3. (a) $y = \dfrac{x^4}{16} + \dfrac{1}{2x^2} = \dfrac{1}{16}x^4 + \dfrac{1}{2}x^{-2} \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{4}x^3 - x^{-3} \quad \Rightarrow$

$1 + (dy/dx)^2 = 1 + \left(\frac{1}{4}x^3 - x^{-3}\right)^2 = 1 + \frac{1}{16}x^6 - \frac{1}{2} + x^{-6} = \frac{1}{16}x^6 + \frac{1}{2} + x^{-6} = \left(\frac{1}{4}x^3 + x^{-3}\right)^2.$

Thus, $L = \int_1^2 \left(\frac{1}{4}x^3 + x^{-3}\right) dx = \left[\frac{1}{16}x^4 - \frac{1}{2}x^{-2}\right]_1^2 = \left(1 - \frac{1}{8}\right) - \left(\frac{1}{16} - \frac{1}{2}\right) = \frac{21}{16}.$

(b) $S = \int_1^2 2\pi x\left(\frac{1}{4}x^3 + x^{-3}\right) dx = 2\pi \int_1^2 \left(\frac{1}{4}x^4 + x^{-2}\right) dx = 2\pi\left[\frac{1}{20}x^5 - \frac{1}{x}\right]_1^2$

$= 2\pi\left[\left(\frac{32}{20} - \frac{1}{2}\right) - \left(\frac{1}{20} - 1\right)\right] = 2\pi\left(\frac{8}{5} - \frac{1}{2} - \frac{1}{20} + 1\right) = 2\pi\left(\frac{41}{20}\right) = \frac{41}{10}\pi$

5. $y = e^{-x^2} \quad \Rightarrow \quad dy/dx = -2xe^{-x^2} \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + 4x^2e^{-2x^2}$. Let $f(x) = \sqrt{1 + 4x^2e^{-2x^2}}$. Then

$$L = \int_0^3 f(x)\,dx \approx S_6 = \frac{(3-0)/6}{3}\left[f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + f(3)\right] \approx 3.292287$$

6. $S = \int_0^3 2\pi y\,ds = \int_0^3 2\pi e^{-x^2}\sqrt{1 + 4x^2e^{-2x^2}}\,dx$. Let $g(x) = 2\pi e^{-x^2}\sqrt{1 + 4x^2e^{-2x^2}}$. Then

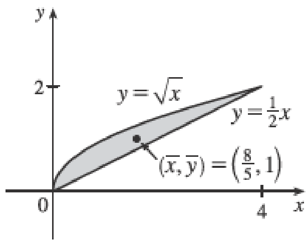
$$S = \int_0^3 g(x)\,dx \approx S_6 = \frac{(3-0)/6}{3}\left[g(0) + 4g(0.5) + 2g(1) + 4g(1.5) + 2g(2) + 4g(2.5) + g(3)\right] \approx 6.648327.$$

7. $y = \int_1^x \sqrt{\sqrt{t} - 1}\,dt \quad \Rightarrow \quad dy/dx = \sqrt{\sqrt{x} - 1} \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + \left(\sqrt{x} - 1\right) = \sqrt{x}.$

Thus, $L = \int_1^{16} \sqrt{\sqrt{x}}\,dx = \int_1^{16} x^{1/4}\,dx = \frac{4}{5}\left[x^{5/4}\right]_1^{16} = \frac{4}{5}(32 - 1) = \frac{124}{5}.$

8. $S = \int_1^{16} 2\pi x\,ds = 2\pi \int_1^{16} x \cdot x^{1/4}\,dx = 2\pi \int_1^{16} x^{5/4}\,dx = 2\pi \cdot \frac{4}{9}\left[x^{9/4}\right]_1^{16} = \frac{8\pi}{9}(512 - 1) = \frac{4088}{9}\pi$

11. $A = \int_0^4 \left(\sqrt{x} - \frac{1}{2}x\right) dx = \left[\frac{2}{3}x^{3/2} - \frac{1}{4}x^2\right]_0^4 = \frac{16}{3} - 4 = \frac{4}{3}$



$\overline{x} = \frac{1}{A}\int_0^4 x\left(\sqrt{x} - \frac{1}{2}x\right) dx = \frac{3}{4}\int_0^4 \left(x^{3/2} - \frac{1}{2}x^2\right) dx$

$= \frac{3}{4}\left[\frac{2}{5}x^{5/2} - \frac{1}{6}x^3\right]_0^4 = \frac{3}{4}\left(\frac{64}{5} - \frac{64}{6}\right) = \frac{3}{4}\left(\frac{64}{30}\right) = \frac{8}{5}$

$\overline{y} = \frac{1}{A}\int_0^4 \frac{1}{2}\left[\left(\sqrt{x}\right)^2 - \left(\frac{1}{2}x\right)^2\right] dx = \frac{3}{4}\int_0^4 \frac{1}{2}\left(x - \frac{1}{4}x^2\right) dx = \frac{3}{8}\left[\frac{1}{2}x^2 - \frac{1}{12}x^3\right]_0^4 = \frac{3}{8}\left(8 - \frac{16}{3}\right) = \frac{3}{8}\left(\frac{8}{3}\right) = 1$

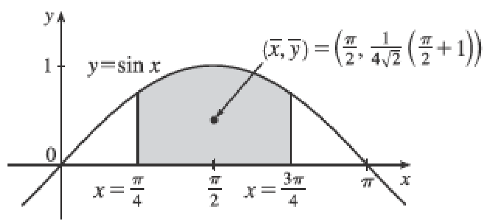
Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{8}{5}, 1\right)$.

12. From the symmetry of the region, $\overline{x} = \frac{\pi}{2}$. $\quad A = \int_{\pi/4}^{3\pi/4} \sin x\,dx = \left[-\cos x\right]_{\pi/4}^{3\pi/4} = \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) = \sqrt{2}$

$\overline{y} = \dfrac{1}{A}\displaystyle\int_{\pi/4}^{3\pi/4} \frac{1}{2}\sin^2 x\,dx = \dfrac{1}{A}\displaystyle\int_{\pi/4}^{3\pi/4} \frac{1}{4}(1 - \cos 2x)\,dx$

$= \dfrac{1}{4\sqrt{2}}\left[x - \frac{1}{2}\sin 2x\right]_{\pi/4}^{3\pi/4}$

$= \dfrac{1}{4\sqrt{2}}\left[\frac{3\pi}{4} - \frac{1}{2}(-1) - \frac{\pi}{4} + \frac{1}{2}\cdot 1\right] = \frac{1}{4\sqrt{2}}\left(\frac{\pi}{2} + 1\right)$



Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{\pi}{2}, \frac{1}{4\sqrt{2}}\left(\frac{\pi}{2} + 1\right)\right) \approx (1.57, 0.45)$.

19. $f(x) = \begin{cases} \frac{\pi}{20}\sin\left(\frac{\pi}{10}x\right) & \text{if } 0 \le x \le 10 \\ 0 & \text{if } x < 0 \text{ or } x > 10 \end{cases}$

(a) $f(x) \ge 0$ for all real numbers x and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{10} \frac{\pi}{20}\sin\left(\frac{\pi}{10}x\right)dx = \frac{\pi}{20}\cdot\frac{10}{\pi}\left[-\cos\left(\frac{\pi}{10}x\right)\right]_0^{10} = \frac{1}{2}(-\cos\pi + \cos 0) = \frac{1}{2}(1+1) = 1$$

Therefore, f is a probability density function.

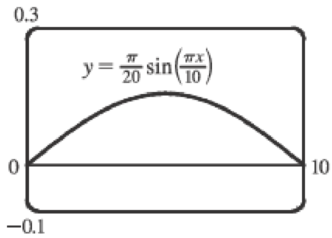
(b) $P(X < 4) = \int_{-\infty}^{4} f(x)\,dx = \int_0^4 \frac{\pi}{20}\sin\left(\frac{\pi}{10}x\right)dx = \frac{1}{2}\left[-\cos\left(\frac{\pi}{10}x\right)\right]_0^4 = \frac{1}{2}\left(-\cos\frac{2\pi}{5} + \cos 0\right)$

$$\approx \tfrac{1}{2}(-0.309017 + 1) \approx 0.3455$$

(c) $\mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{10} \frac{\pi}{20}x\sin\left(\frac{\pi}{10}x\right)dx$

$= \int_0^{\pi} \frac{\pi}{20}\cdot\frac{10}{\pi}u(\sin u)\left(\frac{10}{\pi}\right)du \qquad \left[u = \frac{\pi}{10}x,\ du = \frac{\pi}{10}\,dx\right]$

$= \frac{5}{\pi}\int_0^{\pi} u\sin u\,du \overset{82}{=} \frac{5}{\pi}\left[\sin u - u\cos u\right]_0^{\pi} = \frac{5}{\pi}[0 - \pi(-1)] = 5$



This answer is expected because the graph of f is symmetric about the line $x = 5$.

20. $P(250 \le X \le 280) = \displaystyle\int_{250}^{280} \frac{1}{15\sqrt{2\pi}} \exp\left(\frac{-(x-268)^2}{2\cdot 15^2}\right)dx \approx 0.673$. Thus, the percentage of pregnancies that last between 250 and 280 days is about 67.3%.

21. (a) The probability density function is $f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{8}e^{-t/8} & \text{if } t \ge 0 \end{cases}$

$P(0 \le X \le 3) = \int_0^3 \frac{1}{8}e^{-t/8}\,dt = \left[-e^{-t/8}\right]_0^3 = -e^{-3/8} + 1 \approx 0.3127$

(b) $P(X > 10) = \int_{10}^{\infty} \frac{1}{8}e^{-t/8}\,dt = \lim\limits_{x\to\infty}\left[-e^{-t/8}\right]_{10}^{x} = \lim\limits_{x\to\infty}\left(-e^{-x/8} + e^{-10/8}\right) = 0 + e^{-5/4} \approx 0.2865$

(c) We need to find m such that $P(X \ge m) = \frac{1}{2} \;\Rightarrow\; \int_m^{\infty} \frac{1}{8}e^{-t/8}\,dt = \frac{1}{2} \;\Rightarrow\; \lim\limits_{x\to\infty}\left[-e^{-t/8}\right]_m^x = \frac{1}{2} \;\Rightarrow$

$\lim\limits_{x\to\infty}\left(-e^{-x/8} + e^{-m/8}\right) = \frac{1}{2} \;\Rightarrow\; e^{-m/8} = \frac{1}{2} \;\Rightarrow\; -m/8 = \ln\frac{1}{2} \;\Rightarrow\; m = -8\ln\frac{1}{2} = 8\ln 2 \approx 5.55$ minutes.