Upper bound on $k$-tuple domination numbers of graphs

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Abstract

In a graph $G$, a vertex is said to dominate itself and all vertices adjacent to it. For a positive integer $k$, the $k$-tuple domination number $\gamma_{\times k}(G)$ of $G$ is the minimum size of a subset $D$ of $V(G)$ such that every vertex in $G$ is dominated by at least $k$ vertices in $D$. To generalize/improve known upper bounds for the $k$-tuple domination number, this paper establishes that for any positive integer $k$ and any graph $G$ of $n$ vertices and minimum degree $\delta$:

$$\gamma_{\times k}(G) \leq \frac{\ln(\delta - k + 2) + \ln \tilde{d}_{k-1} + 1}{\delta - k + 2} n,$$

where $\tilde{d}_m = \frac{1}{n} \sum_{i=1}^{n} \binom{d_i + 1}{m}$ with $d_i$ the degree of the $i$th vertex of $G$.

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The concept of domination in graph theory is a natural model for many location problems in operations research. In a graph $G$, a closed neighbor of a vertex $v$ is either $v$ itself or a vertex adjacent to $v$. A vertex $v$ is said to dominate all of its closed neighbors. A

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A dominating set of $G$ is a subset $D$ of $V(G)$ such that every vertex in $V(G)$ is dominated by at least one vertex in $D$. The domination number $\gamma(G)$ of $G$ is the minimum cardinality of a dominating set of $G$. Domination and its variations have been extensively studied in the literature, see [3, 8, 9].

Among the variations of domination, the $k$-tuple domination was introduced in [6], also see [8, page 189]. For a fixed positive integer $k$, a $k$-tuple dominating set of a graph $G$ is a subset $D$ of $V(G)$ such that every vertex in $V(G)$ is dominated by at least $k$ vertices of $D$. The $k$-tuple domination number $\gamma_{\times k}(G)$ of $G$ is the minimum cardinality of a $k$-tuple dominating set of $G$. For the case when $G$ has no $k$-tuple dominating set, $\gamma_{\times k}(G)$ is defined to be $\infty$. Notice that 1-tuple domination is the usual domination, 2-tuple domination was called double domination in [6], and 3-tuple domination was called triple domination in [15]. While determining the exact value of $\gamma_{\times k}(G)$ for a graph $G$ is not easy, many studies focus on its upper bounds. For other results on $k$-tuple domination, please see [7, 10, 11, 12, 13].

Suppose $G$ is a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ where vertex $v_i$ is of degree $d_i$. For any integer $m$, define

$$\hat{d}_m = \frac{1}{n} \sum_{i=1}^{n} \binom{d_i}{m}$$

and

$$\tilde{d}_m = \hat{d}_m + \hat{d}_{m-1} = \frac{1}{n} \sum_{i=1}^{n} \binom{d_i + 1}{m}.$$ 

Notice that $\hat{d}_{-1} = 0, \hat{d}_0 = \hat{d}_0 = 1$ and $\hat{d}_1$ is the average degree of $G$. We also use $\delta$ to denote the minimum degree of $G$.

Alon and Spencer [1], Arnautov [2] and Payan [14] independently proved the following fundamental result:

$$\gamma_{\times 1}(G) \leq \frac{\ln(\delta + 1) + 1}{\delta + 1} n = \frac{\ln(\delta + 1) + \ln \hat{d}_0 + 1}{\delta + 1} n. \quad (1)$$

Harant and Henning [5] gave an upper bound for the double domination number:

$$\gamma_{\times 2}(G) \leq \frac{\ln \delta + \ln(\hat{d}_1 + 1) + 1}{\delta} n = \frac{\ln \delta + \ln \tilde{d}_1 + 1}{\delta} n. \quad (2)$$

Rautenbach and Volkmann [15] established an upper bound for the triple domination...
number:
\[ \gamma_{\times 3}(G) \leq \frac{\ln(\delta - 1) + \ln(\hat{d}_2 + \hat{d}_1) + 1}{\delta - 1} n = \frac{\ln(\delta - 1) + \ln \hat{d}_2 + 1}{\delta - 1} n. \] (3)

Trying to generalize the result, Gagarin and Zverovich [4] proved that for any graph \( G \) with \( 3 \leq k \leq \delta + 1 \),
\[ \gamma_{\times k}(G) \leq \frac{\ln(\delta - k + 2) + \ln(\sum_{m=1}^{k-1} (k - m)\hat{d}_m - \hat{d}_1) + 1}{\delta - k + 2} n. \]

The purpose of this paper is to establish a better upper bound, which also unifies formulas (1), (2) and (3). We employ the probabilistic approach used in [1, 2, 4, 5, 14, 15]. In particular, the way of randomly generating a \( k \)-tuple dominating set is the same as in [4], while the derivation of the upper bound is simpler and more efficient.

**Theorem 1** For any graph \( G \) of minimum degree \( \delta \) with \( 1 \leq k \leq \delta + 1 \),
\[ \gamma_{\times k}(G) \leq \frac{\ln(\delta - k + 2) + \ln(\sum_{m=1}^{k-1} (k - m)\hat{d}_m - \hat{d}_1) + 1}{\delta - k + 2} n. \]

**Proof.** Form a set \( A \) by an independent choice of vertices of \( G \), where each vertex is selected with probability \( p \) and \( 0 \leq p \leq 1 \). For \( 0 \leq m \leq k - 1 \), denote \( B_m \) the set of vertices dominated by exactly \( m \) vertices of \( A \). For each \( B_m \), let \( B'_m \) be a set containing \( k - m \) closed neighbors not in \( A \) for every vertex in \( B_m \). Notice that such closed neighbors exist since \( k \leq \delta + 1 \). It is clear that \( |B'_m| \leq (k - m)|B_m| \). Consider the set
\[ D = A \cup \left( \bigcup_{m=0}^{k-1} B'_m \right). \]

Notice that \( D \) is a \( k \)-tuple domination set, since a vertex either is dominated by at least \( k \) vertices in \( A \) or is in some \( B_m \) with \( 0 \leq m \leq k - 1 \). For the later case, the vertex is dominated by at least \( k \) vertices in \( A \cup B'_m \). The expectation of \( |D| \) is
\[ E(|D|) \leq E(|A|) + \sum_{m=0}^{k-1} E(|B'_m|) \leq E(|A|) + \sum_{m=0}^{k-1} (k - m)E(|B_m|). \]
Notice that we have
\[ E(|A|) = \sum_{i=1}^{n} P(v_i \in A) = pn. \]
Also,

\[
E(|B_m|) = \sum_{i=1}^{n} P(v_i \in B_m) = \sum_{i=1}^{n} \binom{d_i + 1}{m} p^m (1 - p)^{d_i + 1 - m} \\
\leq p^m (1 - p)^{\delta + 1 - m} \sum_{i=1}^{n} \binom{d_i + 1}{m} = p^m (1 - p)^{\delta + 1 - m} \tilde{d}_m n,
\]

Then,

\[
E(|D|)/n \leq p + \sum_{m=0}^{k-1} (k - m) p^m (1 - p)^{\delta + 1 - m} \tilde{d}_m \\
= p + (1 - p)^{\delta - k + 2} \sum_{m=0}^{k-1} (k - m) p^m (1 - p)^{k - 1 - m} \tilde{d}_m.
\]

For \(0 \leq m \leq k - 1\), we have

\[
\binom{k - 1}{m} \binom{d_i + 1}{k - 1} = \binom{d_i + 1}{m} \binom{d_i + 1 - m}{d_i + 2 - k} \geq \binom{d_i + 1}{m} (d_i + 1 - m) \geq \binom{d_i + 1}{m} (k - m).
\]

Notice that the last two inequalities follow from the fact that \(d_i + 1 \geq \delta + 1 \geq k\). Summing up the inequality above for all \(i\) gives \((k - m) \tilde{d}_m \leq \binom{k - 1}{m} \tilde{d}_{k-1}\). This, together with the inequality \((1 - p)^{\delta - k + 2} \leq e^{-(\delta - k + 2)p}\), gives that

\[
E(|D|)/n \leq p + e^{-(\delta - k + 2)p} \tilde{d}_{k-1} \sum_{m=0}^{k-1} \binom{k - 1}{m} p^m (1 - p)^{k - 1 - m} = p + e^{-(\delta - k + 2)p} \tilde{d}_{k-1}.
\]

Choosing \(p = \frac{\ln(\delta - k + 2) + \ln \tilde{d}_{k-1}}{\delta - k + 2}\) gives that \(E(|D|)\) and hence \(\gamma_{\times k}(G)\) is upper bounded by \(\frac{\ln(\delta - k + 2) + \ln \tilde{d}_{k-1} + 1}{\delta - k + 2}\) as desired. \(\square\)

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**References**


