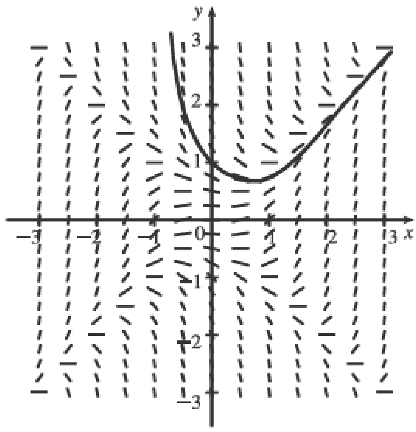
3. (a)



We estimate that when $x = 0.3$, $y = 0.8$, so $y(0.3) \approx 0.8$.

(b) $h = 0.1$, $x_0 = 0$, $y_0 = 1$ and $F(x, y) = x^2 - y^2$. So $y_n = y_{n-1} + 0.1\left(x_{n-1}^2 - y_{n-1}^2\right)$. Thus,

$y_1 = 1 + 0.1\left(0^2 - 1^2\right) = 0.9$, $y_2 = 0.9 + 0.1\left(0.1^2 - 0.9^2\right) = 0.82$, $y_3 = 0.82 + 0.1\left(0.2^2 - 0.82^2\right) = 0.75676$.

This is close to our graphical estimate of $y(0.3) \approx 0.8$.

(c) The centers of the horizontal line segments of the direction field are located on the lines $y = x$ and $y = -x$. When a solution curve crosses one of these lines, it has a local maximum or minimum.

5. $y' = xe^{-\sin x} - y\cos x \;\Rightarrow\; y' + (\cos x)\,y = xe^{-\sin x}$ $(\star)$. This is a linear equation and the integrating factor is $I(x) = e^{\int \cos x\,dx} = e^{\sin x}$. Multiplying $(\star)$ by $e^{\sin x}$ gives $e^{\sin x}\,y' + e^{\sin x}(\cos x)\,y = x \;\Rightarrow\; \left(e^{\sin x}\,y\right)' = x \;\Rightarrow$ $e^{\sin x}\,y = \frac{1}{2}x^2 + C \;\Rightarrow\; y = \left(\frac{1}{2}x^2 + C\right)e^{-\sin x}$.

7. $2ye^{y^2}y' = 2x + 3\sqrt{x} \;\Rightarrow\; 2ye^{y^2}\dfrac{dy}{dx} = 2x + 3\sqrt{x} \;\Rightarrow\; 2ye^{y^2}\,dy = \left(2x + 3\sqrt{x}\right)dx \;\Rightarrow$ $\int 2ye^{y^2}\,dy = \int\left(2x + 3\sqrt{x}\right)dx \;\Rightarrow\; e^{y^2} = x^2 + 2x^{3/2} + C \;\Rightarrow\; y^2 = \ln(x^2 + 2x^{3/2} + C) \;\Rightarrow$ $y = \pm\sqrt{\ln(x^2 + 2x^{3/2} + C)}$

10. $(1 + \cos x)y' = (1 + e^{-y})\sin x \;\Rightarrow\; \dfrac{dy}{1 + e^{-y}} = \dfrac{\sin x\,dx}{1 + \cos x} \;\Rightarrow\; \displaystyle\int \frac{dy}{1 + 1/e^y} = \int \frac{\sin x\,dx}{1 + \cos x} \;\Rightarrow$ $\displaystyle\int \frac{e^y\,dy}{1 + e^y} = \int \frac{\sin x\,dx}{1 + \cos x} \;\Rightarrow\; \ln|1 + e^y| = -\ln|1 + \cos x| + C \;\Rightarrow\; \ln(1 + e^y) = -\ln(1 + \cos x) + C \;\Rightarrow$ $1 + e^y = e^{-\ln(1 + \cos x)} \cdot e^C \;\Rightarrow\; e^y = ke^{-\ln(1 + \cos x)} - 1 \;\Rightarrow\; y = \ln[ke^{-\ln(1 + \cos x)} - 1]$. Since $y(0) = 0$, $0 = \ln[ke^{-\ln 2} - 1] \;\Rightarrow\; e^0 = k\left(\frac{1}{2}\right) - 1 \;\Rightarrow\; k = 4$. Thus, $y(x) = \ln[4e^{-\ln(1 + \cos x)} - 1]$. An equivalent form is $y(x) = \ln\dfrac{3 - \cos x}{1 + \cos x}$.

11. $xy' - y = x\ln x \;\Rightarrow\; y' - \dfrac{1}{x}y = \ln x$. $I(x) = e^{\int(-1/x)\,dx} = e^{-\ln|x|} = \left(e^{\ln|x|}\right)^{-1} = |x|^{-1} = 1/x$ since the condition $y(1) = 2$ implies that we want a solution with $x > 0$. Multiplying the last differential equation by $I(x)$ gives $\dfrac{1}{x}y' - \dfrac{1}{x^2}y = \dfrac{1}{x}\ln x \;\Rightarrow\; \left(\dfrac{1}{x}y\right)' = \dfrac{1}{x}\ln x \;\Rightarrow\; \dfrac{1}{x}y = \displaystyle\int \frac{\ln x}{x}\,dx \;\Rightarrow\; \dfrac{1}{x}y = \frac{1}{2}(\ln x)^2 + C \;\Rightarrow$ $y = \frac{1}{2}x(\ln x)^2 + Cx$. Now $y(1) = 2 \;\Rightarrow\; 2 = 0 + C \;\Rightarrow\; C = 2$, so $y = \frac{1}{2}x(\ln x)^2 + 2x$.

14. $\dfrac{d}{dx}(y) = \dfrac{d}{dx}\left(e^{kx}\right) \;\Rightarrow\; y' = ke^{kx} = ky = \dfrac{\ln y}{x} \cdot y$, so the orthogonal trajectories must have $y' = -\dfrac{x}{y\ln y} \;\Rightarrow$ $\dfrac{dy}{dx} = -\dfrac{x}{y\ln y} \;\Rightarrow\; y\ln y\,dy = -x\,dx \;\Rightarrow\; \int y\ln y\,dy = -\int x\,dx \;\Rightarrow\; \frac{1}{2}y^2\ln y - \frac{1}{4}y^2$ [parts with $u = \ln y$, $dv = y\,dy$] $= -\frac{1}{2}x^2 + C_1 \;\Rightarrow\; 2y^2\ln y - y^2 = C - 2x^2$.

16. (a) Let $t = 0$ correspond to 1990 so that $P(t) = 5.28e^{kt}$ is a starting point for the model. When $t = 10$, $P = 6.07$.

So $6.07 = 5.28e^{10k} \Rightarrow 10k = \ln \frac{6.07}{5.28} \Rightarrow k = \frac{1}{10} \ln \frac{6.07}{5.28} \approx 0.01394$. For the year 2020, $t = 30$, and

$P(30) = 5.28e^{30k} \approx 8.02$ billion.

(b) $P = 10 \Rightarrow 5.28e^{kt} = 10 \Rightarrow \dfrac{10}{5.28} = e^{kt} \Rightarrow kt = \ln \dfrac{10}{5.28} \Rightarrow t = 10 \dfrac{\ln \frac{10}{5.28}}{\ln \frac{6.07}{5.28}} \approx 45.8$ years, that is,

in $1990 + 45 = 2035$.

(c) $P(t) = \dfrac{K}{1 + Ae^{-kt}} = \dfrac{100}{1 + Ae^{-kt}}$, where $A = \dfrac{100 - 5.28}{5.28} \approx 17.94$. Using $k = \dfrac{1}{10} \ln \dfrac{6.07}{5.28}$ from part (a), a model is

$P(t) \approx \dfrac{100}{1 + 17.94e^{-0.01394t}}$ and $P(30) \approx 7.81$ billion, slightly lower than our estimate of 8.02 billion in part (a).

(d) $P = 10 \Rightarrow 1 + Ae^{-kt} = \frac{100}{10} \Rightarrow Ae^{-kt} = 9 \Rightarrow e^{-kt} = 9/A \Rightarrow -kt = \ln(9/A) \Rightarrow$

$t = -\dfrac{1}{k} \ln \dfrac{9}{A} \approx 49.47$ years (that is, in 2039), which is later than the prediction of 2035 in part (b).

19. Let P represent the population and I the number of infected people. The rate of spread dI/dt is jointly proportional to I and to $P - I$, so for some constant k, $\dfrac{dI}{dt} = kI(P - I) = (kP)I\left(1 - \dfrac{I}{P}\right)$. From Equation 9.4.7 with $K = P$ and k replaced by kP, we have $I(t) = \dfrac{P}{1 + Ae^{-kPt}} = \dfrac{I_0 P}{I_0 + (P - I_0)e^{-kPt}}$.

Now, measuring t in days, we substitute $t = 7$, $P = 5000$, $I_0 = 160$ and $I(7) = 1200$ to find k:

$1200 = \dfrac{160 \cdot 5000}{160 + (5000 - 160)e^{-5000 \cdot 7 \cdot k}} \Leftrightarrow 3 = \dfrac{2000}{160 + 4840e^{-35{,}000k}} \Leftrightarrow 480 + 14{,}520e^{-35{,}000k} = 2000 \Leftrightarrow$

$e^{-35{,}000k} = \dfrac{2000 - 480}{14{,}520} \Leftrightarrow -35{,}000k = \ln \dfrac{38}{363} \Leftrightarrow k = \dfrac{-1}{35{,}000} \ln \dfrac{38}{363} \approx 0.00006448$. Next, let

$I = 5000 \times 80\% = 4000$, and solve for t: $4000 = \dfrac{160 \cdot 5000}{160 + (5000 - 160)e^{-k \cdot 5000 \cdot t}} \Leftrightarrow 1 = \dfrac{200}{160 + 4840e^{-5000kt}} \Leftrightarrow$

$160 + 4840e^{-5000kt} = 200 \Leftrightarrow e^{-5000kt} = \dfrac{200 - 160}{4840} \Leftrightarrow -5000kt = \ln \dfrac{1}{121} \Leftrightarrow$

$t = \dfrac{-1}{5000k} \ln \dfrac{1}{121} = \dfrac{1}{\frac{1}{7} \ln \frac{38}{363}} \cdot \ln \dfrac{1}{121} = 7 \cdot \dfrac{\ln 121}{\ln \frac{363}{38}} \approx 14.875$. So it takes about 15 days for 80% of the population to be infected.

21. $\dfrac{dh}{dt} = -\dfrac{R}{V}\left(\dfrac{h}{k + h}\right) \Rightarrow \displaystyle\int \frac{k + h}{h}\, dh = \int \left(-\frac{R}{V}\right) dt \Rightarrow \int \left(1 + \frac{k}{h}\right) dh = -\frac{R}{V} \int 1\, dt \Rightarrow$

$h + k \ln h = -\dfrac{R}{V} t + C$. This equation gives a relationship between h and t, but it is not possible to isolate h and express it in terms of t.

22. $dx/dt = 0.4x - 0.002xy$, $dy/dt = -0.2y + 0.000008xy$

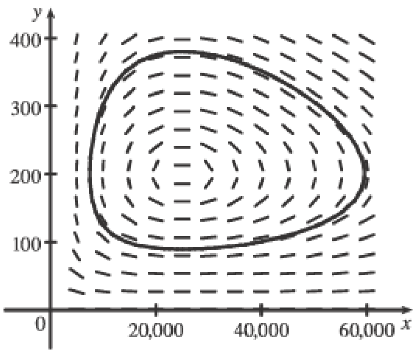
(a) The xy terms represent encounters between the birds and the insects. Since the y-population increases from these terms and the x-population decreases, we expect y to represent the birds and x the insects.

(b) x and y are constant $\Rightarrow$ $x' = 0$ and $y' = 0$ $\Rightarrow$

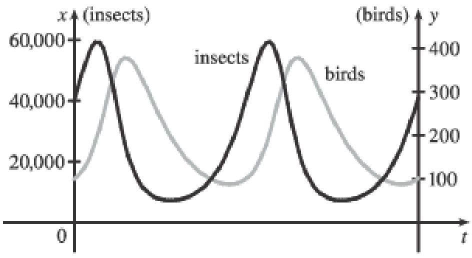
$$\left.\begin{cases} 0 = 0.4x - 0.002xy \\ 0 = -0.2y + 0.000008xy \end{cases}\right\} \Rightarrow \begin{cases} 0 = 0.4x(1 - 0.005y) \\ 0 = -0.2y(1 - 0.00004x) \end{cases} \Rightarrow y = 0 \text{ and } x = 0 \text{ (zero populations)}$$

or $y = \frac{1}{0.005} = 200$ and $x = \frac{1}{0.00004} = 25{,}000$. The non-trivial solution represents the population sizes needed so that there are no changes in either the number of birds or the number of insects.

(c) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-0.2y + 0.000008xy}{0.4x - 0.002xy}$

(d) 

At $(x, y) = (40{,}000, 100)$, $dx/dt = 8000 > 0$, so as t increases we are proceeding in a counterclockwise direction. The populations increase to approximately $(59{,}646, 200)$, at which point the insect population starts to decrease. The birds attain a maximum population of about 380 when the insect population is 25,000. The populations decrease to about $(7370, 200)$, at which point the insect population starts to increase. The birds attain a minimum population of about 88 when the insect population is 25,000, and then the cycle repeats.

(e) 

Both graphs have the same period and the bird population peaks about a quarter-cycle after the insect population.