1 Harmonic Functions

1.1 The origins of Laplace equation and Poisson equation

The Laplace equation has the form

$$\triangle u = 0$$

The Poisson equation has the form

$$\triangle u = f$$

1. Electrostatics

   $$\triangle \phi = -4\pi\rho$$

   where ϕ is the electric potential and ρ the charge density.

2. Incompressible, irrotational flow.

   In fluid dynamics, the primitive variables are ρ (density), v (velocity), and e (internal energy). The basic physical laws are the conservation of mass, momentum and energy. If there is not much heat transfer, we may drop the energy equation. The entropy is treated as a constant and the pressure p is only a function of ρ. The governing equations are

   $$\rho_t + \nabla \cdot (\rho v) = 0$$

   $$(\rho v)_t + \nabla \cdot (\rho v \otimes v) + \nabla p = 0.$$

   For incompressible flow, we have

   $$\frac{d}{dt}\rho := \rho_t + v \cdot \nabla \rho = 0.$$

   In this case, the first equation is reduced to

   $$\nabla \cdot v = 0$$

   Using the first equation, the second equation is reduced,

   $$v_t + v \cdot \nabla v + \frac{\nabla p}{\rho} = 0.$$

   Since p is only a function of ρ, we may define a potential such that $\nabla \Phi = (\nabla p(\rho))/\rho$. By taking $\nabla \times$, and using $v \cdot \nabla v = \nabla(v^2)/2 + (\nabla \times v) \times v$, we get

   $$\omega_t + \nabla \times (\omega \times v) = 0.$$

   Using

   $$\nabla \times (A \times B) = A\nabla \cdot B - B\nabla \cdot A + (B \cdot \nabla)A - (A \cdot \nabla)B$$

   we get

   $$\omega_t + (v \cdot \nabla)\omega - (\omega \cdot \nabla)v = 0.$$

   If $\omega(x, 0) = 0$, then so is $\omega(x, t)$ for all $t > 0$.

   So, the incompressible, irrotational flow has the following governing equation

   $$\nabla \cdot v = 0, \ \nabla \times v = 0.$$

   In particular, if the flow is in 2 dimension, then v satisfies Cauchy-Riemann equation.

3. A complex valued function $f(z) = u(x+iy) + iv(x+iy)$ is said differentiable at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

If f is differentiable in a domain D, then u and v satisfy the Cauchy-Riemann equation

$$u_x = v_y, \; u_y = -v_x$$

By eliminating one unknown, we obtain that both u and v satisfy the Laplace equation in D. To see this, we take limit $z = z_0 + h$ with $h \in \mathbb{R}$, we get

$$f'(z_0) = u_x + iv_x$$

On the other hand, we may take $z = z_0 + ik$ with $k \in \mathbb{R}$. We get

$$f'(z) = \frac{1}{i}(u_y + iv_y)$$

Hence, we have $u_x = v_y$ and $u_y = -v_x$.

4. Let $u_{i,j}$ be the probability density that a particle located at site (i, j). The random motion of the particle is governed by

$$u_{i,j} = \frac{1}{4}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right)$$

Divide the boundary into two parts $C_1 \cup C_2$. The randam particle starts from C_1. When the particle jumps to the boundary C_2 it stops. The boundary condition becomes

$$u = 1 \text{ on } C_1, \; u = 0 \text{ on } C_2,$$

Another situation is that $u(i_0, j_0) = 1$, we assume $u(i, j) = 0$ at boundary points. We look for stationary solution.

1.2 Maximal principle

Theorem 1.1 *If u is a C^2 solution of*

$$\triangle u = 0 \textit{ in } D$$

where D is bounded, then

$$\max_{x \in D} u(x) = \max_{x \in \partial D} u(x)$$

Proof. Let us first assume $\triangle u > 0$ in D. If maximum of u occurs at x_0 inside D, then $\triangle u(x_0) < 0$. This is a contradiction. Next, let $v = u + \epsilon |x|^2$. Then $\triangle v > 0$. Hence, we have

$$\max_{x \in D} v(x) = \max_{x \in \partial D} v(x)$$

This yields

$$\max_{x \in D} u(x) \le \max_{x \in \partial D} v(x) = \max_{x \in \partial D} v(x) \le \max_{x \in \partial D} u(x) + \epsilon l^2$$

where l is the diameter of D. We take $\epsilon \to 0$ to complete the proof. ▮

Corollary 1.1 (Uniqueness) *The solution of the Dirichlet boundary value problem for the Poisson equation is unique.*

This is valid for the general elliptic equation.

1.3 Special solution for special domains

1.3.1 Rectangles

The domain is $[0,a]\times[0,b]$. The equation is

$$\triangle u = 0.$$

The boundary condition is

$$u(0,y)=j(y),\ u_x(a,y)=k(y),\ u(x,0)=g(y),\ u(x,b)=h(y)$$

Use separation of variable

1.3.2 Disk

$$\triangle u = 0 \text{ for } r<a$$

$$u(a,\theta)=h(\theta)$$

We write $u=R(r)\Theta(\theta)$, we get

$$\begin{aligned} 0 &= u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta} \\ &= R''\Theta+\frac{1}{r}R'\Theta+\frac{1}{r^2}\Theta'' \end{aligned}$$

We get

$$\begin{aligned} \Theta''+\lambda\Theta &= 0 \\ r^2R''+rR'-\lambda R &= 0 \end{aligned}$$

The boundary for Θ is

$$\Theta(\theta+2\pi)=\Theta(\theta)$$

Thus, $\lambda=n^2$ and $\Theta=A\cos n\theta+B\sin n\theta$. The equation for R is of the form $R(r)=r^\alpha$. We get $\alpha=\pm n$ for $\lambda=n^2$ with $n\neq 0$. For $n=0$, the two independent solutions are $R=C$ and $R=D\log r$. For smooth solution in D, the solution r^{-n} and $\log r$ are excluded. Thus, general smooth solutions are of the form

$$\begin{aligned} u &= \frac{1}{2}A_0+\sum_{n=1}^{\infty} r^n\left(A_n\cos n\theta+B_n\sin n\theta\right) \\ &= \sum_{n=0}^{\infty} c_n r^n e^{in\theta} \end{aligned}$$

To fit the boundary condition, we need to require

$$h(\theta)=\frac{1}{2}A_0+\sum_{n=1}^{\infty} a^n\left(A_n\cos n\theta+B_n\sin n\theta\right).$$

Thus,

$$\begin{aligned} A_n &= \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\cos n\phi\, d\phi \\ B_n &= \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\sin n\phi\, d\phi \end{aligned}$$

Use this, we obtain

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)G(r,\theta-\phi)\,d\phi$$

where

$$\begin{aligned} G(r,\theta) &= 1 + 2Re\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{in\theta} \\ &= \frac{a^2-r^2}{a^2-2ar\cos\theta+r^2} \end{aligned}$$

Another way is to write

$$u(x) = \frac{a^2-|x|^2}{2\pi a}\int_{|x'|=a}\frac{u(x')}{|x-x'|^2}\,ds'$$