An Introduction to Wavelets

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Computer Impact to Science and Technology

- Computer impact to science and technology is tremendous
  - Computational Physics, Chemistry, Biology
  - Computational Fluid Dynamics, Mechanics, Electric Magnetism, Engineering
  - Computational Finance
  - Computational Geometry, Algebra, PDEs, etc.
- Computer becomes an important tool for scientists and engineers
  - cheap
  - safe
  - efficient

The Call For Fast Algorithms

- Current computer technology
  - 100 Mega Flops PCs are cheap (10^8 floating point operators per second)
  - 0.5 - 1 Giga Flops (10^9) personal workstations are available
  - 1 Tera Flops (10^{12}) super computers are available
- Smart fast algorithms are needed
  - 3-d combustion chamber:
    - ∗ number of unknowns: $N = 10^3 \times 10^3 \times 10^3 = 10^9$
    - ∗ a N^3 complexity algorithm (e.g. Gaussian elimination) takes 10^15 seconds (or approximately $10^7 - 10^8$ years)
  - weather and climate prediction, biosphere modeling

    - number of knowns: $N = 400 \times 400 \times 60 \approx 10^7$ (i.e. grid distance is 100 km)
    - each cell needs 10^2 calculation for subgrid parametrization
    - Using spectral method needs $400^2 \times 400 \times 60 \approx 10^{11}$ for Legendre transform
    - iteration times step size: 0.01 day
    - operation counts per day: several hundreds 10^12 flops
    - Tera flops machines are needed
    - Fast algorithms are needed
- protein simulation
    - minimal size: 1500 atoms
    - find global minimum of a potential defined in R^{3000}
- Solid state physics
- Signal, image processing, pattern recognition (environmental images, medical images, entertainmental images, geological signals)
- Nondestructive detection: geological, industrial or medical applications
- Finance

How do we model real world problems

- Partial Differential Equations
- Integral Equations
- Optimization problems
- Stochastic models
- Statistical models
- Many other models

Key steps for fast algorithms

- Efficient representation of functions and operators
  - Fourier methods
  - spline methods
  - wavelet methods
  - adaptive mesh refinement
- Efficient operations
  - fast Fourier transform
  - fast matrix-vector multiplication
- Fast convergence for iterative approximation
  - multi-grid methods
  - wavelet methods

Efficient representation for functions and operators

- The class of functions we want to deal with
  - $L^2(R)$ functions:

$$L^2(R) = \{u : R \to R \mid \int u(x)^2 \, dx < \infty\}$$

$$\|u\|_{L^2} \equiv \left(\int u(x)^2 \, dx \right)^{1/2}$$

  - Sobolev spaces

$$H^m(R) = \{u : R \to R \mid u, u', \cdots, u^{(m)} \in L^2\}$$

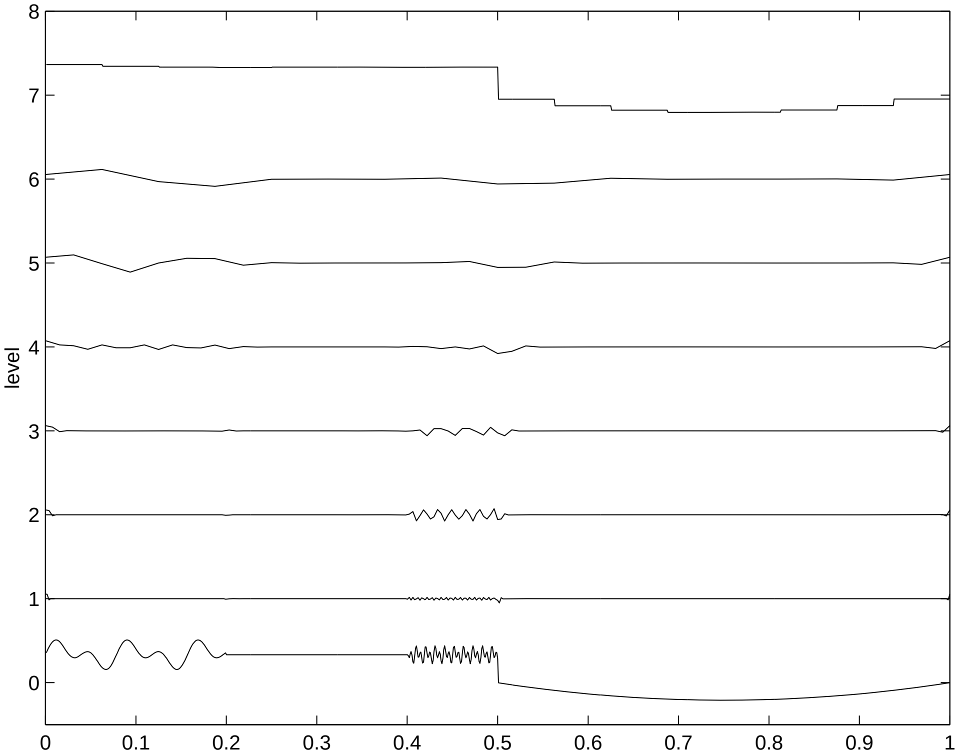
$$\|u\|_{H^m} = \|u\|_{L^2} + \cdots + \|u^{(m)}\|_{L^2}$$

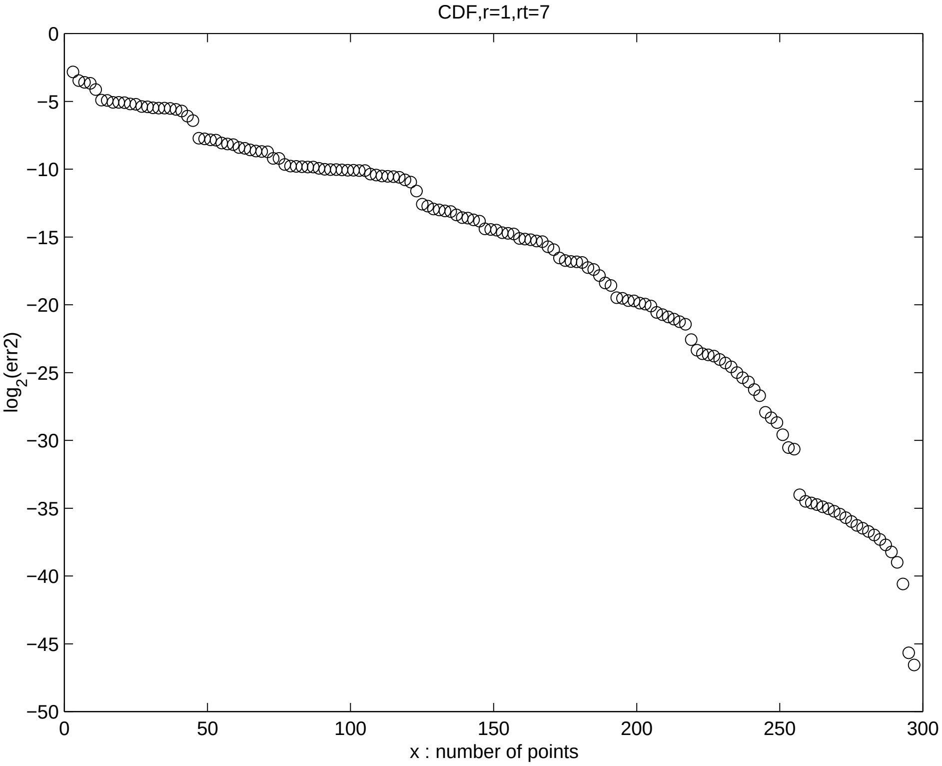
  - Piecewise smooth functions
  - Besov spaces

- The operators we want to handle
  - differential operators
  - integral operators
  - pseudo-differential operators

Examples

- A function

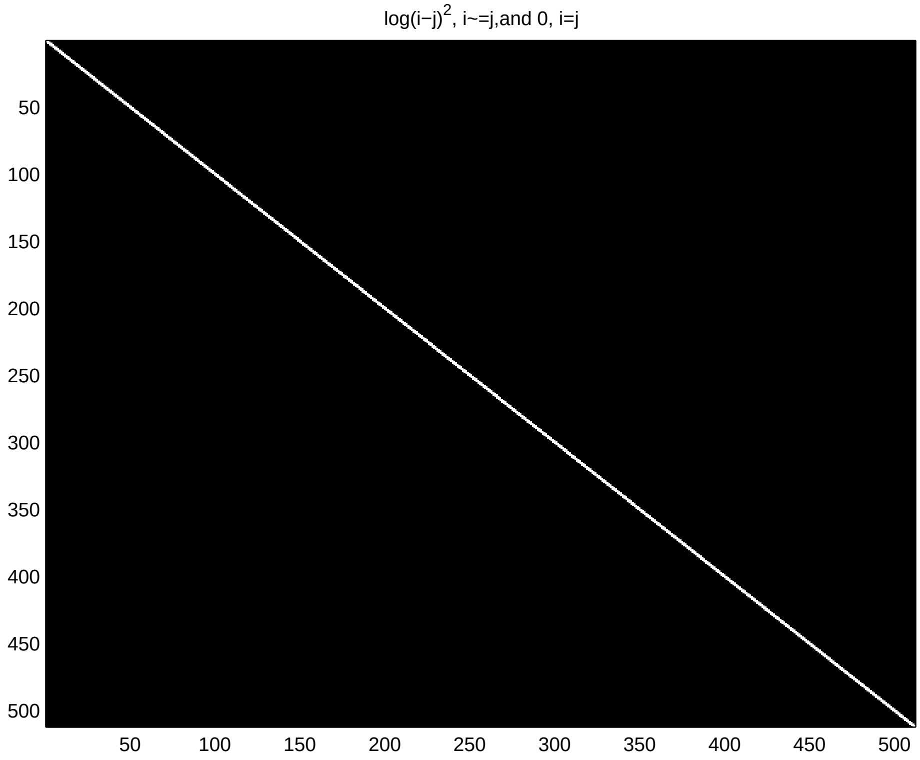


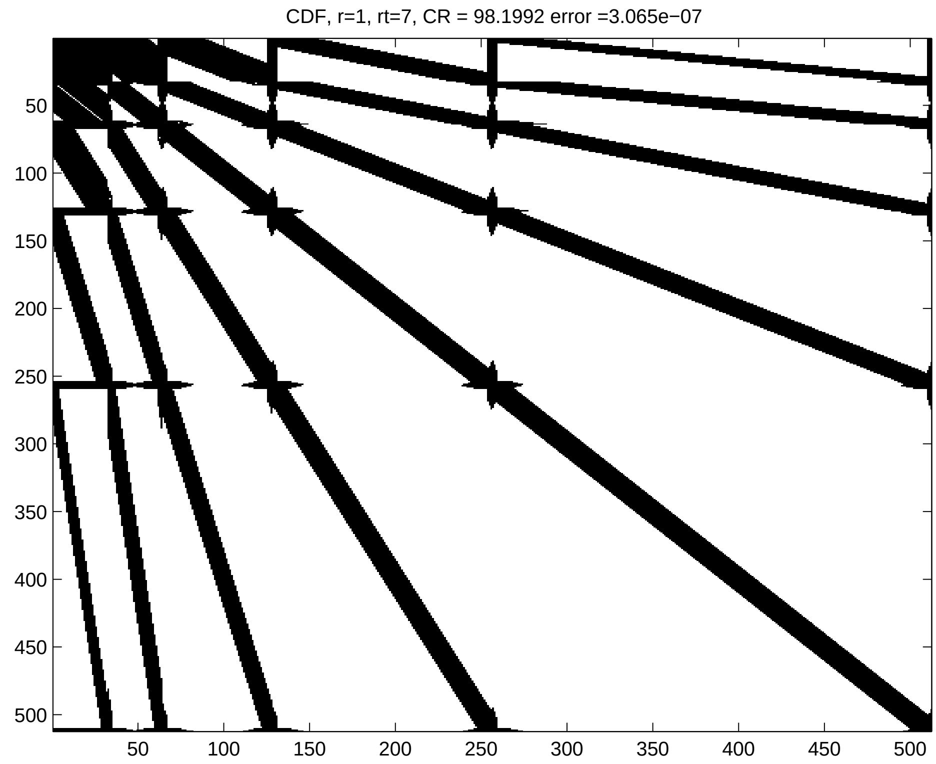


$NPT = 2^{10} = 1024$. N_2: number of data needed to have $\|u - \tilde{u}\|_{L^2} \leq \epsilon$

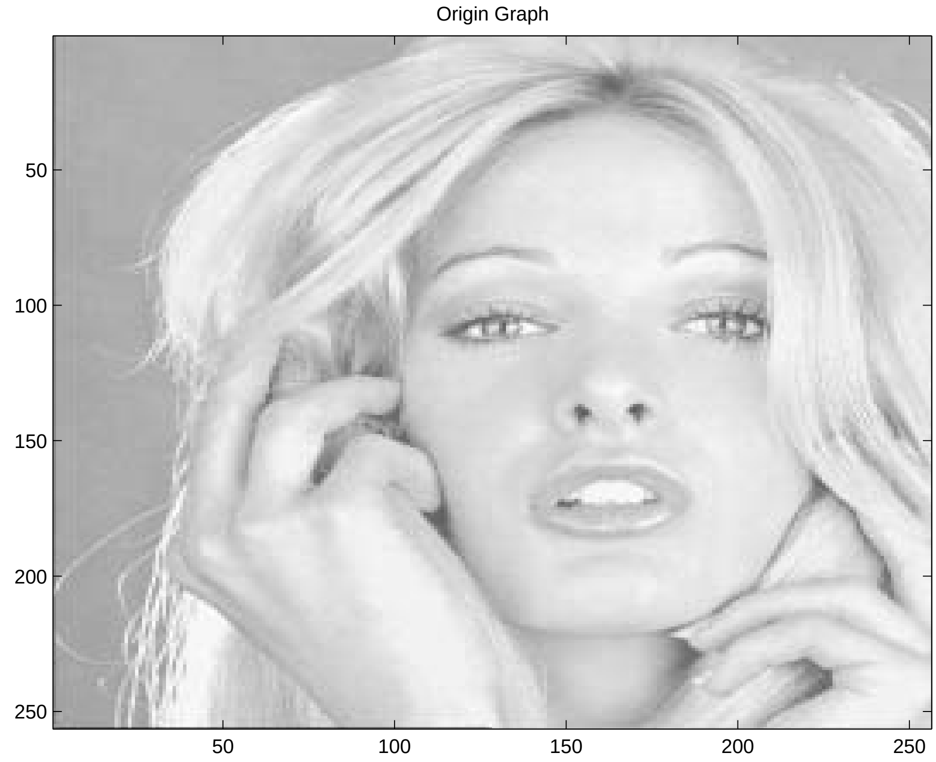
- An integral operator

$$G(i,j) = \begin{cases} \log|i-j|^2 & \text{for } i \neq j \\ 0 & \text{for } i = j \end{cases}$$

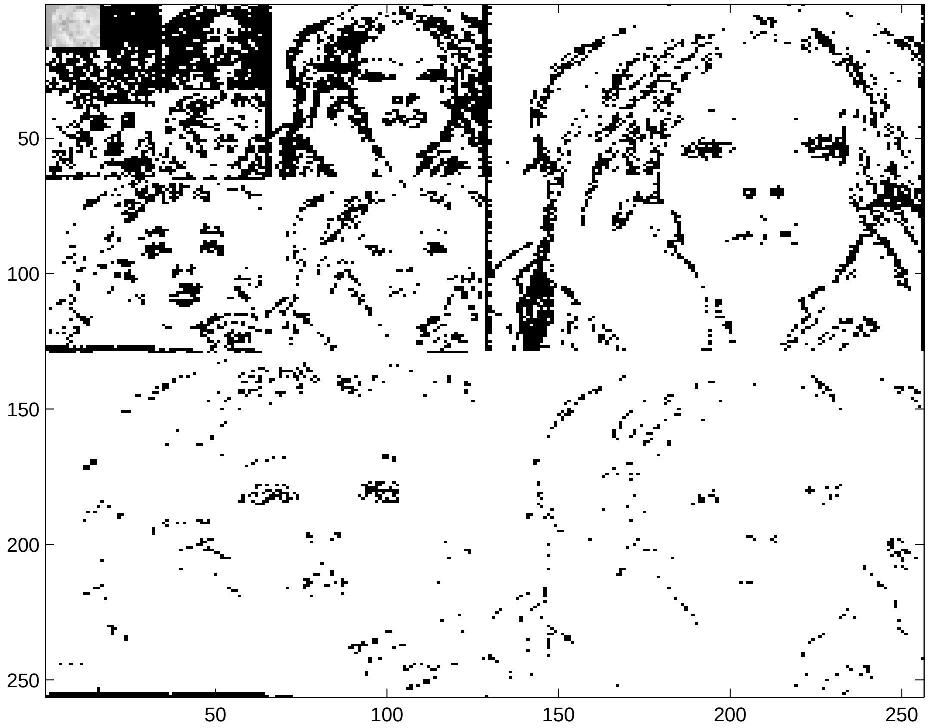




- An image compression




Reconstruction Graph


CDF,r=1,rt=7,NPT=256*256, NZ=8158, CR=8.0333

Preliminaries, Notations

- **$L^2(R)$ as a Hilbert space**
  - Inner product
    $$(u, v) := \int_{-\infty}^{\infty} u(x)\overline{v(x)}\, dx$$
  - Cauchy-Schwartz: $|(u, v)| \leq \|u\|\|v\|$
  - Angle $\cos\theta := (u, v)/(\|u\| \cdot \|v\|)$
  - Completeness (allowing taking limits)
- **ℓ^2 as a Hilbert space**
  - $\ell^2 := \{(c_k)_{k\in Z} \mid \sum |c_k|^{<}\infty\}$
  - Inner product
    $$(c, d) := \sum_k c_k\overline{d_k},\ \|c\|^2 := (c, c)$$
  - Cauchy-Schwartz: $|(c, d)| \leq \|c\|\|d\|$
  - Angle $\cos\theta := (c, d)/(\|c\| \cdot \|d\|)$
  - Completeness (allowing taking limits)
- **Operator norm**
  - Let $A : X \to Y$ be a linear operator, X, Y are Hilbert spaces
  - the operator norm $\|A\|$ is defined by
    $$\|A\| := \sup_{x\in X} \|Ax\|/\|x\|$$
  - the operator norm satisfies
    $$\|Ax\| \leq \|A\| \cdot \|x\|$$

A Simple Example: The Box Function and Haar Basis

- Approximate $u \in L^2(R)$ by box functions
- Approximate $u \in L^2(R)$ by Haar basis

Approximation by box functions

- Let
  $$\phi(x) = 1_{[0,1)} = \begin{cases} 1 & x \in [0,1) \\ 0 & \text{otherwise} \end{cases}$$
  be the box function
- Let
  $$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k),$$
  so that
  $$\int |\phi_{j,k}(x)|^2 \, dx = 1$$
- Given $u \in L^2(R)$ and a resolution level j, we approximate u by $Q_j u$ defined by
  $$\begin{aligned} Q_j u(x) \quad &:= \quad \sum_k (u, \phi_{j,k})\phi_{j,k} \\ &= \quad \bar{u}_{j,k} \quad \text{for } 2^{-j}k \le x < 2^{-j}(k+1) \end{aligned}$$
  where $\bar{u}_{j,k} = 2^{j/2}(u, \phi_{j,k})$ is the cell average of u over the interval $[2^{-j}k, 2^{-j}(k+1))$.
- Let $V_j = \text{span}\{\phi_{j,k}\}_{k \in Z}$, then
  - $V_{j-1} \subset V_j$
  - Q_j is a projection onto V_j (i.e. $Q_j^2 = Q_j$).
- Stability issue:
  - Orthogonal basis property of $\{\phi_{j,k}\}_{k \in Z}$ in V_j:
    $$\|\sum_k c_k \phi_{j,k}\|^2 = \sum_k |c_k|^2$$
  - Uniform boundedness of $\|Q_j\|$
- Approximation issue:
  - A function $u \in L^2(R)$ can be approximated by $Q_j u$:
    $$\|u - Q_j u\|_{L^2} \to 0 \quad \text{as } j \to 0$$
  - **Theorem** (Homework): If $u \in H^1(R)$, then
    $$\|u - Q_j u\|_{L^2} = O(2^{-j})$$

Approximation by Haar Basis

- Change of basis in V_j:
  - Let $\phi = 1_{[0,1)}$, $\phi_{j,k} = 2^{j/2}\phi(2^j x - k)$ and $V_j = \text{ span } \{\phi_{j,k} \mid k \in Z\}$. Then

    $$\phi_{j-1,k} = \frac{1}{\sqrt{2}}\left(\phi_{j,2k} + \phi_{j,2k+1}\right)$$

  - Define Haar function ψ by

    $$\begin{aligned} \psi(x) &:= 1_{[0,1)}(2x) - 1_{[0,1)}(2x-1) \\ &= \begin{cases} 1 & x \in [0,1/2) \\ -1 & x \in [1/2,1) \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

    $\psi_{j,k} := 2^{j/2}\psi(2^j x - k)$ and $W_j = \text{ span } \{\psi_{j,k} \mid k \in Z\}$. Then

    $$\psi_{j-1,k} = \frac{1}{\sqrt{2}}\left(\phi_{j,2k} - \phi_{j,2k+1}\right)$$

  - $\{\psi_{j-1,i}\}_{i \in Z}$ is an orthonormal basis in W_j
  - $V_{j-1} \perp W_{j-1}$
  - $V_j = V_{j-1} \bigoplus W_{j-1}$
  - The change-of-basis:

    $$\{\phi_{j,k}\}_{k \in Z} \mapsto \{\phi_{j-1,k}, \psi_{j-1,k}\}_{k \in Z}$$

    is orthogonal
  - The projection Q_j can be written as

    $$\begin{aligned} Q_j u &= \sum_k (u, \phi_{j,k})\phi_{j,k} \\ &= \sum_k (u, \phi_{j-1,k})\phi_{j-1,k} + \sum_k (u, \psi_{j-1,k})\psi_{j-1,k} \\ &= Q_{j-1}u + (Q_j - Q_{j-1})u \end{aligned}$$

- Discrete wavelet transform
  - Let

    $$\begin{aligned} c_{j,k} &:= (u, \phi_{j,k}) \quad \text{(local averaging)} \\ d_{j,k} &:= (u, \psi_{j,k}) \quad \text{(local differencing)} \end{aligned}$$

    Then (decomposition)

    $$\begin{aligned} c_{j-1,k} &= \frac{1}{\sqrt{2}} \left(c_{j,2k} + c_{j,2k+1} \right) \\ d_{j-1,k} &= \frac{1}{\sqrt{2}} \left(c_{j,2k} - c_{j,2k+1} \right) \end{aligned}$$

  - Perform this transform recursively, we obtain the transform

    $$T_j : (c_{j,k})_{k \in Z} \mapsto (c_{0,k}, d_{0,k}, d_{1,k}, \cdots, d_{j-1,k})_{k \in Z}$$

    which is called a discrete wavelet transform
  - The inverse transform T_j^{-1} is obtained recursively by (reconstruction)

    $$\begin{aligned} c_{j,2k} &= \sqrt{2} \left(c_{j-1,k} + d_{j-1,k} \right) \\ c_{j,2k+1} &= \sqrt{2} \left(c_{j-1,k} - d_{j-1,k} \right) \end{aligned}$$

  - The transformation T_j is the representation of the following change-of-basis in V_j, namely, $\{\phi_{j,k}\}_{k \in Z} \mapsto \{\phi_{0,k}, \psi_{0,k}, \psi_{j-1,k}\}_{k \in Z}$.

- Advantages
  - Fast transform: the transform T and T^{-1} are of O(N) steps, where $N = 2^j$.

    $$2^{j-1} + 2^{j-2} + \cdots + 2^0 = 2^j$$

  - Stability: T_j is orthogonal, hence $\|T_j\| = 1$ for all j.
  - Approximation:
    - For any $u \in L^2$,

      $$\begin{aligned} u &= \lim_{j\to\infty} Q_j u \\ &= \lim_{j\to\infty} \sum_{\ell=-\infty}^{j} (Q_\ell - Q_{\ell-1})u \\ &= \sum_{j=-\infty}^{\infty} \sum_{k\in Z} (u, \psi_{j,k})\psi_{j,k} \end{aligned}$$

    - In other word,

      $$L^2(R) = \bigoplus_{j=-\infty}^{\infty} W_j$$

      and $\{\psi_{j,k}\}_{j,k\in Z}$ are orthonormal basis in $L^2(R)$.
  - Efficiency:
    - If u is smooth near $x_{j,k} := 2^{-j}k$, then

      $$d_{j,k} \approx 2^{-j/2} u'(x_{j,k}) 2^{-j}$$

    - More efficient, namely, many $d_{j,k}$ can be neglected.

- Remarks:
  - The concept of multi-resolution analysis
    1. Nested property: $\cdots \subset V_{-1} \subset V_0 \subset V_1 \cdots$
    2. $\bigcap_{j\in Z} V_j = \{0\}$, $\overline{\bigcup_{j\in Z} V_j} = L^2(R)$
    3. dilation invariant: $u \in V_0$ if and only if $u(2^j \cdot) \in V_j$
    4. translation invariant: $u \in V_0$ if and only if $u(\cdot - k) \in V_0$ for all $k \in Z$.
  - The projection and multi-resolution decomposition
    1. Q_j is a projection onto V_j
    2. Require $\|Q_j\|$ are uniformly bounded
    3. decomposition: let $W_{j-1} = (Q_j - Q_{j-1})L^2(R)$, then $V_j = V_{j-1} + W_{j-1}$. Hence

$$L^2(R) = \bigoplus_{j=-\infty}^{\infty} W_j$$

  - Basis in V_j and W_j.
    1. V_j is generated by $\{\phi_{j,k}\}_{k\in Z}$ for some ϕ
    2. W_j is generated by $\{\psi_{j,k}\}_{k\in Z}$ for some ψ
    3. orthonormal basis property:

$$\begin{aligned} \sum_k |c_{j,k}|^2 &= \|\sum_k c_{j,k}\phi_{j,k}\|^2 \\ \sum_k |d_{j,k}|^2 &= \|\sum_k d_{j,k}\psi_{j,k}\|^2 \end{aligned}$$

  - Multi-resolution Basis in $L^2(R)$

$$\{\psi_{j,k}\}_{j,k\in Z}$$

    form an orthogonal, multi-resolution basis in $L^2(R)$.

Approximation of functions

Three classes of approximation methods

- by functions which are localized in physical space
  - by splines (piecewise polynomials) or finite elements
    - piecewise linear functions
    - piecewise polynomial functions
  - by Lagrange refinable functions (fundamentals)
- by functions which are localized in frequency space
  - Fourier methods (trigonometric orthogonal polynomials)
  - by orthogonal polynomials
    - Legendre polynomials for $C[-1,1]$
    - Chebyshev polynomials for $C[-1,1], w(x) = 1/\sqrt{1-x^2}$
    - Laguerre polynomials for $C[0,\infty)$, $w(x) = e^{-x}$.
    - Hermite polynomials for $L^2(-\infty,\infty), w(x) = e^{-x^2/2}dx$
- by functions which are "localized" in physical and frequency domains
  - wavelets
  - Window Fourier
  - spectral elements (h-p version finite elements)

Brief History of Approximation Theory

- Approximation by polynomials
  - Newton, Lagrange, Hermite: divided difference, interpolation, Taylor expansion
  - Weistrass: all continuous functions on closed interval can be approximated by polynomials
  - (early 20 century) best approximation theory in L^2, in Sup norm: Find the distance of f and the space spanned by polynomials of order d. Jackson, Chebyshev, Kolmogorov
- Approximation by orthogonal polynomials
  - Fourier (1807) Fourier expansion (approximation of functions by trigonometric functions)
  - (1800-1900) Approximation of functions by orthogonal polynomials: Hermite, Legendre, Chebyshev, etc. (closely related to PDEs)
- Approximation by Splines (piecewise polynomials)
  - 1940-1980 Spline theory on real line (Schoenberg, de Boor)
  - 1980-1990 Box spline (for high dimension) (de Boor, Dahmen, Michelli)
- Approximation by finite elements
  - 1930: Courant (by piecewise linear nodal function)
  - 1950: mechanical engineer
  - 1960-80: mathematical theory (conforming, nonconforming)
- Approximation by Wavelets
  - Mathematical side: (see Meyer's book)
    - 1910 Haar

    - 1930s Littlewood, Paley, Lusin
    - 1960-1980: atomic decomposition, Calderòn (1964)
    - 1982 Strömberg's wavelets
    - 1986: Meyer wavelet
    - 1987: Daubechies' orthogonal wavelet
    - 1989: Mallat's multi-resolution analysis
    - 1989: Biorthogonal wavelet
    - 1991-1998: merge with spline community (Dahmen, Michelli)
  - Signal processing
    - 1977: Quadratic Mirror Filter (Estaban, Galland)
    - 1983-1984: Grossmann, Morlet
  - Quantum mechanics:
    - Coherent state Aslaksen (1968), Klauder, Paul (1985)

- Reference: [Mey1],[Daub1]

Approximation of Functions by Splines

- piecewise linear approximation
- B-spline approximation
- The projection operator Q_j

Piecewise linear approximation

- Let $\phi = 1_{[0,1)} * 1_{[0,1)}$ be the hat function, where

  $$u * v := \int u(x-y)v(y)\,dy$$

  (Homework): Prove that $u * v = v * u$, $(u * v)' = (u') * v = u * (v')$.

- Let $V_j = \text{span}\{\phi_{j,k} := 2^{j/2}\phi(2^j \cdot -k)\}_{k\in Z}$, then

  $V_j = \{$functions in C^0 and are linear on each $[2^{-j}k, 2^{-j}(k+1))\}$

- Approximate function $u \in L^2(R)$ by a projection $Q_j u$, where Q_j can be chosen for example by
  - orthogonal projection: $(u - Q_j u) \perp V_j$.
  - nodal projection

    $$\begin{aligned} Q_j u &= \sum_k 2^{-j/2} u(x_{j,k})\phi_{j,k} \\ &= \sum_k (u, 2^{-j,k}\delta(x - x_{j,k})\phi_{j,k} \end{aligned}$$

    where $x_{j,k} := 2^{-j}k$, $\delta(x - x_{j,k})$ is the delta function centered at $x_{j,k}$

- Stability issue:
  - Riesz basis property (Homework): There exist $A > 0$, $B < \infty$ such that for any $u = \sum_k c_k\phi_{j,k} \in V_j$ we have

    $$A\sum_k |c_k|^2 \leq \|\sum_k c_k\phi_{j,k}\|^2 \leq B\sum_k |c_k|^2$$

    with A, B being independent of j.
  - $\|Q_j\|$ are uniformly bounded (Homework)

- Approximation issue (Homework): If $u \in H^2(R)$, then

  $$\|u - Q_j u\|_{L^2} = O(2^{-2j})$$

Piecewise polynomial (B-spline) Approximation

- Let $\phi = 1_{[0,1)} * \cdots * 1_{[0,1)}$ be the B-spline of order m.
  (Homework): Let $\phi^{[m]}$ be the B-spline of order m. Show that $\frac{d}{dx}\phi^{[m]}(x) = \phi^{[m-1]}(x) - \phi^{[m-1]}(x)$.

- Define

$$\begin{aligned} V_j &:= \text{span } \{\phi_{j,k}\}_{k \in Z} \\ &= \{\text{functions in } C^{m-1} \text{ and are} \\ &\qquad \text{polynomial of degree } m-1 \text{ on each } [2^{-j}k, 2^{-j}(k+1))\} \end{aligned}$$

- Stability issue:
  - Riesz basis property:

$$A \sum_k |c_k|^2 \le \| \sum_k c_k \phi_{j,k} \|^2 \le B \sum_k |c_k|^2$$

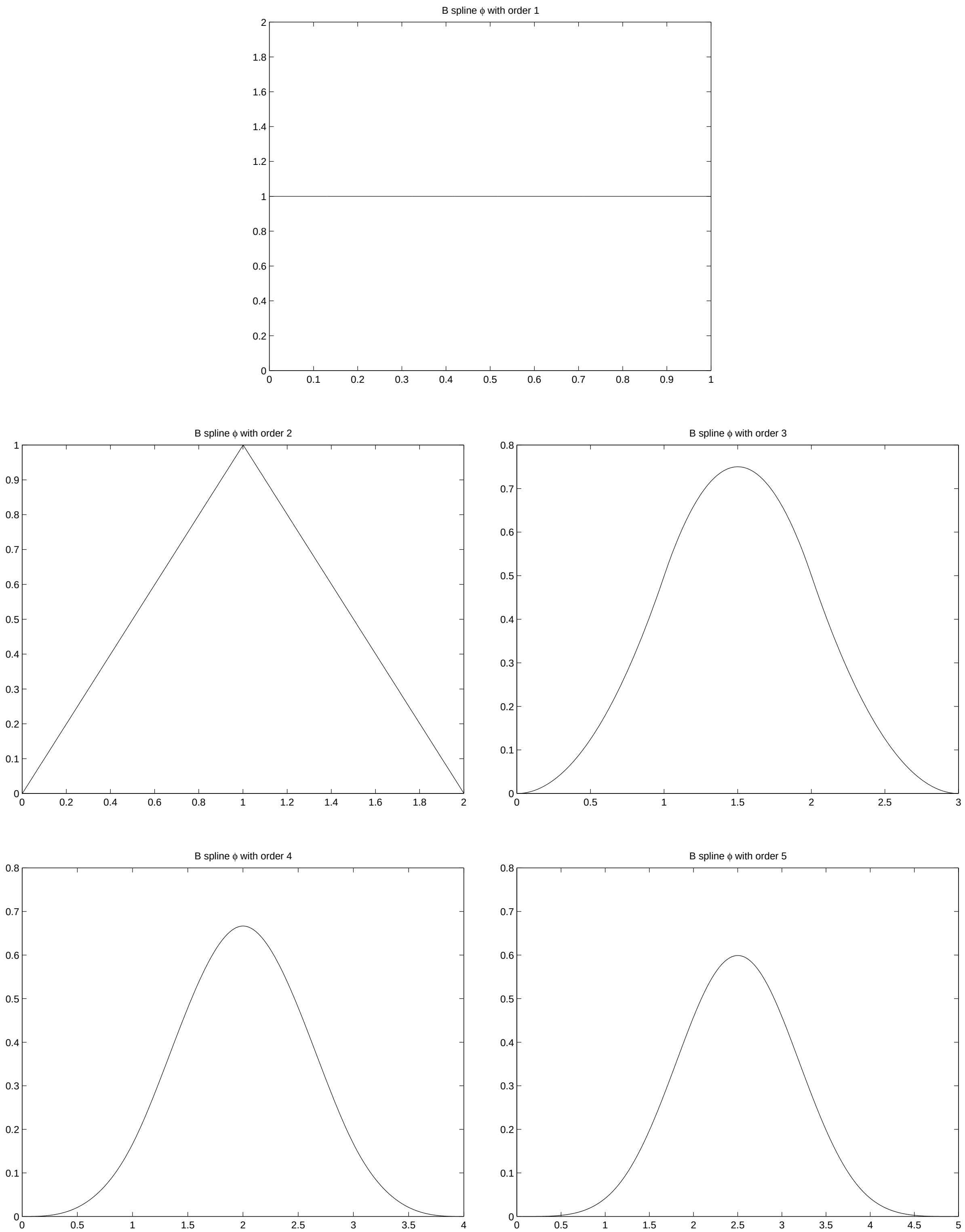
  - We approximate $u \in L^2(R)$ by $Q_j u$ in V_j, where Q_j is a projection onto V_j. We require $\|Q_j\|$ to be uniformly bounded. For example:
    - orthogonal projection: $Q_j u = \sum_k c_k \phi_{j,k}$ with c_k satisfying

$$\sum_k (\phi_{j,i}, \phi_{j,k}) c_k = (u, \phi_{j,i}), \ \forall i$$

- Approximation issue:
  Theorem: If $u \in H^m(R)$, then

$$\|u - Q_j u\|_{L^2} = O(2^{-m\ell})$$

  where Q_j is some uniformly bounded projection of u.



B-spline for $m = 1, 2, 3, 4, 5$. Support of $\phi^{[m]} = [0, m]$

The Projection Operator Q_j

- A operator $Q_j : L^2 \to V_j$ is a projection (i.e. $Q_j^2 = Q_j$).

- Representation of Q_j:
  Fact: Q_j can be represented by

  $$Q_j u = \sum_k (u, \tilde{\phi}_{j,i})\, \phi_{j,i}$$

  for some function $\tilde{\phi}$, where $\tilde{\phi}_{j,i}(x) := 2^{j/2}\tilde{\phi}(2^j x - k)$.

- In previous examples
  - Case 1: box function $\phi = 1_{[0,1)}$, the projection we demonstrate has $\tilde{\phi} = 1_{[0,1)}$.
  - Case 2: hat function $\phi = 1_{[0,1)} * 1_{[0,1)}$, we have introduced two projections:
    - $\tilde{\phi} = \delta$
    - $\tilde{\phi} =$? (will be constructed later)

- The function $\tilde{\phi}$ is called a dual function of ϕ. It satisfies

  $$\int \phi(x)\tilde{\phi}(x-k)\, dx = \delta_{0,k}$$

  or equivalently,

  $$(\phi_{j,i}, \tilde{\phi}_{j,k}) = \delta_{i,k}, \ \forall j$$

- The space

  $$\tilde{V}_j := \text{ span } \{\tilde{\phi}_{j,i}\}_{i \in Z}$$

  is called a dual space of V_j.

Refinable function (or Scaling function)

- Goal: We want to construct a function ϕ such that $V_j :=$ span $\{\phi_{j,k}\}_{k\in Z}$ constitutes a multi-resolution analysis.

- A function ϕ is called refinable if there exist coefficients $\{h_k\}_{k\in Z}$ such that

$$\phi(x) = 2\sum_k h_k \phi(2x-k)$$

- We define the generating function of $\{h_k\}_{k\in Z}$ by

$$h(z) = \sum_{k=-\infty}^{\infty} h_k z^k$$

  It is necessarily that $h(1) = 1$. $h(z)$ is called the mask of ϕ.

- Fact: If $h(1) = 1$ then ϕ exists as a distribution. (by Fourier method)

- Three examples:
  - $h(z) = \left(\frac{1+z}{2}\right)^K$, the corresponding ϕ is the B-spline of order K.
  - $h(z) = \left(\frac{1+z}{2}\right)^K G_K(z)/G_K(z^2)$, the corresponding refinable function $\tilde{\phi}$ is the representation function associated with the orthogonal projection onto the spline spaces
  - $h(z) = z^{-K}\left(\frac{1+z}{2}\right)^{2K} Q_K(z)$ the corresponding refinable function is the Lagrange interpolating function. Here,

$$Q_K(z) = \sum_{n=0}^{K-1} \begin{pmatrix} K+n-1 \\ n \end{pmatrix} \left(\frac{2-z-z^{-1}}{4}\right)^n$$

- Construction of refinable function

- Properties of refinable function
  - support, decay
  - regularity
- Approximation power

B-spline is refinable:

- *The* r^{th} *order B-spline* $\phi^{[r]}$ *is refinable and the corresponding mask is* $\left(\frac{1+z}{2}\right)^r$.

  1. Suppose the mask of $\phi^{[r]}$ is $h^{[r]}$ with coefficients $h_k^{[r]}$.
  2. From definition, $\phi^{[r]} = \phi^{[1]} * \cdots * \phi^{[1]}$, we have

  $$\begin{aligned}
  \phi^{[r]}(x) &= \phi^{[r-1]} * \phi^{[1]}(x) \\
  &= \int \phi^{[r-1]}(y)\phi^{[1]}(x-y)dy \\
  &= 2\int \sum_k h_k^{[r-1]}\phi^{[r-1]}(2y-k)\left(\phi^{[1]}(2x-2y)+\phi^{[1]}(2x-2y-1)\right. \\
  &= 2\sum h_k^{[r-1]}\int \phi^{[r-1]}(2y-k)\phi^{[1]}(2x-2y)dy \\
  &\quad +2\sum h_k^{[r-1]}\int \phi^{[r-1]}(2y-k)\phi^{[1]}(2x-2y-1)dy \\
  &= \sum h_k^{[r-1]}\int \phi^{[r-1]}(y)\phi^{[1]}(2x-k-y)dy \\
  &\quad +\sum h_k^{[r-1]}\int \phi^{[r-1]}(y)\phi^{[1]}(2x-k-1-y)dy \\
  &= \sum h_k^{[r-1]}\phi^{[r]}(2x-k)+\sum h_k^{[r-1]}\phi^{[r]}(2x-k-1) \\
  &= \sum (h_k^{[r-1]}+h_{k-1}^{[r-1]})\phi^{[r]}(2x-k).
  \end{aligned}$$

  3. This implies $h_k^{[r]} = (h_k^{[r-1]}+h_{k-1}^{[r-1]})/2$.
  4. It is easy to see that $h^{[1]}(z) = \left(\frac{1+z}{2}\right)$.

- Support $\phi^{[r]} = [0, r]$

- $\phi^{[r]} \in C^{r-1-\epsilon}$ for any $\epsilon > 0$.

- We may define $\phi^{[0]} = \delta$ which satisfies

  $$\delta(x) = 2\delta(2x)$$

  i.e. $h^{[0]} = 1$.

- Riesz basis property: for $r \geq 1$, $\phi_{0,k}$ constitute a Riesz basis in $V_0 :=$ span $\{\phi_{0,k}\}_{k \in Z}$, i.e. there exists two constants $A > 0$ and $B < \infty$ such that

  $$A \sum_k |c_k|^2 \leq \| \sum_k c_k \phi_{0,k}^{[r]} \|^2 \leq B \sum_k |c_k|^2$$

Orthogonal Project Q_j and the corresponding $\tilde{\phi}$

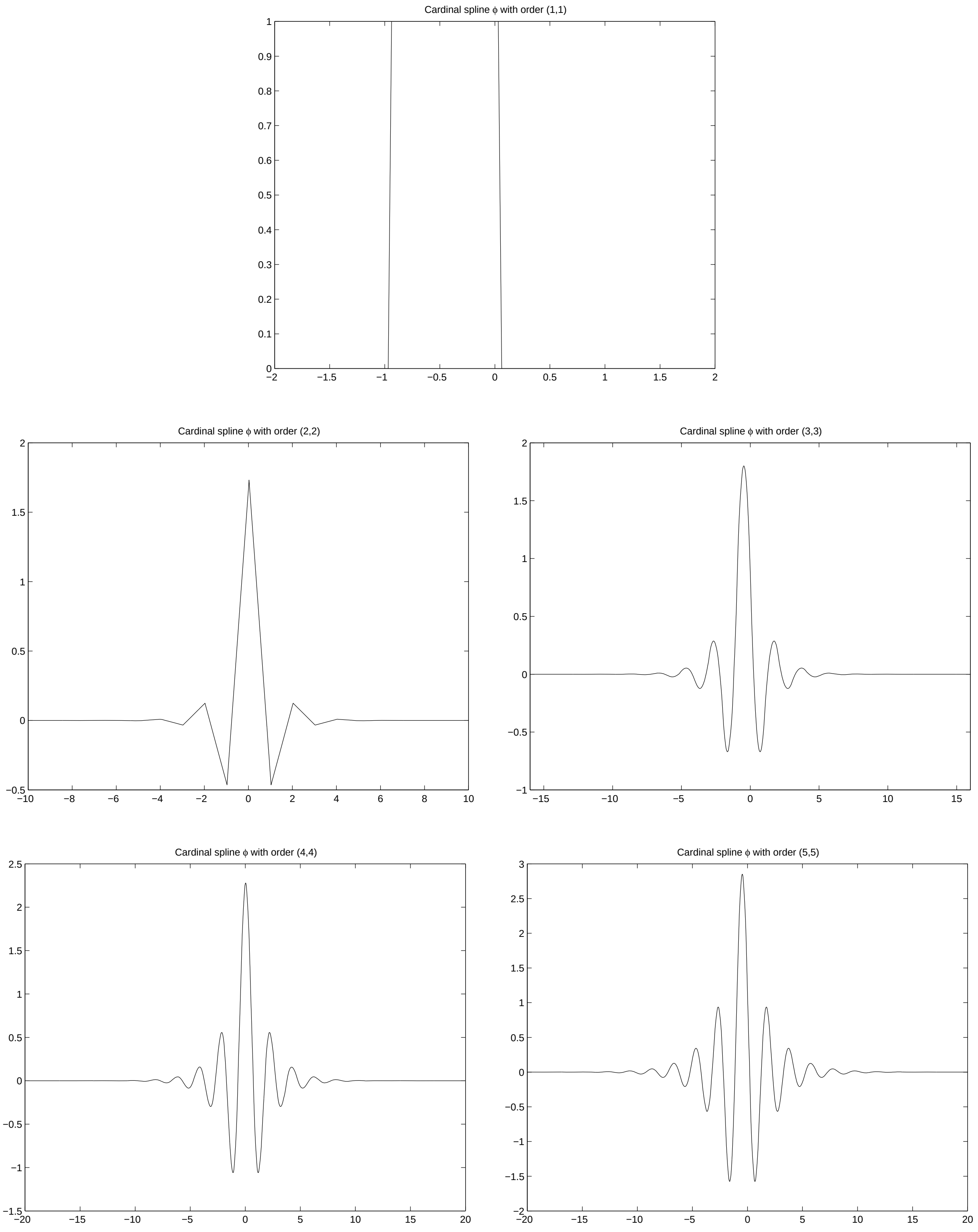
- Question:
  1. Let V_0 be the space spanned by B-spline of order K.
  2. Let Q_j be the orthogonal projection onto V_j.
  3. We represent Q_j by
$$Q_j u = \sum_k (u, \tilde{\phi}_{j,k}) \phi_{j,k}$$
  4. Is $\tilde{\phi}$ refinable? What is the corresponding mask?

- **Claim**: Yes, $\tilde{\phi}$ is refinable and the corresponding mask $\tilde{h}(z)$ is given by
$$\tilde{h}(z) = \left(\frac{1+z}{2}\right)^K \frac{G_K(z)}{G_K(z^2)}$$
where $G_K(z) = \sum G_i^{[K]} z^i$ and
$$G_i^{[K]} := \int \phi^{[K]}(x)\phi^{[K]}(x-i)\,dx = \phi^{[2K]}(K-i)$$

- Proof:
  1. From $(Q_{j-1}u - u) \perp V_{j-1}$, we have
$$\sum (u, \tilde{\phi}_{j-1,k})(\phi_{j-1,k}, \phi_{j-1,i}) = (u, \phi_{j-1,i})$$
  2. Take $u = \phi_{j,0}$, plug into the above equation, using
$$\begin{aligned}\tilde{\phi}_{j-1,k} &= \frac{1}{\sqrt{2}} \sum \tilde{h}_{2k-l} \tilde{\phi}_{j,l} \\ \phi_{j-1,i} &= \frac{1}{\sqrt{2}} \sum \tilde{h}_{2i-m} \tilde{\phi}_{j,m}\end{aligned}$$
     we obtain
$$\sum G_{i-k} \tilde{h}_{2k} = \sum h_{2i-m} G_m$$
     This is equivalent to
$$G(z^2)\tilde{h}(z) = h(z)G(z)$$
     by the following lemma

3. Lemma:

$$\begin{aligned}
\sum_i (\sum_j a_{i-j} b_j) z^i &= a(z)b(z) \\
\sum_i (\sum_j a_{i+j} b_j) z^i &= a(z)b(z^{-1}) \\
\sum_i (\sum_j a_{2i-j} b_j) z^{2i} &= \frac{1}{2}(a(z)b(z) + a(-z)b(-z)) \\
\sum_i (\sum_j a_{2i+1-j} b_j) z^{2i+1} &= \frac{1}{2}(a(z)b(z) - a(-z)b(-z)) \\
\sum_i (\sum_j a_{i-2j} b_{2j}) z^i &= a(z)\frac{1}{2}(b(z) + b(-z))
\end{aligned}$$



The function $\tilde{\phi}$ (for K = 1,2,3,4,5) corresponding the orthogonal projection Q_j.

Construction of refinable function

- **Cascade algorithm**

$$\begin{aligned}\phi^n(x) &= 2\sum_k h_k \phi^{n-1}(2x-k)\\ \phi^0 &= 1_{[0,1)}\end{aligned}$$

- **Subdivision scheme**

  1. Find $\{\phi(k)\}_{k\in Z}$ by solving the eigen system:

  $$\phi(i) = 2\sum_k h_{2i-k}\phi(k)$$

  2. Find the value of ϕ at $x_{j+1,k}$ points by the subdivision scheme

  $$\phi(x_{j+1,i}) = S_h(\phi(x_{j,\cdot})_i$$

  where

  $$(S_h b)_i = \sum_k 2h_{i-2k}b_k$$

  Or equivalently,

  $$\phi(2^{-(j+1)}i) = 2\sum_k h_k \phi(2^{-j}i - k)$$

- **Fourier method:**

  1. Taking Fourier transform on the refinable equation, we obtain

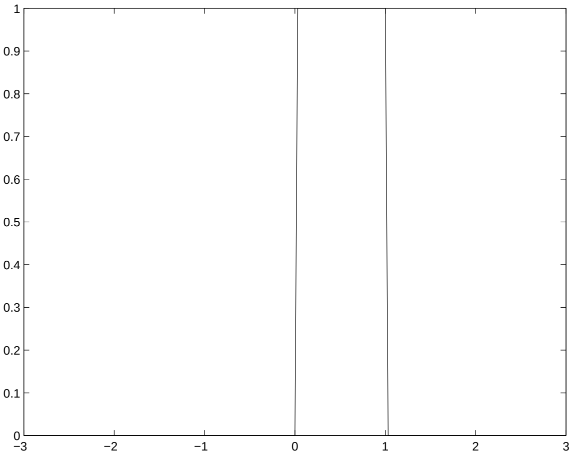
  $$\widehat{\phi}(\xi) = m(\frac{\xi}{2})\widehat{\phi}(\frac{\xi}{2})$$

  where $m(\xi) = h(e^{i\xi})$.

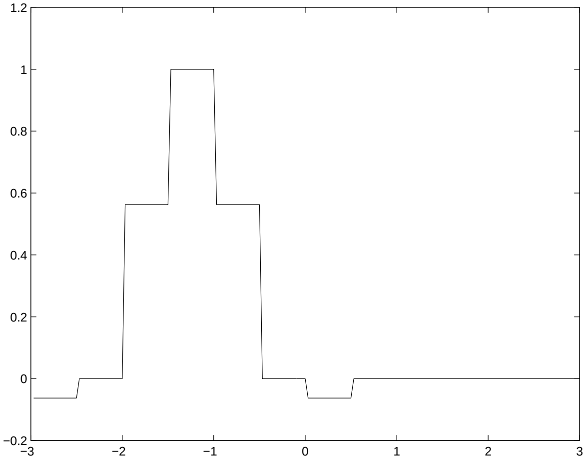
  2. By the normalization $\widehat{\phi}(0) = 1$, we obtain

  $$\widehat{\phi}(\xi) = \prod_{j=1}^{\infty} m\left(\frac{\xi}{2^j}\right)$$

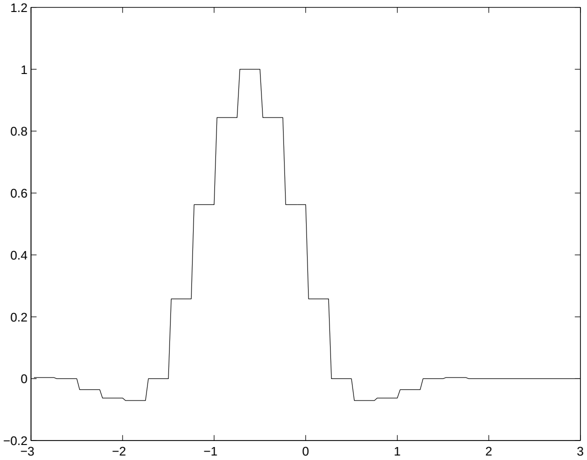
Casecade for h_k=(−1/16,0,9/10,1,9/16,0,−1/16)



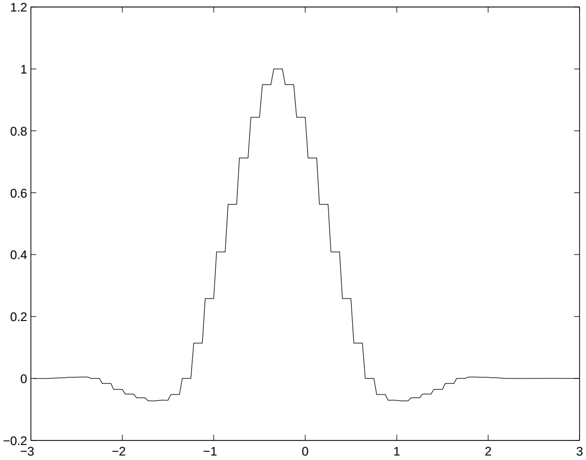
Casecade : Lagrange fundamental of order 2, interative steps = 1



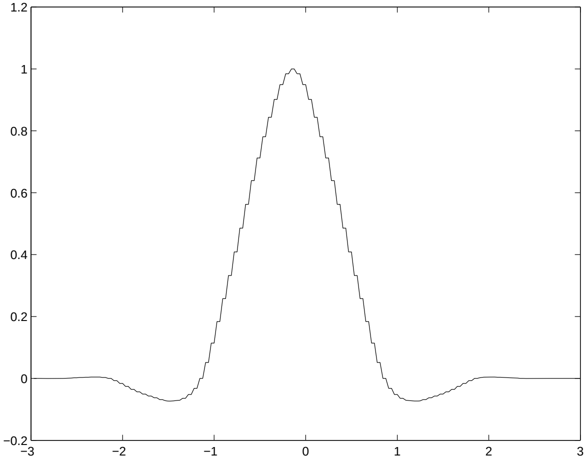
Casecade : Lagrange fundamental of order 2, interative steps = 2

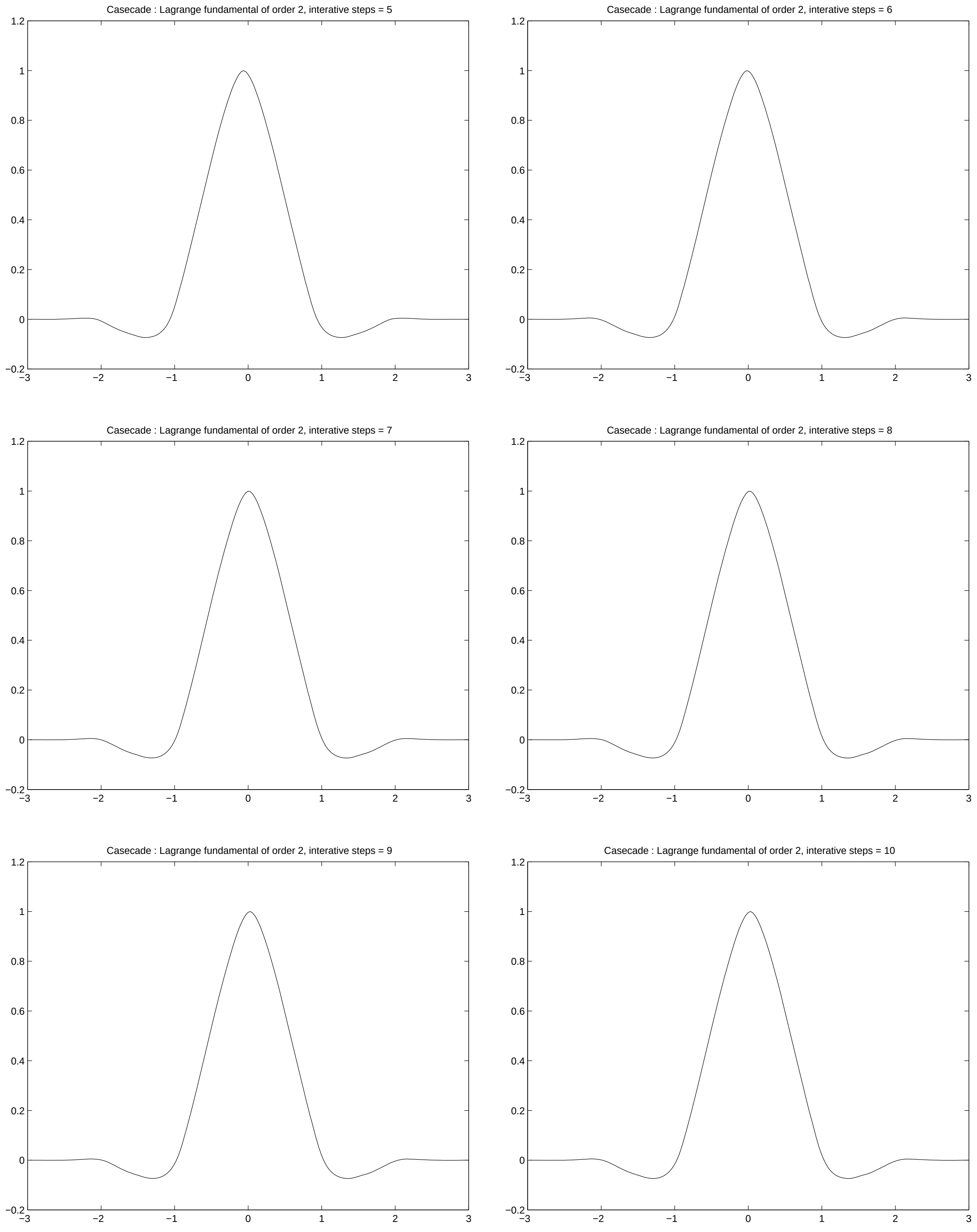


Casecade : Lagrange fundamental of order 2, interative steps = 3



Casecade : Lagrange fundamental of order 2, interative steps = 4





Cascade algorithm, Lagrange interpolation $K = 2$

- **Remark.** Cascade algorithm is equivalent to subdivision algorithm

  1. The subdivision scheme S_h is defined by

     $$(S_h b)_i = \sum_k 2h_{i-2k} b_k$$

     In particular, $2h = S_h \delta_0$.

  2. A cascade algorithm is

     $$\begin{aligned}
     \phi^n(x) &= 2\sum_i h_i \phi^{n-1}(2x-i) \\
     &= \sum_i (S_h\delta_0)_i \phi^{n-1}(2x-i) \\
     &= 2^2 \sum_i h_i \sum_k h_k \phi^{n-2}(2^2x-2i-k) \\
     &= \sum_\ell (\sum_i 2h_{\ell-2i} 2h_i) \phi^{n-2}(2^2x-\ell) \\
     &= \sum_\ell (S_h(2h))_\ell \phi^{n-2}(2^2x-\ell) \\
     &= \sum_\ell S_h^2 \delta_0 \phi^{n-2}(2^2x-\ell)
     \end{aligned}$$

  3. In general, we have

     $$\phi^n(x) = \sum_i (S_h^n \delta_0)_i \phi^0(2^n x - i)$$

Properties

- Convergence and regularity
  - If $h(1) = 1$, then ϕ exists as a distribution
  - Convergence in L^2 is related to the regularity of ϕ.
  - Regularity is related to the the "approximation order" of ϕ
  - If $h(z) = \left(\frac{1+z}{2}\right)^p H(z)$ with $H(1) \neq 0$, then we say that $\phi(z)$ has approximation order p.
  - If ϕ has approximation order p, then $\pi_p \subset V_0$, where π_p be the set of all polynomials of order less than p.
  - The higher the p is, the more regular the ϕ is
- Support and Decay
  - A mask $h(z)$ is called of finite length if there exists an integer M such that $h(k) = 0$ for $|k| > M$
  - ϕ is of finite support if and only if its mask is of finite length
  - ϕ decay exponentially at $x = \pm\infty$ iff h_k decays exponentially at $k = \pm\infty$.
- Riesz basis property:
  Under some mild assumption on h, the corresponding refinable function ϕ satisfies the Riesz basis property:

  $$A \sum_k |c_k|^2 \leq \| \sum_k c_k \phi_{j,k} \|^2 \leq B \sum_k |c_k|^2$$
- Approximation power:
  Theorem. (Strang-Fix, Unser) Suppose ϕ is of pth order and V_j is the span of $\{\phi_{j,k}\}_{k \in Z}$. Let Q_j be any projection from L^2 onto V_j. Then

  $$\|Q_j u - u\|_{L^2} = C_Q 2^{-jp} + O(2^{-j(p+1)})$$
- The main technique is by Fourier method.
- References:

Interpolatory functions

- **Problem**: Fix a scale level j and given $u_{j,k}$ at $x_{j,k}$ for all $k \in Z$, we want to construct a smooth function u such that $u(x_{j,k}) = u_{j,k}$ for all $k \in Z$.

- **Definition.** A refinable function $\tilde{\phi}$ is called interpolatory if $\tilde{\phi}(k) = \delta_{0,k}$ for all $k \in Z$. Or equivalently,

$$(\tilde{\phi}_{j,k}, \delta(\cdot - x_{j,i})) = \delta_{k,i}$$

- **Answer:** Suppose that $\tilde{\phi}$ is interpolatory. Then the answer to the above problem is

$$u = \sum_k u_{j,k} \tilde{\phi}_{j,k}$$

- A refinable function $\tilde{\phi}$ is interpolatory (i.e. $(\tilde{\phi}, \delta(\cdot - k)) = \delta_{0,k}$) iff its mask H satisfies

$$H(z) + H(-z) = 1$$

- Proof.

  1. From $\tilde{\phi}(k) = \delta_{0,k}$ and the refinement equation

  $$\tilde{\phi}(i) = 2 \sum_k \tilde{h}_k \tilde{\phi}(2i - k) = 2\tilde{h}_{2i}$$

     This implies $\tilde{h}_{2i} = 0$ for all integer $i \neq 0$ and $\tilde{h}_0 = \frac{1}{2}$.

  2. $H(z) + H(-z) = 1$ iff its even coefficients $h_{2i} = 0$ for $i \neq= 0$ and $h_0 = \frac{1}{2}$.

Example 1. The Lagrange interpolatory function

- **Definition.**
  1. Initially, define $\tilde{\phi}(k) = \delta_{0,k}$
  2. Using subdivision scheme, define $\tilde{\phi}$ at $x_{j+1,2k+1}$ by Lagrange interpolation using data at $x_{j,k-K+1}, \cdots, x_{j,k+K}$, i.e.

$$\tilde{\phi}(x_{j+1,2k+1}) = \sum_{u} \prod_{\substack{-K < v \leq K \\ v \neq u}} \frac{x_{j+1,2k+1} - x_{j,k+v}}{x_{j,k+u} - x_{j,k+v}} \tilde{\phi}(x_{j,k+u})$$

- The mask
  1. $h(z) = z^{-K}\left(\frac{1+z}{2}\right)^{2K} Q_K(z)$, where

$$Q_K(z) = \sum_{n=0}^{K-1} \binom{K+n-1}{n} \left(\frac{2-z-z^{-1}}{4}\right)^{n}$$

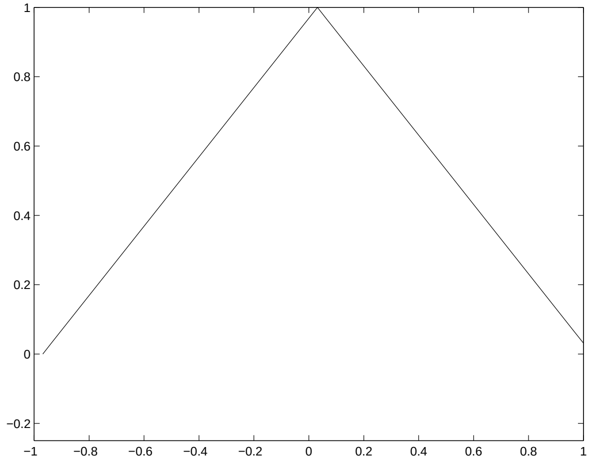
  2. Example:
     - $K = 1$: $2h = 2^{-2}(1,2,1)$
     - $K = 2$: $2h = 2^{-4}(-1,0,9,16,9,0,-1)$

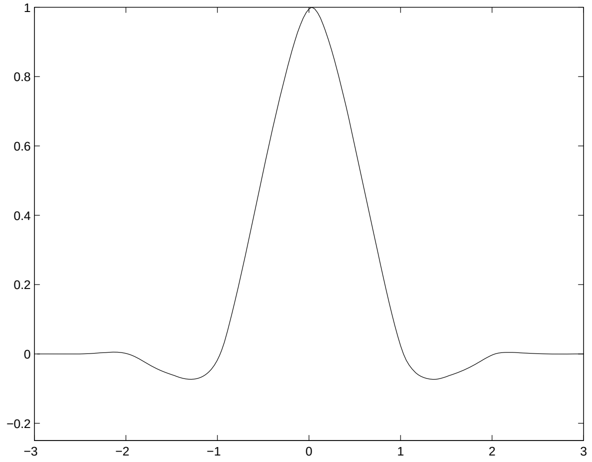
- Property:
  1. *The Lagrange interpolation mask has the smallest length among all interpolatory mask of order* $2K$.
  2. Support of $\tilde{\phi}$ of order $2K$ is $[-3K, 3K]$.
  3. Regularity: the order of differentiability is linearly proportional to K

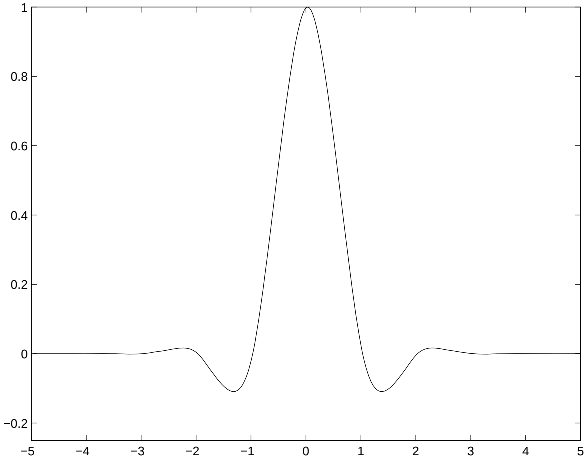
- Riesz basis property

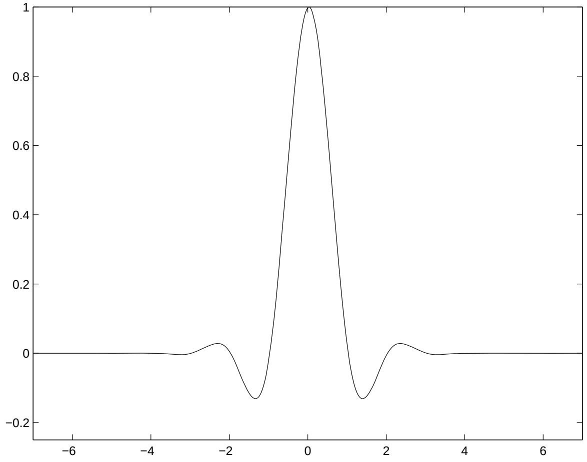
- Approximation power: *If* Q_j *is any projection onto* V_j *which is spanned by the Lagrange interpolant* $\tilde{\phi}$ *of order* $2K$, *then*

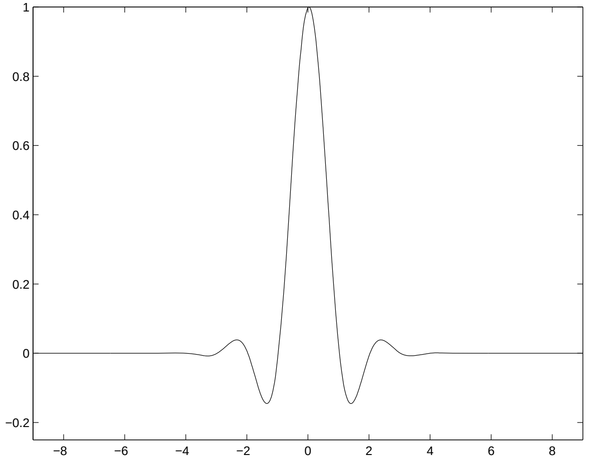
$$\|u - Q_j u\| = C_{\tilde{\phi}} 2^{-2Kj} + O(2^{-(2K+1)j})$$


Lagrange fundamentals with order 1


Lagrange fundamentals with order 2

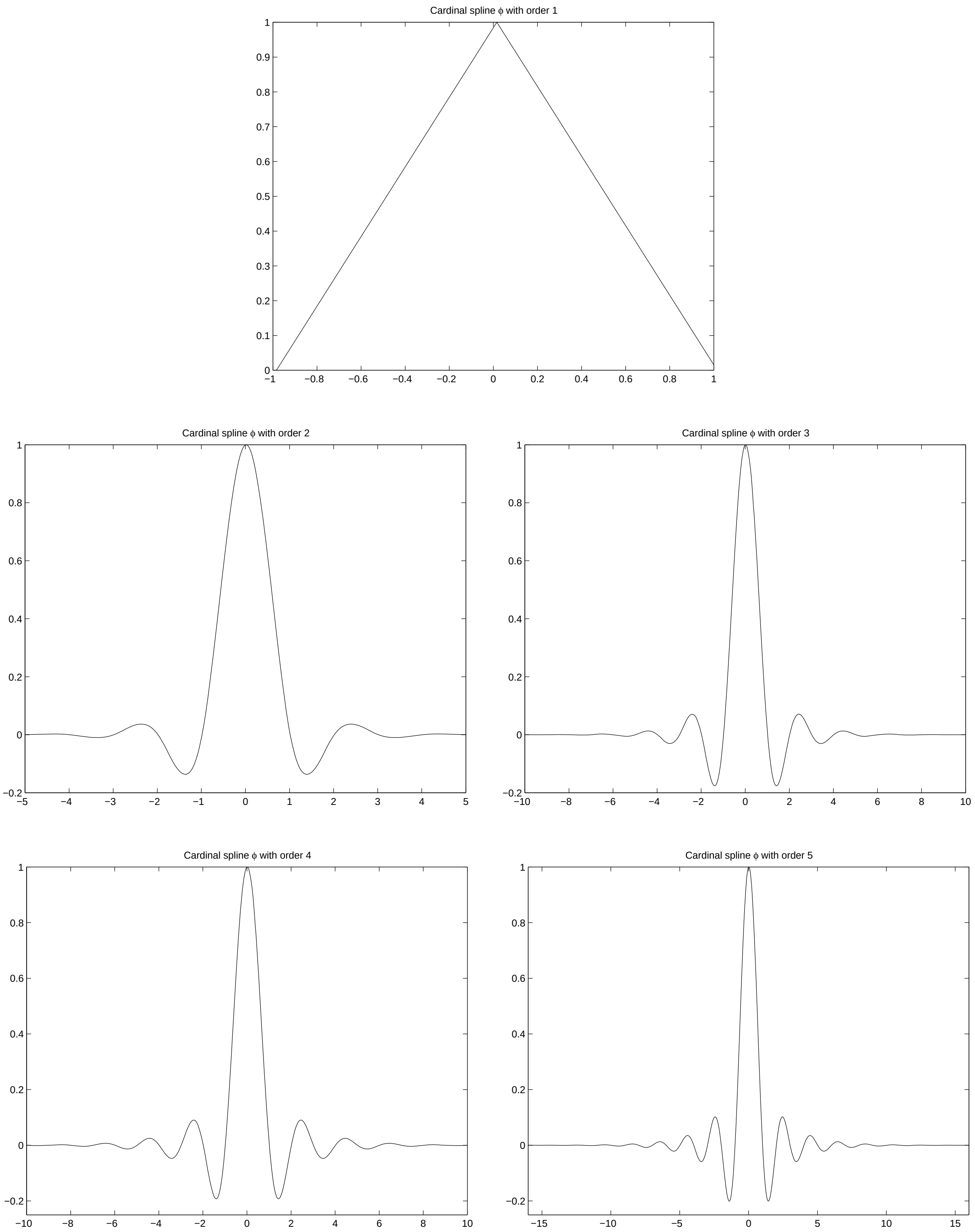

Lagrange fundamentals with order 3


Lagrange fundamentals with order 4


Lagrange fundamentals with order 5

Example 2. The Cardinal Spline

- **Problem.** Given $\{u_{j,i}\}_{i \in Z}$, we want to find a function u such that
  1. $u(x_{j,i}) = u_{j,i}$ for all $i \in Z$
  2. In each interval $(x_{j,i}, x_{j,i+1})$, $u(\cdot)$ is a polynomial of degree $2K - 1$
- **Answer:**
  1. There exists a refinable function $\tilde{\phi}$ (called the cardinal spline) such that the above u is given by
  $$u = \sum_{k} u_{j,k} \tilde{\phi}_{j,k}$$
  2. The corresponding mask $\tilde{h}(z)$ is given by
  $$z^{-K} \left(\frac{1+z}{2} \right)^{2K} \frac{G_K(z)}{G_K(z^2)}$$
- **Property:**
  1. $\tilde{\phi}$ can reproduce polynomial of order less than $2K$
  2. The order of $\tilde{\phi}$ is $2K$
  3. The regularity of $\tilde{\phi}$ is $C^{2K-1-\epsilon}$ for any small $\epsilon > 0$
  4. The support of $\tilde{\phi}$ is ∞, but $\tilde{\phi}$ decays exponentially fast
  5. Dual property:
  $$(\tilde{\phi}, \delta(\cdot - k)) = \delta_{0,k}$$
  6. Riesz basis property
  7. Approximation power



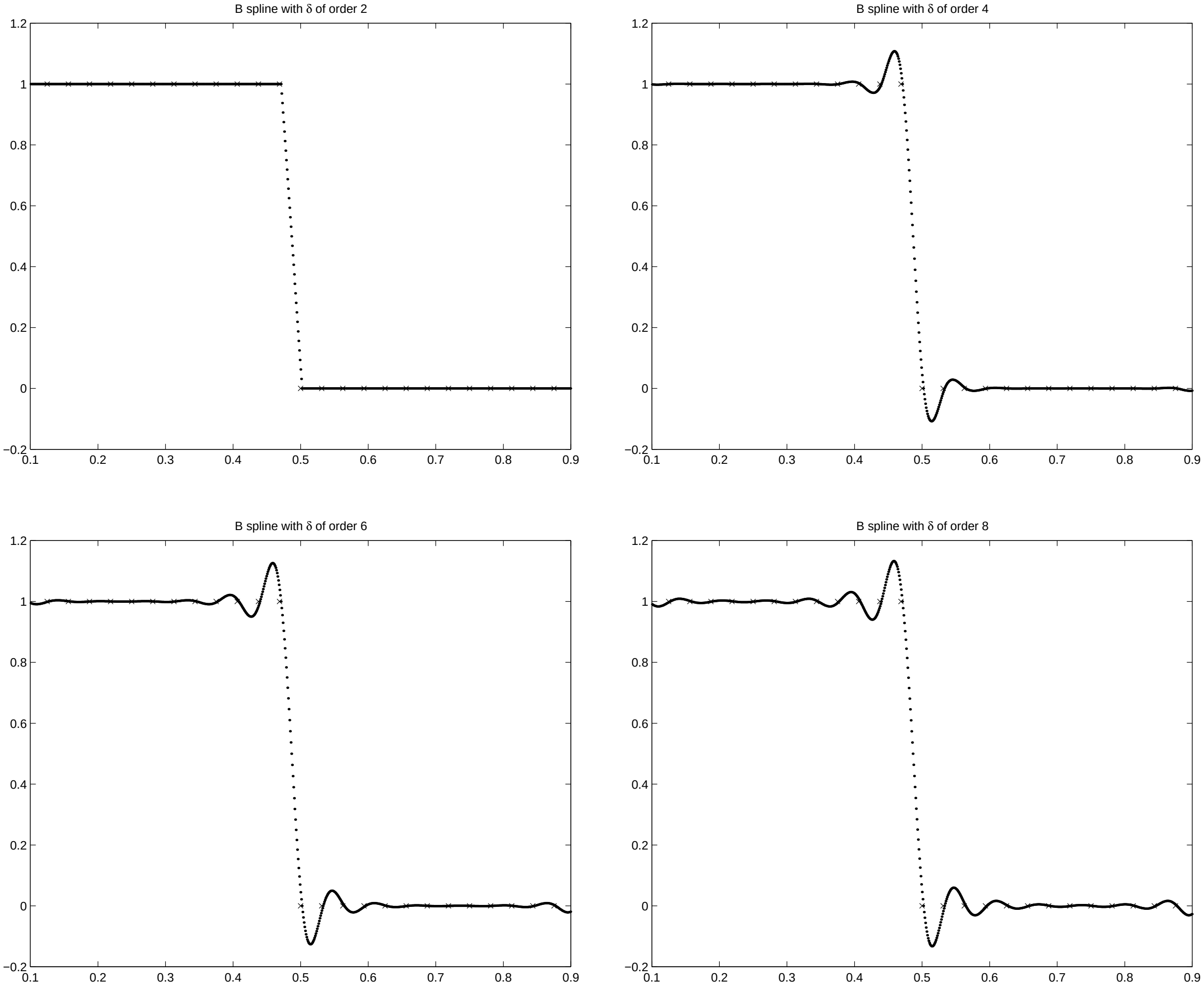
The cardinal splines for $K = 1, 2, 3, 4, 5$

Gibbs phenomena from higher order polynomial approximation

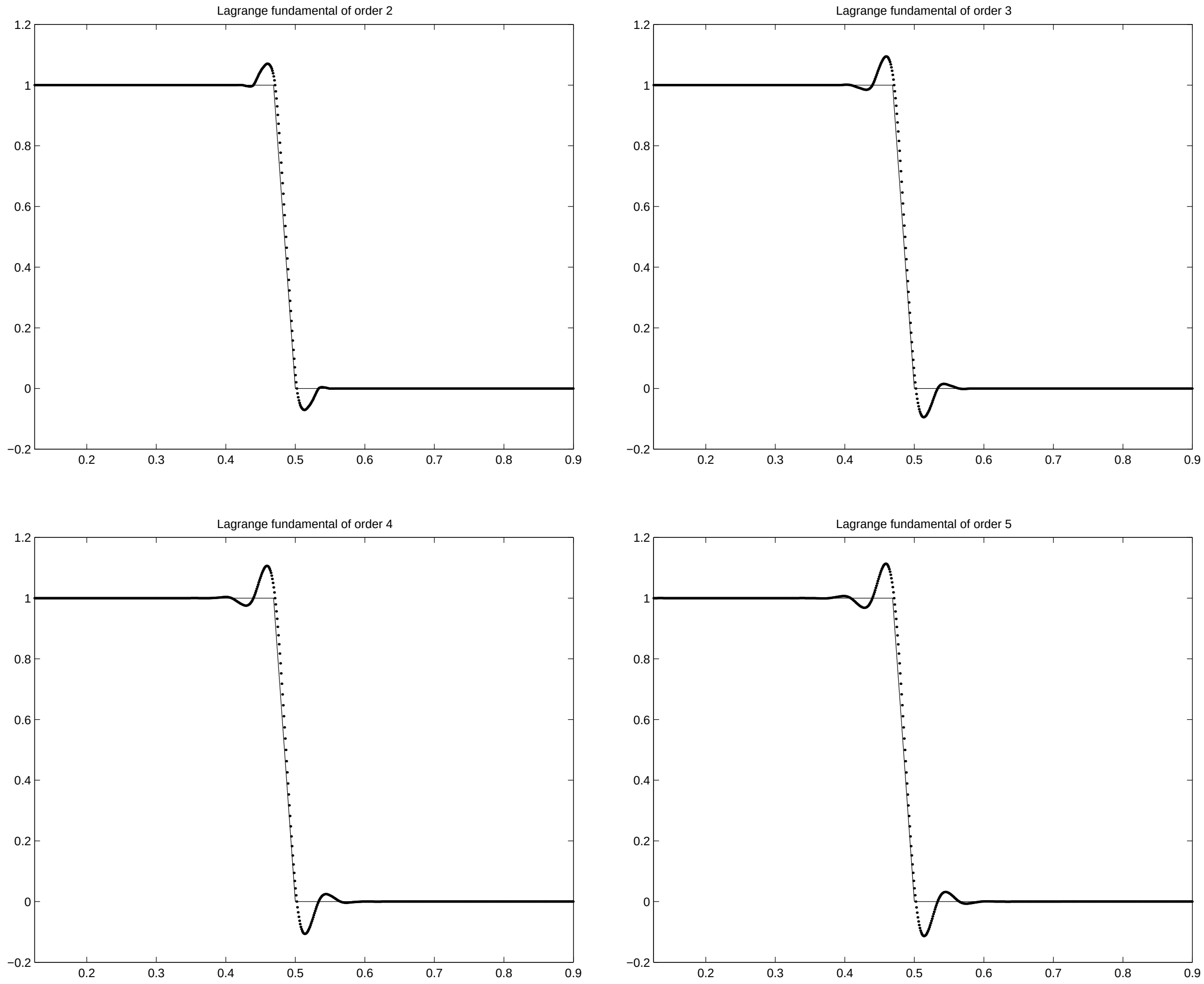
Consider the Heviside function

$$u(x) = \begin{cases} 1 & \text{for } x < 1/2 \\ 0 & \text{for } x > 1/2 \end{cases}$$

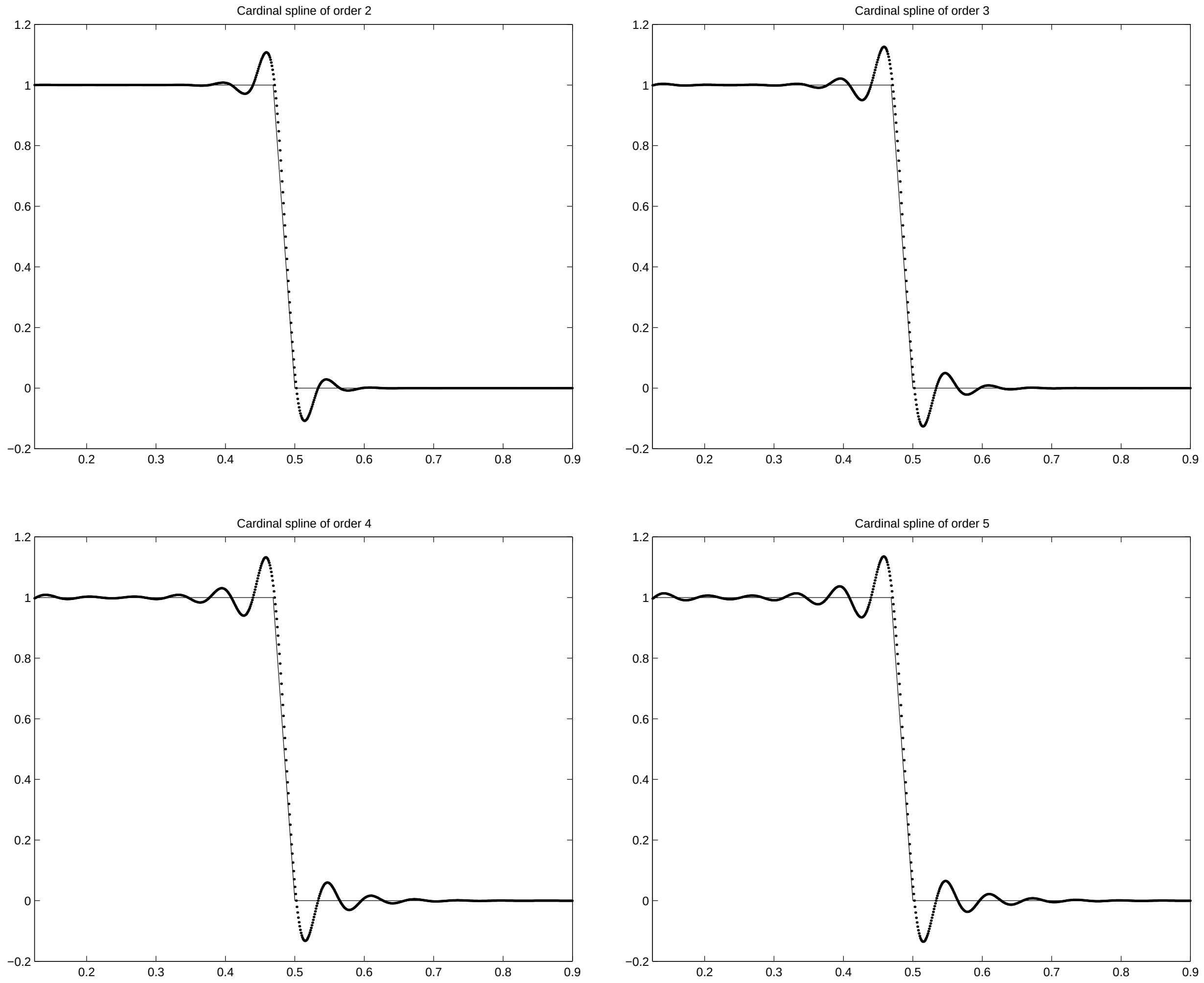
We interpolate u at $x_{j,k}$ with $j = 5$ by using (1) splines, (2) Lagrange interpolatory refinable function, and (3) the cardinal splines.



The Gibbs phenomena by using high-order splines. $(K = 2, 3, 4, 5)$



The Gibbs phenomena by using high-order Lagrange interpolants. $(K = 2, 3, 4, 5)$



The Gibbs phenomena by using high-order cardinal splines. $(K = 2, 3, 4, 5)$

Discrete wavelet transform

A wavelet expansion method decomposes data into fluctuations at various resolutions. It depends on four sets of coefficients $\{h_k\}, \{g_k\}, \{\tilde{h}_k\}$ and $\{\tilde{g}_k\}$.

- Decomposition: $(c_j) \mapsto (c_{j-1}, d_{j-1})$.

$$\begin{cases} \text{the low-pass data: } c_{j-1,i} = \sqrt{2}\sum_k h_k c_{j,2i+k}, \\ \text{the high-pass data: } d_{j-1,i} = \sqrt{2}\sum_k g_k c_{j,2i+1+k}. \end{cases}$$

  discrete wavelet transform:

$$T_J : \ c_J \mapsto (c_0, d_0, d_1, \cdots, d_{J-1})$$

- Reconstruction:

$$c_{j,i} = \sqrt{2}\sum_k \left[\tilde{h}_{i-2k} c_{j-1,k} + \tilde{g}_{i-2k-1} d_{j-1,k}\right].$$

Condition for perfect reconstruction

- Define $h(z) = \sum_k h_k z^k$, $\tilde{h}(z) = \sum_k \tilde{h}_k z^k$.
- Perfect Reconstruction: (Homework)
  - Let $c_j(z) = \sum_k c_{j,k} z^k$
  - $$\begin{aligned} c_j(z) &= \sqrt{2}\tilde{h}(z)c_{j-1}(z^2) + \sqrt{2}\tilde{g}(z) z d_{j-1}(z^2) \\ &= \tilde{h}(z)\Big(h(z^{-1})c_j(z) + h(-z^{-1})c_j(-z)\Big) \\ &\quad +\tilde{g}(z)\Big(g(z^{-1})d_j(z) - g(-z^{-1}) z d_j(-z)\Big) \end{aligned}$$
  - This gives

$$\begin{aligned} h(z^{-1})\tilde{h}(z) + g(z^{-1})\tilde{g}(z) &= 1 \\ h(-z^{-1})\tilde{h}(z) - g(-z^{-1})\tilde{g}(z) &= 0. \end{aligned}$$

- A formal calculation gives that

$$h(z^{-1})\tilde{h}(z) + h(-z^{-1})\tilde{h}(-z) = 1,$$

$$\begin{aligned} g(z) &= \tilde{h}(-z^{-1})P(z^2) \\ \tilde{g}(z) &= h(-z^{-1})/P(z^2) \end{aligned}$$

for some Laurent series $P(z)$.

- If $h(z), \cdots \tilde{g}(z)$ are finte length, then $P(z) = z^m$.
- Otherwise, $P(z)$ should be in the Wiener class, that is, $P(z) = \sum_k p_k z^k$ with $\sum_k |p_k| < \infty$ and $P(z) \neq 0$ for all $|z| = 1$

General procedure to construct filter banks

- Find a Laurent series H such that

$$H(z) + H(-z) = 1$$

- Factorize $H(z)$ into $h(z)\tilde{h}(z)$, i.e.

$$\begin{aligned} h(z)\tilde{h}(z^{-1}) + h(-z)\tilde{h}(-z^{-1}) &= 1 \\ h(1) = \tilde{h}(1) &= 1 \end{aligned}$$

- Define $g(z)$ and $\tilde{g}(z)$ by

$$\begin{aligned} g(z) &= \tilde{h}(-z^{-1})P(z^2) \\ \tilde{g}(z) &= h(-z^{-1})/P(z^2) \end{aligned}$$

  with a proper function $P(z)$ in the Wiener class.

General Design Principles

- Fast
- Stable
- Efficient

A Trick to construct interpolatory mask

- Choose $H_0(z) = \left(\frac{1+z}{2}\right)$
- $$(H_0(z) + H_0(-z))^n = 1^n$$
- Examples: rewrite $H_0(-z)$ by $H_1(z)$

  1. $n = 2$:

     $$\begin{aligned}(H_0 + H_1)^2 &= H_0^2 + H_0H_1 + H_1H_0 + H_1^2 \\ &= H_0(H_0 + H_1) + H_1(H_1 + H_0)\end{aligned}$$

     Hence $H(z) = H_0(H_0 + H_1)$

  2. $n = 3$:

     $$\begin{aligned}(H_0 + H_1)^3 &= H_0^3 + 3H_0^2H_1 + 3H_1^2H_0 + H_1^3 \\ &= H_0^2(H_0 + 3H_1) + H_1^2(H_1 + 3H_0)\end{aligned}$$

     Hence, $H(z) = H_0^2(H_0 + 3H_1)$

  3. (Homework) Derive the general formula for general n

  4. (Homework) If we choose

     $$H_0 = z^{-1}\left(\frac{1+z}{2}\right)^2$$

     Derive the general formula for H from $(H_0(z) + H_0(-z))^n = 1$

Splitting Tricks

- Daubechies' orthogonal wavelet:

$$\text{orthogonality} \iff h(z) = \tilde{h}(z^{-1})$$

$$\underbrace{Q(z^{-1})\left(\frac{1+z^{-1}}{2}\right)^K}_{\tilde{h}(z)} \underbrace{Q(z)\left(\frac{1+z}{2}\right)^K}_{h(z)} + \tag{1}$$

$$\underbrace{Q(-z^{-1})\left(\frac{1-z^{-1}}{2}\right)^K}_{\tilde{g}(z)} \underbrace{Q(-z)\left(\frac{1-z}{2}\right)^K}_{g(z)} = 1 \tag{2}$$

- Chui-Wang's Semi-orthogonal wavelet (Spline wavelet)

$$\underbrace{\left[\Big(G(z)/G(z^2)\Big)\left(\frac{1+z}{2}\right)^K\right]}_{\tilde{h}(z)} \underbrace{\left(\frac{1+z}{2}\right)^K}_{h(z)} +$$

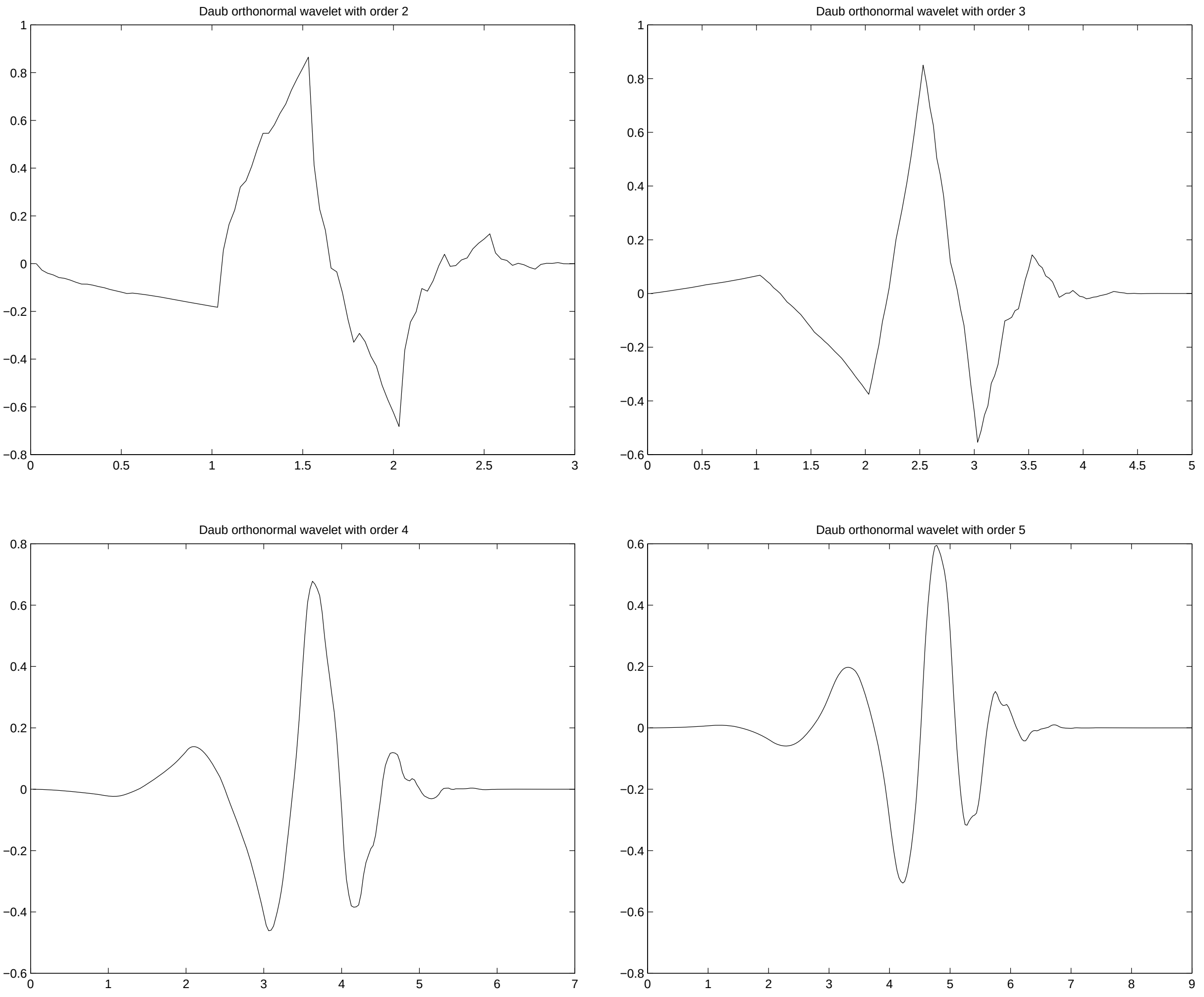
$$\underbrace{\left(\frac{1-z}{2}\right)^K/G(z^2)}_{\tilde{g}(z)} \underbrace{\left[G(-z)\left(\frac{1-z}{2}\right)^K\right]}_{g(z)} = 1$$

- $\{\psi_j^\ell\}_j$ are not orthogonal, but $L^2(R) = \bigoplus_\ell W^\ell$

- Cohen-Daubechies-Feauveau biorthogonal wavelet

$$\underbrace{Q_K(z)\left(\frac{1+z}{2}\right)^{\tilde{d}}}_{\tilde{h}(z)} \underbrace{\left(\frac{1+z}{2}\right)^d}_{h(z)} + \underbrace{\left(\frac{1-z}{2}\right)^d}_{\tilde{g}(z)} \underbrace{\left(\frac{1-z}{2}\right)^{\tilde{d}} Q_K(-z)}_{g(z)} = 1$$

where $d + \tilde{d} = 2K$.



Daubechies orthogonal wavelets, $K = 2, 3, 4, 5$

Multi-resolution Analysis

- From the masks $h(z)$ and $\tilde{h}(z)$, we can define two sets of multi-resolution analysis:

$$
\begin{aligned}
\cdots &\subset V_{j-1} \subset V_j \subset \cdots \subset L^2(R) \\
\cdots &\subset \tilde{V}_{j-1} \subset \tilde{V}_j \subset \cdots \subset L^2(R)
\end{aligned}
$$

  - the refinable functions ϕ and $\tilde{\phi}$:

$$
\begin{aligned}
\phi(x) &= 2\sum_k h_k \phi(2x-k), \\
\tilde{\phi}(x) &= 2\sum_k \tilde{h}_k \tilde{\phi}(2x-k),
\end{aligned}
$$

  - The two sequences of nested spaces:

$$V_j = \text{ span } \{\phi_{j,k}\}_{k\in Z}, \quad \tilde{V}_j = \text{ span } \{\tilde{\phi}_{j,k}\}_{k\in Z}$$

  - $V_j \to L^2(R)$ and $V_j \to L^2(R)$ as $j \to \infty$
  - Riesz basis properties of ϕ and $\tilde{\phi}$:

$$
\begin{aligned}
A\sum_k |c_k|^2 &\leq \|\sum_k c_k \phi_{j,k}\|^2 \leq B\sum_l |c_k|^2 \\
\tilde{A}\sum_k |c_k|^2 &\leq \|\sum_k c_k \tilde{\phi}_{j,k}\|^2 \leq \tilde{B}\sum_l |c_k|^2
\end{aligned}
$$

- ϕ and $\tilde{\phi}$ are dual to each other:
$$(\phi_{j,i}, \tilde{\phi}_{j,k}) = \delta_{i,k}$$
- Proof.
  1. If is sufficient to show that
$$(\phi(\cdot), \tilde{\phi}(\cdot - k)) = \delta_{0,k}$$
  2. We use cascade algorithm starting with ϕ^0 and $\tilde{\phi}^0$ being the box function. Hence they are dual to each other.
  3. Suppose ϕ^{n-1} and $\tilde{\phi}$ are dual to each other, from the cascade algorithm
$$\begin{aligned}
(\phi^n(\cdot), \tilde{\phi}(\cdot - i)) &= \int \phi^n(x)\tilde{\phi}^n(x-i)\,dx \\
&= \int (\sum_k 2h_k\phi^{n-1}(2x-k))\,(\sum_\ell 2\tilde{h}_\ell\tilde{\phi}^{n-1}(2(x-i) - \\
&= \int \sum_k \sum_\ell 2h_k\tilde{h}_\ell\delta_{k,2i+\ell} \\
&= \int \sum_\ell 2h_{2i+\ell}\tilde{h}_\ell \\
&= \delta_{0,i}
\end{aligned}$$
     In the last equality, we have used
$$h(z)\tilde{h}(z) + h(-z)\tilde{h}(-z) = 1$$
  4. By letting $n \to \infty$, we obtain the duality of ϕ and $\tilde{\phi}$.

- The projections Q_j and $\tilde{Q}_j$.
  - We define
$$\begin{aligned}
Q_j u &= \sum_k (u, \tilde{\phi}_{j,k})\phi_{j,k} \\
\tilde{Q}_j u &= \sum_k (u, \phi_{j,k})\tilde{\phi}_{j,k}
\end{aligned}$$
  - Both $Q_j - Q_{j-1}$ and $\tilde{Q}_j - \tilde{Q}_{j-1}$ are projections too

- Wavelets and multi-resolution decomposition
  - Define wavelet ψ and dual wavelet $\tilde{\psi}$ by

$$\begin{aligned} \psi(x) &= 2\sum_k g_{k-1}\phi(2x-k) \\ \tilde{\psi}(x) &= 2\sum_k \tilde{g}_{k-1}\tilde{\phi}(2x-k) \end{aligned}$$

  - Let $W_j =$ span $\{\psi_{j,k}\}_{k\in Z}$ and $\tilde{W}_j =$ span $\{\tilde{\psi}_{j,k}\}_{k\in Z}$ Then

$$W_{j-1} \subset V_j, \quad \tilde{W}_{j-1} \subset \tilde{V}_j$$

    (Here, we use the condition: $P(z)$ is in the Wiener class)
  - We have

$$\begin{aligned} g(z) = \tilde{h}(-z)P(z^2) &\iff W_{j-1} \perp \tilde{V}_{j-1} \\ \tilde{g}(z) = h(-z)/P(z^2) &\iff \tilde{W}_{j-1} \perp V_{j-1} \end{aligned}$$

  - Proof.
    1. We only need to show that $(\psi, \tilde{\phi}(\cdot - i)) = 0$.
    2. We use the definition for $\tilde{\phi}$ and ψ and the duality of ϕ and $\tilde{\phi}$ below:

$$\begin{aligned} (\psi, \tilde{\phi}(\cdot - i)) &= \int \psi(x)\,\tilde{\phi}(x-i)\,dx \\ &= \int (\sum_k 2g_{k-1}\phi(2x-k))\,(\sum_\ell 2\tilde{h}_\ell\tilde{\phi}(2(x-i)-\ell))\,dx \\ &= \sum_k\sum_\ell 2g_{k-1}\tilde{h}_\ell\delta_{k,2i+\ell} \\ &= \sum_\ell 2g_{2i+\ell-1}\tilde{h}_\ell \\ &= 0 \end{aligned}$$

    In the last formula, we have used

$$g(z)\tilde{h}(z) - g(-z)\tilde{h}(-z) = 0.$$

  - $(\psi_{j,k}, \tilde{\psi}_{j,\ell}) = \delta_{k,\ell}$. This follows from

$$g(z)\tilde{g}(z) + g(-z)\tilde{g}(-z) = h(z)\tilde{h}(z) + h(-z)\tilde{h}(-z) = 1$$

- Multi-resolution decomposition
  - V_j and $\tilde{V}_j$ can be decomposed into

    $$V_j = V_{j-1} + W_{j-1}, \ \tilde{V}_j = \tilde{V}_{j-1} + \tilde{W}_{j-1}$$

    Or

    $$\begin{aligned} Q_j u &= \sum_i (u, \tilde{\phi}_{j,i}) \phi_{j,i} \\ &= \sum_i (u, \tilde{\phi}_{j-1,i}) \phi_{j-1,i} + \sum_i (u, \tilde{\psi}_{j-1,i}) \psi_{j-1,i} \\ \tilde{Q}_j u &= \sum_i (u, \phi_{j,i}) \tilde{\phi}_{j,i} \\ &= \sum_i (u, \phi_{j-1,i}) \tilde{\phi}_{j-1,i} + \sum_i (u, \psi_{j-1,i}) \tilde{\psi}_{j-1,i} \end{aligned}$$

  - Performing this decomposition recursively, we obtain

    $$\begin{aligned} V_j &= V_0 + W_0 + \cdots + W_{j-1} \\ \tilde{V}_j &= \tilde{V}_0 + \tilde{W}_0 + \cdots + \tilde{W}_{j-1} \end{aligned}$$

    or

    $$\begin{aligned} Q_j u &= \sum_i (u, \tilde{\phi}_{0,i}) \phi_{0,i} + \sum_{\ell=0}^{j-1} (u, \tilde{\psi}_{\ell,i}) \psi_{\ell,i} \\ \tilde{Q}_j u &= \sum_i (u, \phi_{0,i}) \tilde{\phi}_{0,i} + \sum_{\ell=0}^{j-1} (u, \psi_{\ell,i}) \tilde{\psi}_{\ell,i} \end{aligned}$$

- We can decompose $L^2(R)$ into

  $$\begin{aligned} L^2(R) &= \bigoplus_{j=-\infty}^{\infty} W_j \\ L^2(R) &= \bigoplus_{j=-\infty}^{\infty} \tilde{W}_j \end{aligned}$$

and

$$
\begin{aligned}
u &= \sum_{j=-\infty}^{\infty} (u, \tilde{\psi}_{j,i}) \psi_{j,i} \\
u &= \sum_{j=-\infty}^{\infty} (u, \psi_{j,i}) \tilde{\psi}_{j,i}
\end{aligned}
$$

- Riesz basis property

$$\begin{aligned}\gamma \sum_{j,i} |\tilde{c}_{j,i}|^2 &\leq \|\sum_{j,i} \tilde{c}_{j,i}\psi_{j,i}\|^2 \leq \Gamma \sum_{j,i} |c_{j,i}|^2 \\ \tilde{\gamma} \sum_{j,i} |c_{j,i}|^2 &\leq \|\sum_{j,i} c_{j,i}\psi_{j,i}\|^2 \leq \tilde{\Gamma} \sum_{j,i} |c_{j,i}|^2\end{aligned}$$

- Approximation power
  1. If $h(z)$ has order d, then $\|Q_j u - u\| \leq O(2^{-jd})$ for smooth u
  2. If $\tilde{h}(z)$ has order $\tilde{d}$, then $\|\tilde{Q}_j u - u\| \leq O(2^{-j\tilde{d}})$ for smooth u

Splitting

We start from the two identities:

- Orthogonal splitting:
  - We start from the identity:

$$z^{-K}\left(\frac{1+z}{2}\right)^{2K} Q_K(z) + (-z)^{-K}\left(\frac{1-z}{2}\right)^{2K} Q_K(-z) = 1$$

  - Orthogonality means $\psi_{j,i}$ forms an orthonormal basis in $L^2(R)$
  - This is equivalent to $\psi = \tilde{\psi}$, or $h(z) = \tilde{h}(z^{-1})$.
  - We can split

$$\begin{aligned} Q_K(z) &= Q(z)Q(z^{-1}) \\ z^{-K}\left(\frac{1+z}{2}\right)^{2K} &= \left(\frac{1+z}{2}\right)^{K}\left(\frac{1+z^{-1}}{2}\right) \end{aligned}$$

- Semi-orthogonal splitting:
  - We start from:

$$z^{-K}\left(\frac{1+z}{2}\right)^{2K} \frac{G_K(z)}{G_K(z^2)} + (-z)^{-K}\left(\frac{1-z}{2}\right)^{2K} \frac{G_K(-z)}{G_K(z^2)} = 1$$

  - Semi-orthogonality means that $V_j = \tilde{V}_j$, $W_j = \tilde{W}_j$, but $\{\psi_{j,i}\}_{i \in Z}$ are not orthogonal in W_j.
  - The condition is Q_j is the orthogonal projection onto V_j. Hence

$$\begin{aligned} \tilde{h}(z) &= z^{-K}\left(\frac{1+z}{2}\right)^{K} \frac{G_K(z)}{G_K(z^2)} \\ h(z) &= \left(\frac{1+z}{2}\right)^{K} \end{aligned}$$

Meaning of wavelet coefficients

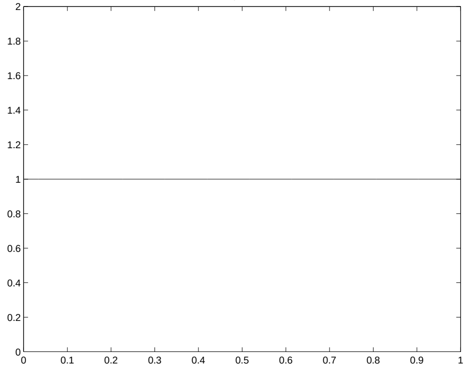
- $c_{j,i} = (u, \phi_{j,i})$: local "averaging" of u at resolution 2^{-j} around $x_{j,i}$

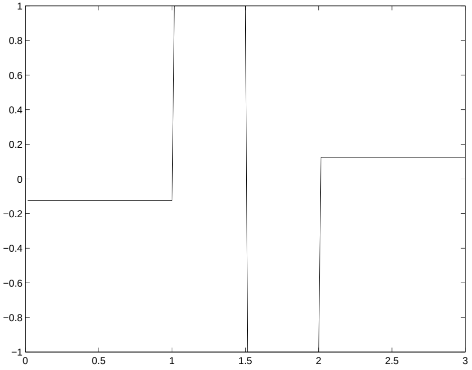
- $d_{j,i} = (u, \phi_{j,i})$: local "differencing" of u at resolution 2^{-j} around $x_{j,i}$

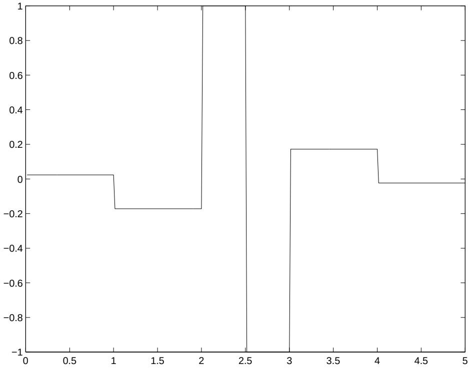
The two important parameters are

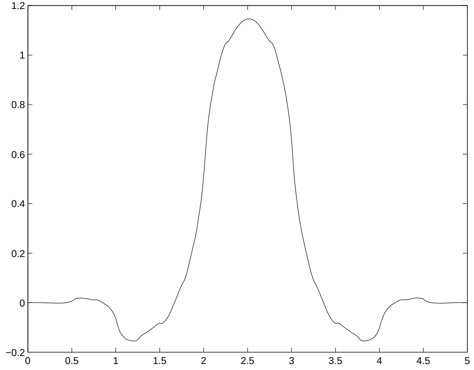
- averaging order: d (the order of h)

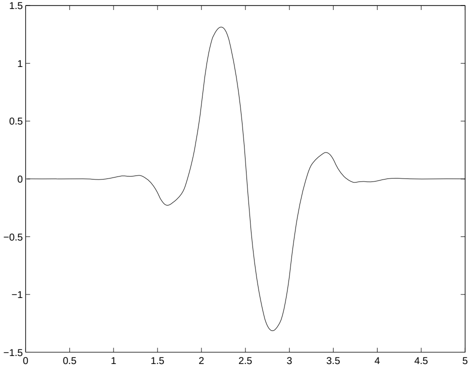
- differencing order: $\tilde{d}$ (the order of $\tilde{h}$). This is usually called the number of vanishing moments in the literatures.

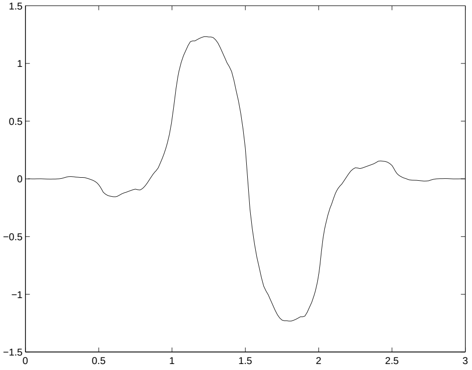

B spline ϕ with order 1

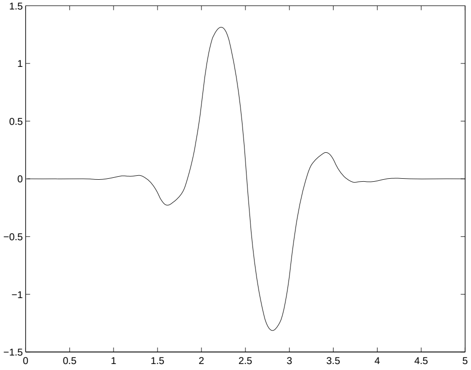

CDF : psi order (1,3);

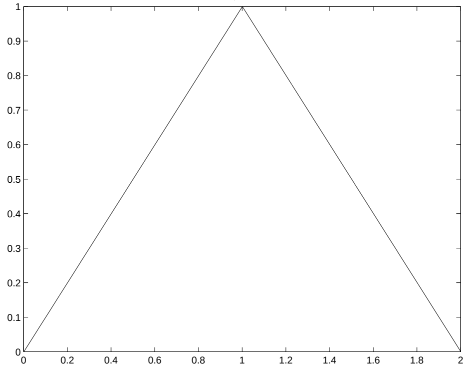

CDF : psi order (1,5);


CDF : tphi order (1,3);

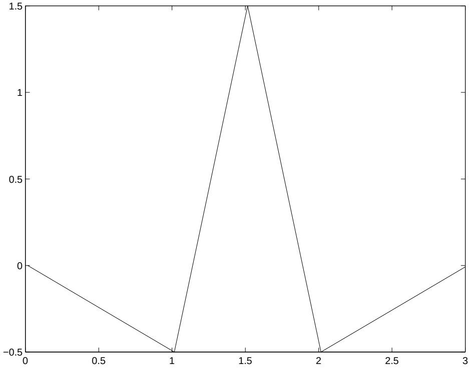

CDF : tpsi order (1,5);


CDF : tpsi order (1,3);

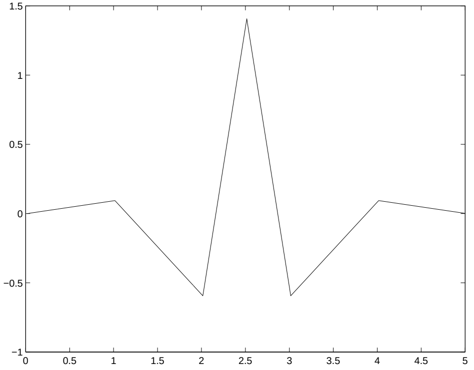

CDF : tpsi order (1,5);



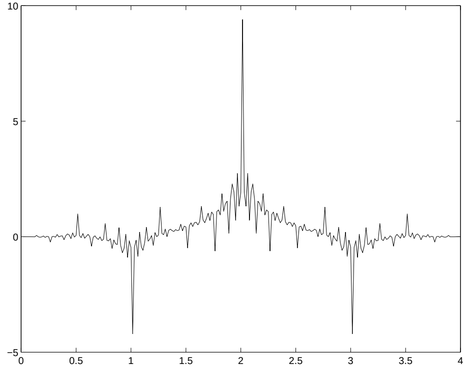
B spline ϕ with order 2



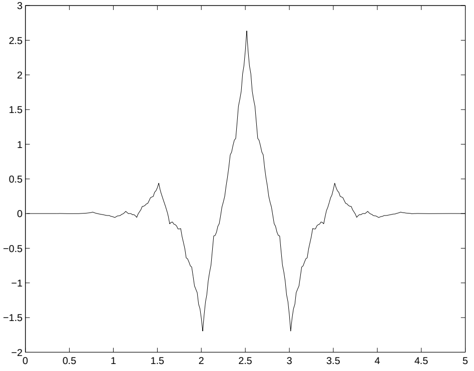
CDF : psi order (2,2);



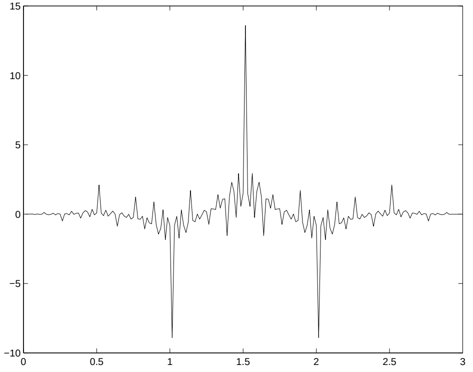
CDF : psi order (2,4);



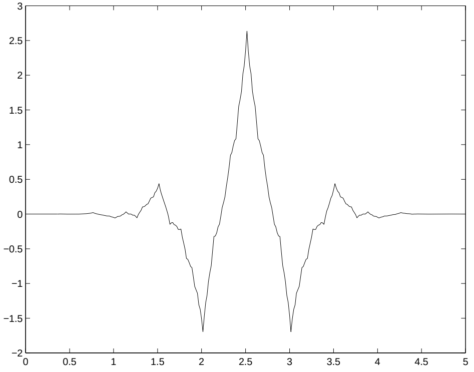
CDF : tphi order (2,2);



CDF : tpsi order (2,4);

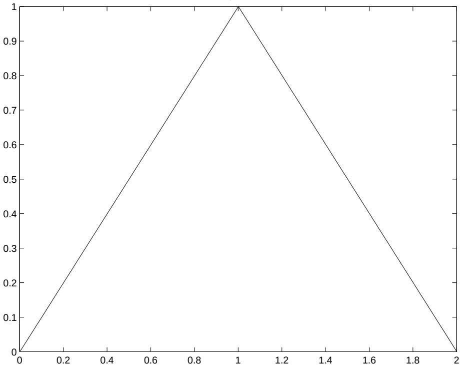


CDF : tpsi order (2,2);

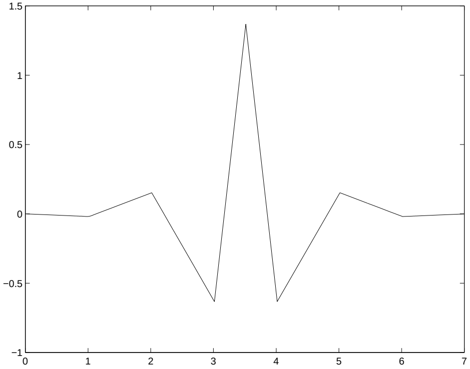


CDF : tpsi order (2,4);

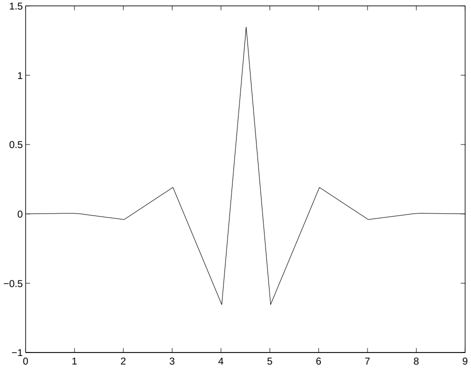
B spline ϕ with order 2



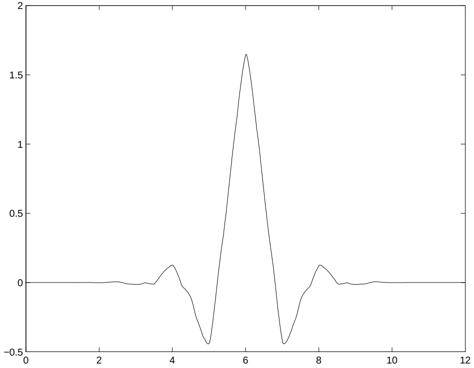
CDF : psi order (2,6);



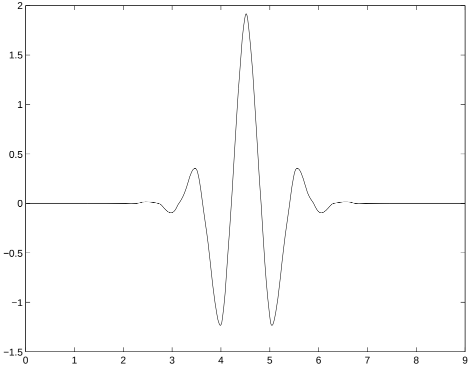
CDF : psi order (2,8);



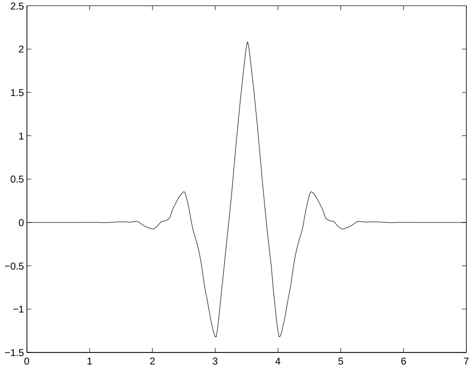
CDF : tphi order (2,6);



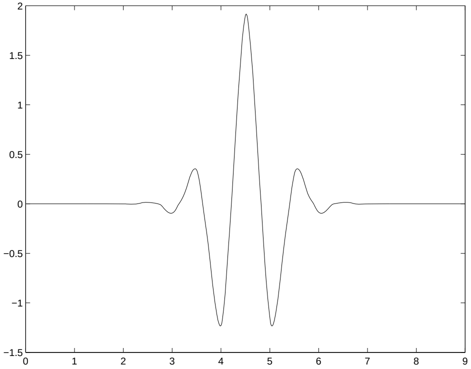
CDF : tpsi order (2,8);



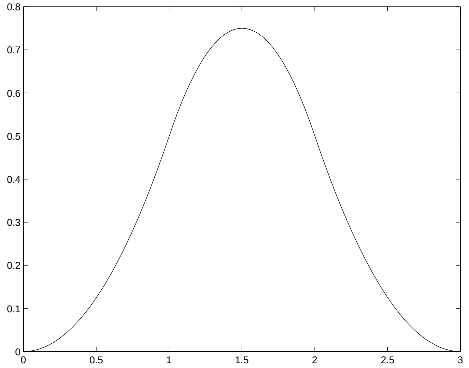
CDF : tpsi order (2,6);



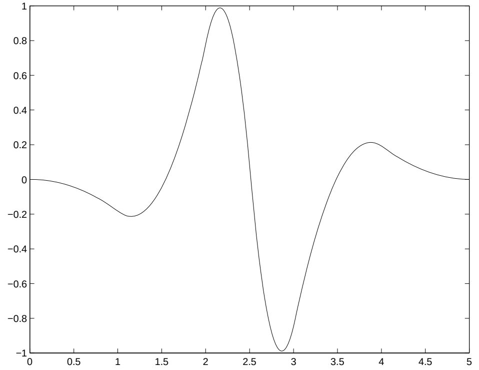
CDF : tpsi order (2,8);



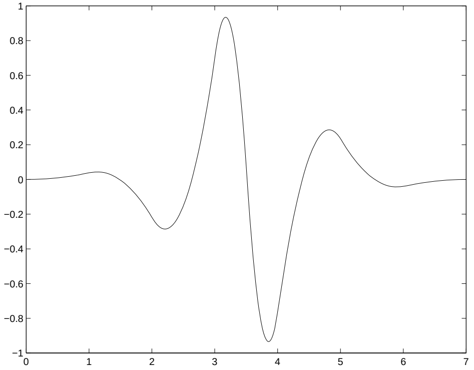
B spline ϕ with order 3



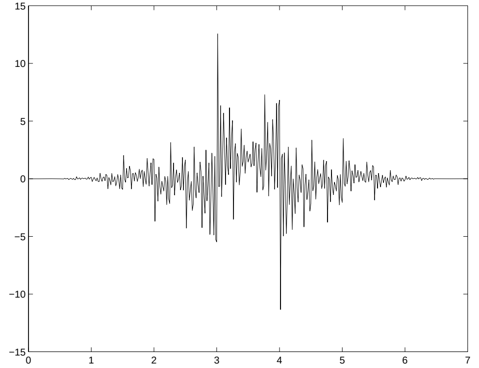
CDF : psi order (3,3);



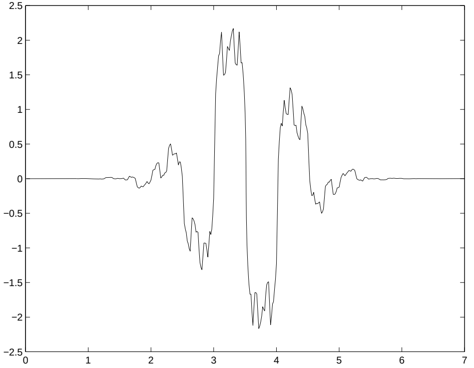
CDF : psi order (3,5);



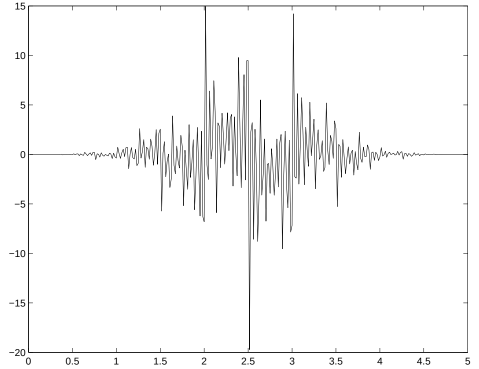
CDF : tphi order (3,3);



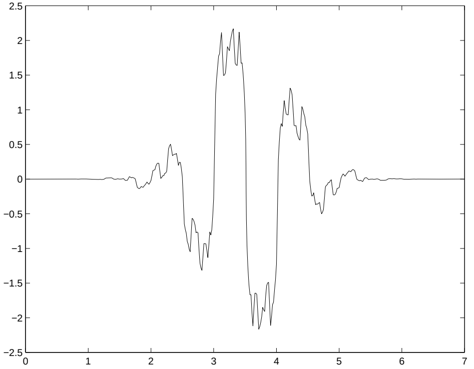
CDF : tpsi order (3,5);



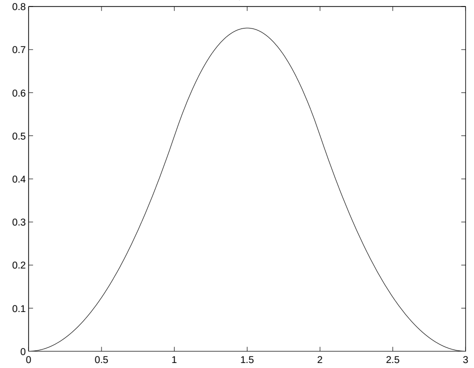
CDF : tpsi order (3,3);



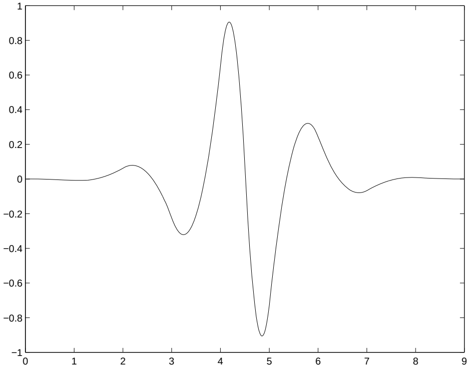
CDF : tpsi order (3,5);



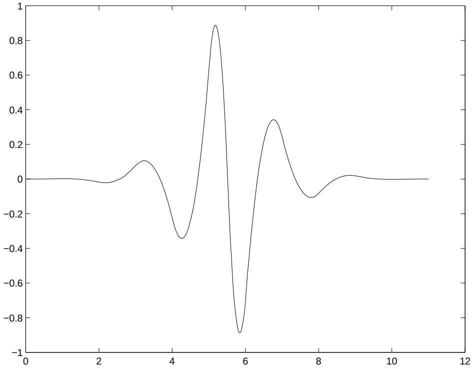
B spline ϕ with order 3



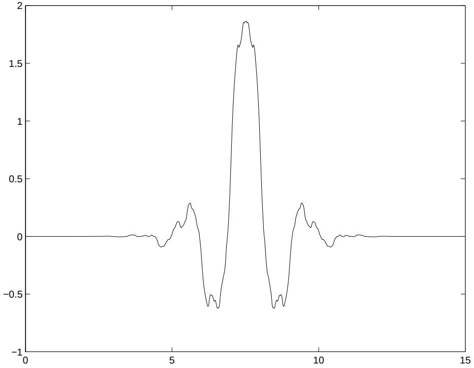
CDF : psi order (3,7);



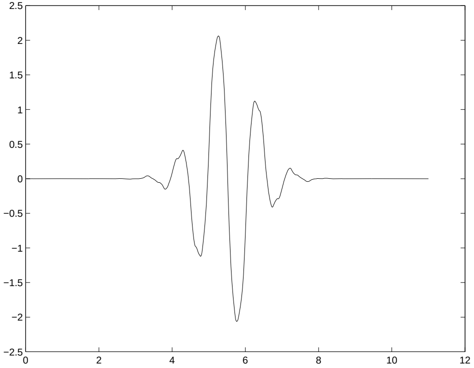
CDF : psi order (3,9);



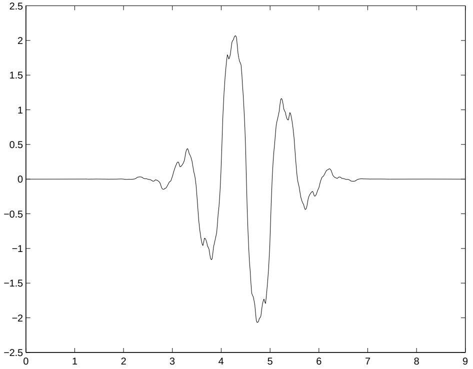
CDF : tphi order (3,7);



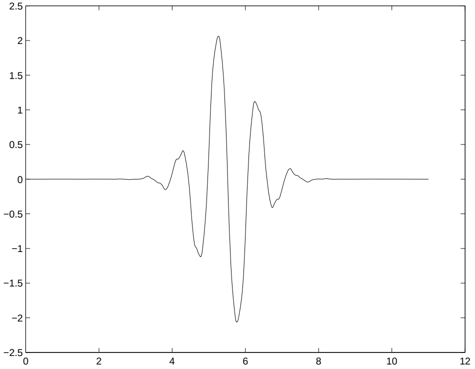
CDF : tpsi order (3,9);

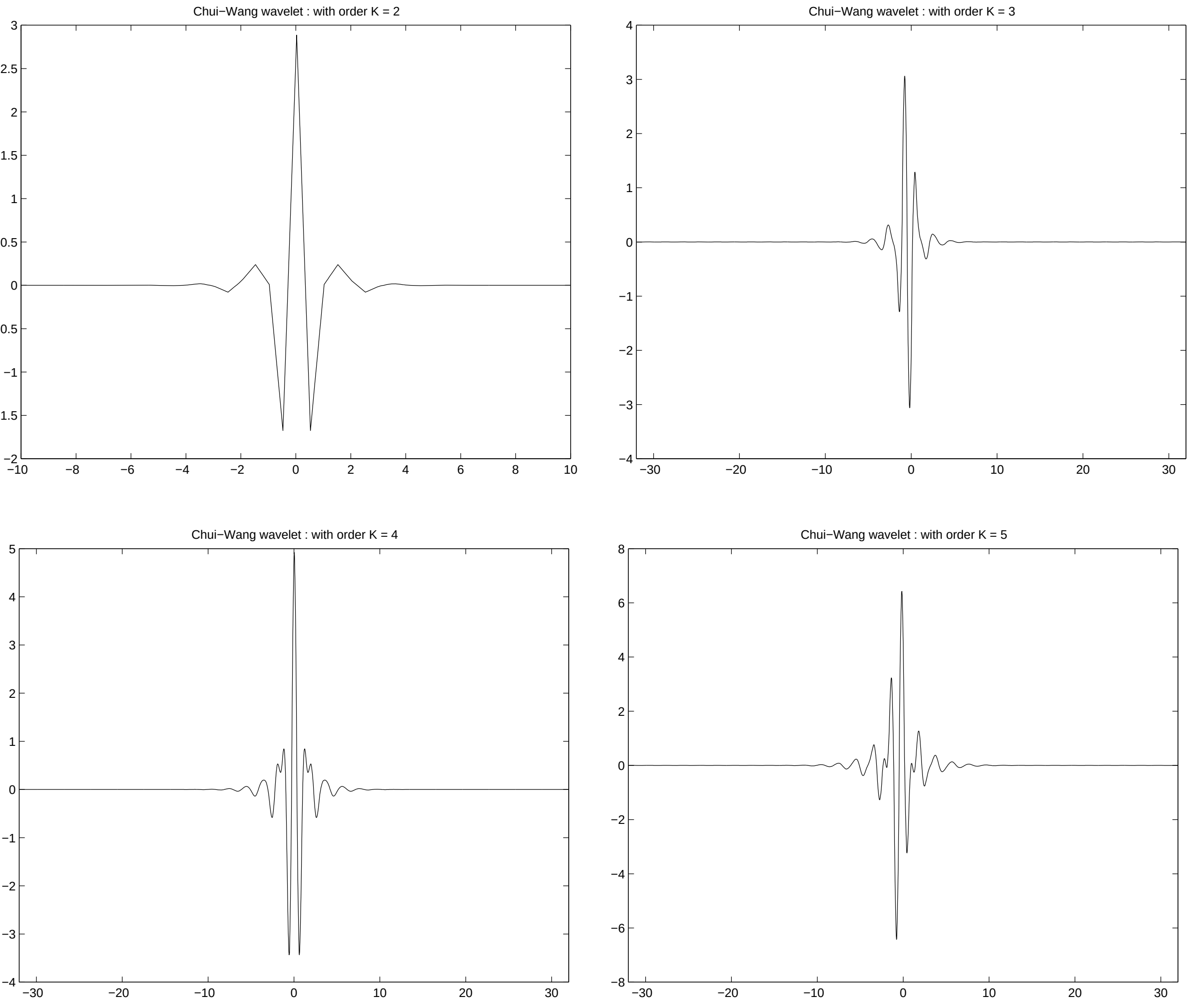


CDF : tpsi order (3,7);



CDF : tpsi order (3,9);





Chui-Wang's wavelets, $K = 2, 3, 4, 5$

Commutation Formula

- Suppose Φ is the refinable function with mask $H(z)$ and ϕ is the refinable function with mask $h(z)$. Then the following are equivalent:

  (i). $H(z) = \left(\frac{1+z^{-1}}{2}\right)^{-1} h(z)$ or $h_k = \frac{H_{k-1}+H_k}{2}$

  (ii). $\frac{d}{dx}\phi_k(x) = \Phi_{k-1}(x) - \Phi_k(x)$

- Suppose

$$\begin{aligned} H(z) &= \left(\frac{1+z^{-1}}{2}\right)^{-1} h(z) \\ \tilde{H}(z) &= \left(\frac{1+z}{2}\right)\tilde{h}(z) \end{aligned}$$

  Suppose h and $\tilde{h}$ are dual to each other. Let $U(x) = \int_{-\infty}^{x} u$ and

$$\begin{aligned} Qu &= \sum_k (u, \phi_k)\tilde{\phi}_k \\ Ru &= \sum_k (U, \Phi_k)\tilde{\Phi}_k \end{aligned}$$

  Then

$$\frac{d}{dx}(Ru) = Qu.$$

- Proof.

  1. We have

$$\begin{aligned} \phi_k' &= \Phi_{k-1} - \Phi_k \\ \tilde{\Phi}_k' &= \tilde{\phi}_k - \tilde{\phi}_{k+1} \end{aligned}$$

  2.

$$(Ru)' = \sum_k (U, \Phi_k)\tilde{\Phi}_k'$$

$$
\begin{aligned}
&= \sum_k (U, \Phi_k)(\tilde{\phi}_k - \tilde{\phi}_{k+1}) \\
&= \sum_k (U, \Phi_k - \Phi_{k-1})\tilde{\phi}_k \\
&= \sum_k (U, -\phi'_k)\tilde{\phi}_k \\
&= Qu
\end{aligned}
$$

- $(U, \Phi_k) = \sum_{m \le k} (u, \phi_m)$

$$
\begin{aligned}
(U, \Phi_k) &= (U, -\sum_{m \le k} \phi'_m) \\
&= \sum_{m \le k} (u, \phi_m)
\end{aligned}
$$

Applications

- Efficient representation for functions
- Study local regularity or singularity of a function
  - Edge (discontinuity) detection of a function
  - Kink (curvature discontinuity) detection of a function
- Efficient representation of integral operators: for instance, we want to evaluate the integral:

  $$\int K(x,y)f(y)\,dy$$

  - K is smooth
  - K is a Calderón operator, e.g. K is the Newtonian potential

  $$K(x,y) = \begin{cases} -\log|x-y| & \text{in 2-d} \\ -\frac{1}{|x-y|} & \text{in 3-d} \end{cases}$$

  Under wavelet transform, K becomes sparse.

- Differential operators $\frac{d^m}{dx^m}$ in wavelet basis
  - sparse
  - diagonal preconditioning
- Signal, image compression, recognition, restoration.

Suggestion of reading:

- **Web Sites:**
  1. wavelet digest: http://www.wavelet.org/wavelet/index.html
  2. multigrid: http://na.cs.yale.edu/mgnet/www/ngnet
  3. Dahmen: http://elc2.igpm.rwth-aachen.de/ dahmen/
- **Books:**
  1. Daubechies: Ten Lectures on Wavelets (SIAM)
  2. Cohen:
  3. Chui: An Introduction to Wavelets (Academic Press)
  4. Chui:
  5. Strang and Nguyen: Wavelets and Filter Banks
  6. Meyer:
- **Papers and Books**
  - Basic Notion of Fourier Analysis:
    1. Courant and John: An Introduction to Calculus and Analysis (V. 1, Ch. 8, V. 2, Ch. 4.13)
  - For Biorthogonal wavelet: see the original paper of Cohen-Daubechies-Feauveau [**?**], or Cohen's book
  - For Orthogonal wavelet: either see Daubechies' original paper [**?**] or her Ten Lectures Note [**?**]
  - For semi-orthogonal wavelet: see Chui-Wang's original paper [**?**] or Chui's books [**?**]
  - For Riesz basis property and regularity (L^2-theory): see Villemoes' paper [**?**]
  - Riesz basis property and Stability (general theory): Dahmen
  - Approximation power: see Unser's paper [**?**]

- Spline theory: see Kincaid and Cheney's book on Numerical Analysis. In the Chapter on Approximation Theory, they have three sections on Spline and B-spline. For more technical book, see de Boor's book on Spline.
- Subdivision Schemes: see original paper of [**?**], the paper of Daubechies-Guskov-Sweldens [**?**] and some references therein. See also Dahmen and Michelli's papers.
- Construction of filter banks, see
    1. Sweldens' Lifting scheme [**?**]
    2. Dahmen's Local decomposition paper [**?**]
- Integral integral and fast matrix-vector multiplication: see B-C-R [**?**]
- Application to PDEs: see Dahmen's web site.
- Image processing, see Mallat, Vetterli

99

G. Beylkin, R. Coifman, and V. Rokhlin, Fast wavelet transforms and numerical algorithms, *Comm. Pure Appl. Math.* **44**, (1991) p.141-183 .

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