7. $\left\{\left(1+\frac{3}{n}\right)^{4n}\right\}$ is convergent. Let $y=\left(1+\frac{3}{x}\right)^{4x}$. Then

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}4x\ln(1+3/x)=\lim_{x\to\infty}\frac{\ln(1+3/x)}{1/(4x)}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{\frac{1}{1+3/x}\left(-\frac{3}{x^2}\right)}{-1/(4x^2)}=\lim_{x\to\infty}\frac{12}{1+3/x}=12,\text{ so}$$

$$\lim_{x\to\infty}y=\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^{4n}=e^{12}.$$

9. We use induction, hypothesizing that $a_{n-1}<a_n<2$. Note first that $1<a_2=\frac{1}{3}(1+4)=\frac{5}{3}<2$, so the hypothesis holds for $n=2$. Now assume that $a_{k-1}<a_k<2$. Then $a_k=\frac{1}{3}(a_{k-1}+4)<\frac{1}{3}(a_k+4)<\frac{1}{3}(2+4)=2$. So $a_k<a_{k+1}<2$, and the induction is complete. To find the limit of the sequence, we note that $L=\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}a_{n+1}\ \Rightarrow$ $L=\frac{1}{3}(L+4)\ \Rightarrow\ L=2$.

13. $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim\limits_{n\to\infty}\left[\frac{(n+1)^3}{5^{n+1}}\cdot\frac{5^n}{n^3}\right]=\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^3\cdot\frac{1}{5}=\frac{1}{5}<1$, so $\sum\limits_{n=1}^{\infty}\frac{n^3}{5^n}$ converges by the Ratio Test.

14. Let $b_n=\frac{1}{\sqrt{n+1}}$. Then b_n is positive for $n\geq1$, the sequence $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty}b_n=0$, so the series $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$ converges by the Alternating Series Test.

16. $\lim\limits_{n\to\infty}\frac{n}{3n+1}=\frac{1}{3}$, so $\lim\limits_{n\to\infty}\ln\left(\frac{n}{3n+1}\right)=\ln\frac{1}{3}\neq0$. Thus, the series $\sum\limits_{n=1}^{\infty}\ln\left(\frac{n}{3n+1}\right)$ diverges by the Test for Divergence.

19. $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim\limits_{n\to\infty}\frac{1\cdot3\cdot5\cdot\dots\cdot(2n-1)(2n+1)}{5^{n+1}\,(n+1)!}\cdot\frac{5^n\,n!}{1\cdot3\cdot5\cdot\dots\cdot(2n-1)}=\lim\limits_{n\to\infty}\frac{2n+1}{5(n+1)}=\frac{2}{5}<1$, so the series converges by the Ratio Test.

22. Use the Limit Comparison Test with $a_n=\frac{\sqrt{n+1}-\sqrt{n-1}}{n}=\frac{2}{n\left(\sqrt{n+1}+\sqrt{n-1}\right)}$ (rationalizing the numerator) and $b_n=\frac{1}{n^{3/2}}$. $\lim\limits_{n\to\infty}\frac{a_n}{b_n}=\lim\limits_{n\to\infty}\frac{2\sqrt{n}}{\sqrt{n+1}+\sqrt{n-1}}=1$, so since $\sum\limits_{n=1}^{\infty}b_n$ converges $\left[p=\frac{3}{2}>1\right]$, $\sum\limits_{n=1}^{\infty}a_n$ converges also.

23. Consider the series of absolute values: $\sum\limits_{n=1}^{\infty}n^{-1/3}$ is a p-series with $p=\frac{1}{3}\leq1$ and is therefore divergent. But if we apply the Alternating Series Test, we see that $b_n=\frac{1}{\sqrt[3]{n}}>0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty}b_n=0$, so the series $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\,n^{-1/3}$ converges. Thus, $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\,n^{-1/3}$ is conditionally convergent.

24. $\sum\limits_{n=1}^{\infty}\left|(-1)^{n-1}\,n^{-3}\right|=\sum\limits_{n=1}^{\infty}n^{-3}$ is a convergent p-series $[p=3>1]$. Therefore, $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\,n^{-3}$ is absolutely convergent.

25. $\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{(-1)^{n+1}(n+2)3^{n+1}}{2^{2n+3}}\cdot\frac{2^{2n+1}}{(-1)^n(n+1)3^n}\right|=\frac{n+2}{n+1}\cdot\frac{3}{4}=\frac{1+(2/n)}{1+(1/n)}\cdot\frac{3}{4}\to\frac{3}{4}<1$ as $n\to\infty$, so by the Ratio Test, $\sum\limits_{n=1}^{\infty}\frac{(-1)^n(n+1)3^n}{2^{2n+1}}$ is absolutely convergent.

29. $\sum_{n=1}^{\infty} [\tan^{-1}(n+1) - \tan^{-1} n] = \lim_{n\to\infty} s_n$

$$= \lim_{n\to\infty} [(\tan^{-1} 2 - \tan^{-1} 1) + (\tan^{-1} 3 - \tan^{-1} 2) + \cdots + (\tan^{-1}(n+1) - \tan^{-1} n)]$$

$$= \lim_{n\to\infty} [\tan^{-1}(n+1) - \tan^{-1} 1] = \tfrac{\pi}{2} - \tfrac{\pi}{4} = \tfrac{\pi}{4}$$

42. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}(x-2)^{n+1}}{(n+3)!} \cdot \frac{(n+2)!}{2^n(x-2)^n}\right| = \lim_{n\to\infty} \frac{2}{n+3}|x-2| = 0 < 1$, so the series $\sum_{n=1}^{\infty} \frac{2^n(x-2)^n}{(n+2)!}$ converges for all x. $R = \infty$ and $I = (-\infty, \infty)$.

48. $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ with interval of convergence $[-1, 1]$, so

$\tan^{-1}(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$, which converges when $x^2 \in [-1, 1] \quad \Leftrightarrow \quad x \in [-1, 1]$.

Therefore, $R = 1$.

49. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1 \quad \Rightarrow \quad \ln(1-x) = -\int \frac{dx}{1-x} = -\int \sum_{n=0}^{\infty} x^n \, dx = C - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$.

$\ln(1-0) = C - 0 \quad \Rightarrow \quad C = 0 \quad \Rightarrow \quad \ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{-x^n}{n}$ with $R = 1$.

55. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, so $\frac{e^x}{x} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n-1}}{n!} = x^{-1} + \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$ and

$\int \frac{e^x}{x} \, dx = C + \ln|x| + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$.

56. $(1+x^4)^{1/2} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (x^4)^n = 1 + \left(\tfrac{1}{2}\right)x^4 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(x^4)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(x^4)^3 + \cdots$

$$= 1 + \tfrac{1}{2}x^4 - \tfrac{1}{8}x^8 + \tfrac{1}{16}x^{12} - \cdots$$

so $\int_0^1 (1+x^4)^{1/2} \, dx = \left[x + \frac{1}{10}x^5 - \frac{1}{72}x^9 + \frac{1}{208}x^{13} - \cdots\right]_0^1 = 1 + \frac{1}{10} - \frac{1}{72} + \frac{1}{208} - \cdots$.

This is an alternating series, so by the Alternating Series Test, the error in the approximation

$\int_0^1 (1+x^4)^{1/2} \, dx \approx 1 + \frac{1}{10} - \frac{1}{72} \approx 1.086$ is less than $\frac{1}{208}$, sufficient for the desired accuracy.

Thus, correct to two decimal places, $\int_0^1 (1+x^4)^{1/2} \, dx \approx 1.09$.