v^{K_v}$ is 1-dim'l

V_v: the repr'n space

$G/F \rightsquigarrow G/\mathcal{O}_F \rightsquigarrow G(\mathcal{O}_v)$.

Basic Question:

Given admissible repr $\pi = \bigotimes'_{v \in \Sigma_v} \pi_v$

is it a (cuspidal) auto. repr'n?

eg. $X_{/F} \rightsquigarrow \{ H^i_e(X, \overline{\mathbb{Q}_\ell}) \}_\ell$ it is admissible, in general it's very easy to get admi. ones, but very hard to get cuspidal one.

π	π_v
• L-functions	local L-factors
• functional Eq'n	functional Eq'n
• constant for F.E.	local const. for F.E.

$$\prod_v \text{local} \begin{pmatrix} \text{L-factor} \\ \varepsilon\text{-factor} \end{pmatrix} = \text{global} \begin{matrix} L \\ \varepsilon \end{matrix} \text{ factors}$$

Point: can be read off from the parameters.

Recipe: $\pi_v \rightsquigarrow r \circ \varphi_v$; $L_F \xrightarrow{\rho} GL(V)$, $L_F \to {}^LG \xrightarrow{r} GL(V)$

(local) $\qquad L_F = W_F'$ or $W_F \times SU_2(\mathbb{R})$

omitting some additive character $\qquad \rho_v' = (\rho, N)$ Nilp. op. from Lie algebra.

$$L(\rho_v', s) = \det\left(Id - q_v^{-s} \Phi_v \Big|_{V_N^{I_v}} \right)^{-1}$$

fixed by inertial gp

killed by monodromy

$$\varepsilon(\rho_v', s) = \varepsilon(V)\, q^{-a(V)s}$$

• $a(V) = a(\rho) + \dim(V^I) - \dim(V_N^I)$

• $\varepsilon(V) = \varepsilon(\rho) \cdot \det\left(-\Phi \Big| (V^I / V_N^I)\right)$

local const. for ρ (Dwork, Deligne ...)

$$L(\pi, s, r) = \prod_v L(\pi_v, s, r)$$

expect: Analytic continuation

+ F.E. $L(\pi, s, r) = \varepsilon(\pi, s)\, L(\pi, 1-s, \tilde{r})$.

Properties of Representation and Parameters.

π_v	$\longleftrightarrow$	φ_v
spherical repr (unramified principal series)	<u>unramified</u>	I_v operates trivial + $N=0$
( )	<u>supercuspidal</u> (non-arch)	irred. (& $N=0$) (say for GL_n)
essentially L^2 (L^2 after modified by an central char)	(essentially) <u>discrete series</u>	$\mathrm{Im}\,\varphi_v \not\subset$ any proper F-rat'l Levi
(idea: unitarily induced from discrete series)	essentially <u>tempered</u>	essentially bounded (modified by a char)

⋮

EXAMPLES: GL_2

Easy part: $L_{F_v} = W_{F_v} \times SL_2(\mathbb{C}) \xrightarrow{\varphi_v} GL_2(\mathbb{C})$ (semi-simple)

i. irreducible and $SL_2(\mathbb{C})$ operates trivially

ii. $SL_2(\mathbb{C})$ operates irreducibly already

$\Rightarrow W_{F_v}$ acts via an abelian character iii • $\mu \oplus \nu$, $SL_2(\mathbb{C})$ operate trivially.

(i. ≥ dihedral type say.)

Back. from parameters $\longrightarrow$ repr'n.

iii. $\mu \oplus \nu$: $\mu, \nu : F_v^\times \longrightarrow \mathbb{C}^\times$ (by C.F.T.)

induction : $B : \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \xrightarrow[\mu,\nu]{} \mathbb{C}^\times$

$$\rho(\mu,\nu) = \mathrm{Ind}_B^G(\mu \oplus \nu) = \left\{ \begin{array}{c} f : G \to \mathbb{C} \\ \text{smooth} \end{array} \;\middle|\; f\left(\begin{pmatrix} u & * \\ 0 & v \end{pmatrix} \cdot x\right) = \mu(u)\cdot\nu(v) \cdot \left\| \frac{u}{v} \right\|^{1/2} \cdot f(x) \right\}$$

∞-dim'l repr of G, admissible. quasi-Haar measure $\|\rho_G\|^{1/2}$

- For non-archemedean.
  most of the time $\rho(\mu,\nu)$ is irreducible

  $\rightsquigarrow$ take $\pi(\mu,\nu) = \rho(\mu,\nu)$

  If reducible $\rightsquigarrow$ 2 irreducible subquotient

  one ∞-dim'l

  one finite dim'l $= \pi(\mu,\nu)$,

- Non-arche. $\rho(\mu,\nu)$ reducible $\Leftrightarrow$ $\mu\nu^{-1} = |\cdot|_F^{\pm 1}$

  In this case

  $\sigma(\mu,\nu) =$ up to twist by 1-dim char
  the (a) special repr'n.

ii. $SL_2(\mathbb{C})$. has a special repr $sp_2(\rho, N)$

e_0, e_1

$W : e_i \mapsto \|w\|^i e_0$, $e_0 \xrightarrow{N} e_1 \longrightarrow 0$

General fact for $G = GL_n$:

$$\det(\varphi_v) \xleftrightarrow[\text{CFT}]{} \text{central character}.$$

$\mu\nu^{-1} = |\cdot|^{\pm 1}$ unramified

$\Rightarrow \pi(\mu,\nu) = $ 1-dim'l

$\pi(\|\cdot\|^{\frac{1}{2}}, \|\cdot\|^{\frac{1}{2}}) = $ trivial repr of $GL_2(F_v)$.

For ii) only occur when non-arch.
so the only important thing is for i. 2 dihedral :

$\rightarrow$ Weil representation

K/F sep. quad.
$$\begin{tikzcd} K^\times \arrow[r, "\theta"] \arrow[d, "\wr"] & \mathbb{C}^\times \\ W_K^{ab} \arrow[ur] & \end{tikzcd}$$

auxiliary $\psi : (F_v, +) \to \mathbb{C}^\times$ non-trivial

$\rightsquigarrow \pi(\mathrm{Ind}_{W_K}^{W_F} \theta)$ constructed using the "Weil repr"

$\{ f : K^\times \to \mathbb{C} \mid f(t^{-1}x) = \theta(t) f(x) \ \forall t, x \in K^\times \}$

$\rightsquigarrow r(\theta, \psi)$, $G_1 \circlearrowleft$

$\cap|$ index 2

$G(F_v)$

$\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} f\right)(x)$

$= \psi(u) N_{K/F}(x) f(x)$

$\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f\right)(x)$

$= \underset{\varepsilon(\hat{\sigma}, s)}{\text{unitary part of}} \int_K f(y) \psi_{K/F}(x y^\sigma)\, d_\psi y$

$\pi(\mathrm{Ind}_{W_K}^{W_F} \theta) := \mathrm{Ind}_{G_1}^{G(F_v)} (r(\theta, \psi))$.

- irred. if $\theta \neq {}^\sigma\theta$ by an auto σ of order 2.

This is roughly the matching process goes.

i). called supercuspidal, $\supseteq$ dihedral

ii). called special Steinberg

"=" unless char $K = 2$.

Global Situation: $GL_2/\mathbb{Q}$ say.

- classical holomorphic modular forms
- Maaß forms (Non-holomorphic)

Representation theoretic way:

$\mathcal{F} := \{$ functions on adelic lattices in $\mathbb{C}\} =$

$$GL_2(\mathbb{Q}) \backslash GL_{\mathbb{R}}(\mathbb{C}) \times \prod_p{}' GL_2(\mathbb{A}_{\mathbb{Q},f})$$

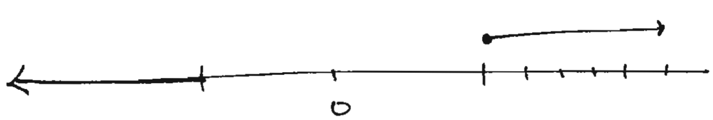
$$GL_{\mathbb{R}}(\mathbb{C}) \cong GL_2(\mathbb{R})$$

$$\mathrm{Hom}_{GL_2(\mathbb{R})}(\pi_{\mathbb{R}}, \mathcal{F}) = \begin{cases} \text{holo. modular forms.} \\ \text{Maaß forms.} \end{cases}$$

canonically, intertwinding operator evaluating at a specific generator of $\pi_{\mathbb{R}}$.

In $\mathcal{F}$, if $\pi = \pi_{\mathbb{R}} \otimes \pi_f$ ($\pi_f = \otimes' \pi_p$) occurs $\rightsquigarrow$ get holo. mod. forms / Maaß forms.

Roughly: For $\pi_{\mathbb{R}}$ = a discrete series rpr of $GL_2(\mathbb{R})$ with a specific H-C module D_{k-1} with a basis indexed by a subset of $\mathbb{Z}$



$$GL_2(\mathbb{R}) \downarrow \mathbb{C} - \mathbb{R} = \mathbb{H}^{\pm}$$

To be continued.

Prof. Ching-Li Chai. 7/30.

Lecture IV. (Final Lecture of Summer School)

parameters continued.

$$\Phi_G = \left\{ \begin{array}{l} \text{(locally) relevant "admissible"} \\ \text{L-homomorphisms } L_F \longrightarrow {}^L G \end{array} \right\} / \mathrm{Inn}(\hat{G})$$

$\{$unramified repr of $G(F)$, F local field$\}$

$\xleftrightarrow[1:1]{}$ $\{$unramified parameters in $\Phi_G\}$

(in general many to 1)

classically in the case of Galois repr'n, these are conjugacy classes of Frob. at unramified places.

ie.

$$\begin{array}{ccc} X & \subseteq & \mathcal{X} \\ {\scriptstyle \text{proper smooth}}\downarrow & & \downarrow{\scriptstyle \text{smooth}} \\ \mathrm{Spec}\, F & \hookrightarrow & \mathrm{Spec}\, \mathcal{O}_{F,S} \end{array}$$

F: number field
(or global field)

ie. Example from alg. geom. are unramified repr. by conjecture, should be automorphic forms.

unramified parameters:

$$\begin{array}{ccc} L_F & \longrightarrow & {}^L G \\ \searrow & & \nearrow \\ & W_F \twoheadrightarrow \{\mathrm{Frob}_F^{\mathbb{Z}}\} & \end{array}$$

1. Factor through W_F
2. Acts trivially for inertial

Taking a generator σ

$\{$unram. parameters$\} \leftrightarrow ({}^LG \rtimes \sigma)_{ss} / \mathrm{Inn}(\hat{G})$

$\leftrightarrow \left(\begin{array}{l} \text{expressible in terms of the maximally} \\ F\text{-split torus } T_F \text{ and its Weyl group.} \end{array} \right)$ $*$

On the other hand, have Satake homomorphism:

$$\mathcal{H} \xrightarrow{\sim} \mathbb{C}[T_d(F)/_F W] = \mathbb{C}[X_*(T_d)]^{FW}$$

$$\overset{\Psi}{f} \mapsto \left(t \mapsto \delta(t)^{1/2} \int_{u \in U(F)} f(tu)\,du = \delta(t)^{1/2} \int_{u \in U(F)} f(ut)\,du \right)$$

where U = unipotent radical of a maximal F-rational parabolic subgroup.

$\mathcal{H} = \mathcal{H}(G;K)$; K: good max. cpt. subgp. $\mathcal{H}$: Hecke algebra

T_d : (split part ? of) max torus T.

$\delta(t)$ sum of positive roots. $|\rho_G|$.

this is the dual version of $*$. So,

taking dual : ~~act~~

$* \leftrightarrow$ 1-dim'l characters $\mathcal{H} \to \mathbb{C}$.

on the other hand. (spherical functions (repr's))

$$\begin{array}{ccc} \mathcal{H} \longrightarrow \mathbb{C} & & \mathrm{Ind}_{P(F)}^{G(F)}(\chi) \quad P\text{: max. parabolic} \\ \updownarrow & & \uparrow \\ \chi : T(F) \longrightarrow \mathbb{C}^\times & & \text{irreducible "most of the time"} \end{array}$$

or $T_d(F)$

This 1-1 correspondence was solved only for local case in case $G = GL_n$. (Recently)

"Importance"

$\pi = \otimes_v \pi_v$ automorphic $\rightsquigarrow \exists S$ finite st. $v \notin S$
π_v is unramified

$\{\pi_v\}_{v \notin S} \longleftrightarrow \{\varphi_v\}_{v \notin S}$ for GL_n, there is only one way to fill in others.

"Functoriality Principle": for packets

$$\begin{tikzcd} {}^L G \arrow[rr, "\alpha"] \arrow[dr] & & {}^L H \arrow[dl] \\ & W_F & \end{tikzcd}$$

homo.

$\pi = \otimes_v \pi_v$ auto. repr of G

$\{\varphi_v\}_{v \notin S} \rightsquigarrow \{\alpha \circ \varphi_v\}_{v \notin S}$

unramified by construction

Question (Weak form): $\exists ?\ \tilde{\pi} = \otimes_v \tilde{\pi}_v$ auto. repr of H st. $\tilde{\varphi}_v = \alpha \circ \varphi_v$ for almost all places v?

An example of local Langlands classification at arch. places:

$F = \mathbb{C}$, max cpt gp has smaller rank $(= \frac{1}{2}\,\mathrm{rk})$

H-C. Thm $\Rightarrow$ no discrete series.

each $\varphi_v \longleftrightarrow$ one repr. a parameter is just

$\mathbb{C}^\times \longrightarrow \hat{G}$

Remark: characters of $\mathbb{C}^\times$ are of the form

$z \mapsto z^\lambda \bar{z}^\mu$, $\lambda - \mu \in \mathbb{Z}$. In fact

$$\begin{tikzcd} \mathbb{C}^\times \arrow[r] \arrow[dr] & \hat{G} \\ & \hat{T} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$\longleftrightarrow z \mapsto z^\lambda \bar{z}^\mu$, $\lambda, \mu \in X^*(T) \otimes \mathbb{C}$
$\lambda - \mu \in X^*(T)$

$\hat{T}$ max. torus

let P_1 = parabolic subgroup of G
(standard) containing T st.

$|\hat{\varphi}||_{\mathfrak{a}_{P_1}} \in$ open Weyl chamber.

get a parameter $\varphi_1 : \mathbb{C}^\times \longrightarrow L_1$ (Levi for P_1)
this corr. to a tempered (repr) of L_1
parameter
then use induction procedure from L_1 to ...
This is Historically how Langlands begin his work.

Global Consideration:

Shimura Varieties

$G/\mathbb{Q}$ connected reductive $/\mathbb{Q}$

$\mathbb{C}^\times = \underset{\sim}{\mathbb{C}}^\times(\mathbb{R})$
as alg. gp over $\mathbb{R}$
consider homomorphism: $h : \underset{\sim}{\mathbb{C}}^\times \underset{/\mathbb{R}}{\longrightarrow} G$ st.

$Ad(h) : \underset{\sim}{\mathbb{C}}^\times \longrightarrow Aut(\mathfrak{g}_{\mathbb{R}})$. require a real Hodge str.

condition: Hodge type $(-1,1)$, $(0,0)$, $(1,-1)$.
(and some other conditions)
$G(\mathbb{R})$-conjugacy class X of h "is" a finite union of bounded symmetric domains
(not all gp have this property)
data (G, X):

$$Sh(G,X) := \text{"}\varprojlim_K\text{"} \; G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K$$

open cpt subgp of $G(\mathbb{A}_f)$.

has canonical structure as alg. v. / $\mathbb{C}$

Thm (...) $Sh(G, X)$ has a natural structure as a (pro) algebraic variety over a number field (reflex field) $E = E(G, X) \hookrightarrow \mathbb{C}$

$Gal(\bar{\mathbb{Q}}/E)$ = the fixer of the $G/\mathbb{C}$ – conjugacy class of μ_h, any $h \in X$.

Example: $G = GL_2$, $X = \mathbb{C} - \mathbb{R}$, we get projective system of modular curves.

$Sh(G, X) \circlearrowleft G(\mathbb{A}_f)$ almost like a homog. space.

but never has an action of alg. gp

only finite Adelic pts $G(\mathbb{A}_f) = \prod_p' \mathbb{Q}_p$.

(does not preserve the finite level, only the whole system is preserved).

$Sh(G, X) \circlearrowleft G(\mathbb{A}_f)$ gives rise to Hecke corr.

$/E = E(G, X)$ on each $Sh(G, X)/K$.

$\rightsquigarrow$ Can use these symmetry to study these alg. v.

For number theory: Pick $H^\cdot$ any coh. theory

$$\circlearrowright H^\cdot(Sh(G, X)) := \varinjlim H(Sh(G, X)/K)$$

$G(\mathbb{A}_f) \leftarrow$ big gp action

$\to$ decomposition according to repr of $G(\mathbb{A}_f)$.

$$= \bigoplus_{\pi_f} W_{\pi_f} \otimes V_{\pi_f}$$

finite dim — repr. space of π_f.

For instance, if $H^\bullet = H^\bullet(\ , \mathbb{Q}_\ell)$
then have $\mathrm{Gal}(\bar{\mathbb{Q}}/E) \curvearrowright W_{\pi_f}$ and
$G(\mathbb{A}_f) \curvearrowright V_{\pi_f}$.

Hodge structures appear in W_{π_f}.

How to get information of the cohomology?

Look at arch places: try to "compute"

take $IH^\bullet(Sh(G,X)_{/\mathbb{C}}, V_\tau)$ for $\tau : G \longrightarrow GL(V_\tau)$

cpx version

$\uparrow S$ thm of Looijenga, Saper-Stern

$H^\bullet_{(2)}(Sh(G,X)(\mathbb{C}), V_\tau)$

this can be done for such inductive system because those maps are (finite) étale.

essential piece of L^2-cochains:

$$\left(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}) \otimes L^2(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes V_\tau\right)^{K_\infty}$$

take cohomology $= H^\bullet((\mathfrak{g}, K_\infty), V_\tau \otimes L^2(G(\mathbb{Q})\backslash G(\mathbb{A})))$

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})) = \bigoplus_\pi m_\pi V_\pi$$

the question reduces to compute

$$H^\bullet(\mathfrak{g}, K ; V_\tau \otimes \pi_{\mathbb{R}} \otimes \pi_f) \quad \text{can taken out}$$

to compute $H^\bullet(\mathfrak{g}, K ; V_\tau \otimes \pi_{\mathbb{R}})$ is purely Hodge-th:

(Kuga) $\square = \tau(C) - \pi(C)$, C Casimir op.

Laplacian $= dd^* + d^*d$

so $H^\bullet(\cdots) = 0$ unless $\square = 0$. but $\square = 0$ then get

$$H^\bullet(\cdots) = \left(\Lambda^\bullet(\mathfrak{g}/K) \otimes V_\tau \otimes L^2(\cdots)\right)^K.$$

Importance of these:

Get Galois Repr that we know something about. (via automorphic repr) eg. Eichler-Shimura relations through Hecke relations.

"secret thoughts":

try to $\Big\{$ search for Galois repr among these (Taniyama-Shimura ...)

Build up (approximate) Galois repr to get a "given" one. (or just a piece)

- There is a recipe for computing the IH / Lie alg. coh. of a real packet.

$$\begin{array}{l} -\mu_h \rightsquigarrow {}^L\hat{G} \xrightarrow{\text{"}-\mu\text{"}} \mathrm{Aut}(\tau_\mu) \\ \psi \uparrow \leftarrow \text{using } \tau \nearrow \\ W_{\mathbb{R}}^\times \times SL(2,\mathbb{C}) \end{array}$$

both can be read off combinatorially.

$\nearrow$ will give the all like Hodge, Lefschetz structures.

END