7.1

2. (a) $f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$ for any y in B. The domain of f^{-1} is B and the range of f^{-1} is A.

(b) See the steps in (5).

(c) Reflect the graph of f about the line $y = x$.

6. The horizontal line $y = 0$ (the x-axis) intersects the graph of f in more than one point. Thus, by the Horizontal Line Test, f is not one-to-one.

20. (a) f is 1-1 because it passes the Horizontal Line Test.

(b) Domain of $f = [-3, 3] =$ Range of f^{-1}. Range of $f = [-1, 3] =$ Domain of f^{-1}.

(c) Since $f(0) = 2$, $f^{-1}(2) = 0$.

(d) Since $f(-1.7) \approx 0$, $f^{-1}(0) = -1.7$.

24. $y = f(x) = \dfrac{4x-1}{2x+3} \quad \Rightarrow \quad y(2x+3) = 4x - 1 \quad \Rightarrow \quad 2xy + 3y = 4x - 1 \quad \Rightarrow \quad 3y + 1 = 4x - 2xy \quad \Rightarrow$

$3y + 1 = (4 - 2y)x \quad \Rightarrow \quad x = \dfrac{3y+1}{4-2y}$. Interchange x and y: $y = \dfrac{3x+1}{4-2x}$. So $f^{-1}(x) = \dfrac{3x+1}{4-2x}$.

38. $f(0) = 2 \quad \Rightarrow \quad f^{-1}(2) = 0$, and $f(x) = x^3 + 3\sin x + 2\cos x \quad \Rightarrow \quad f'(x) = 3x^2 + 3\cos x - 2\sin x$ and $f'(0) = 3$.

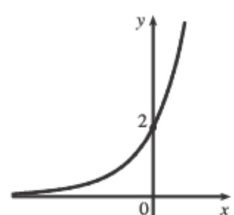
Thus, $(f^{-1})'(2) = \dfrac{1}{f'(f^{-1}(2))} = \dfrac{1}{f'(0)} = \dfrac{1}{3}$.

7.2

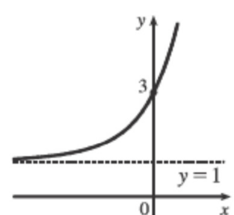
2. (a) The number e is the value of a such that the slope of the tangent line at $x = 0$ on the graph of $y = a^x$ is exactly 1.

(b) $e \approx 2.71828$ (c) $f(x) = e^x$

10. We start with the graph of $y = e^x$ (Figure 12), vertically stretch by a factor of 2, and then shift 1 unit upward. There is a horizontal asymptote of $y = 1$.



$y = 2e^x$



$y = 1 + 2e^x$

16. (a) The sine and exponential functions have domain $\mathbb{R}$, so $g(t) = \sin(e^{-t})$ also has domain $\mathbb{R}$.

(b) The function $g(t) = \sqrt{1 - 2^t}$ has domain $\{t \mid 1 - 2^t \geq 0\} = \{t \mid 2^t \leq 1\} = \{t \mid t \leq 0\} = (-\infty, 0]$.

24. By (3), if $a > 1$, $\lim\limits_{x \to -\infty} a^x = 0$, so $\lim\limits_{x \to -\infty} (1.001)^x = 0$.

48. $y = \dfrac{e^x}{x} \quad \Rightarrow \quad y' = \dfrac{x \cdot e^x - e^x \cdot 1}{x^2} = \dfrac{e^x(x-1)}{x^2}$.

At $(1, e)$, $y' = 0$, and an equation of the tangent line is $y - e = 0(x - 1)$, or $y = e$.

88. Let $r(t) = ae^{bt}$ with $a = 450.268$ and $b = 1.12567$, and $n(t) =$ population after t hours. Since $r(t) = n'(t)$, $\int_0^3 r(t)\,dt = n(3) - n(0)$ is the total change in the population after three hours. Since we start with 400 bacteria, the population will be

$$n(3) = 400 + \int_0^3 r(t)\,dt = 400 + \int_0^3 ae^{bt}\,dt = 400 + \frac{a}{b}\left[e^{bt}\right]_0^3 = 400 + \frac{a}{b}\left(e^{3b} - 1\right)$$
$$\approx 400 + 11{,}313 = 11{,}713 \text{ bacteria}$$

7.3

2. (a) The natural logarithm is the logarithm with base e, denoted $\ln x$.

(b) The common logarithm is the logarithm with base 10, denoted $\log x$.

(c) See Figure 3.

4. (a) $\ln(1/e) = \ln 1 - \ln e = 0 - 1 = -1$

(b) $\log_{10}\sqrt{10} = \log_{10} 10^{1/2} = \frac{1}{2}$ by (2).

26. (a) $e^{2x+3} - 7 = 0 \;\Rightarrow\; e^{2x+3} = 7 \;\Rightarrow\; 2x + 3 = \ln 7 \;\Rightarrow\; 2x = \ln 7 - 3 \;\Rightarrow\; x = \frac{1}{2}(\ln 7 - 3)$

(b) $\ln(5 - 2x) = -3 \;\Rightarrow\; 5 - 2x = e^{-3} \;\Rightarrow\; 2x = 5 - e^{-3} \;\Rightarrow\; x = \frac{1}{2}(5 - e^{-3})$

40. (a) $v(t) = ce^{-kt} \;\Rightarrow\; a(t) = v'(t) = -kce^{-kt} = -kv(t)$

(b) $v(0) = ce^0 = c$, so c is the initial velocity.

(c) $v(t) = ce^{-kt} = c/2 \;\Rightarrow\; e^{-kt} = \frac{1}{2} \;\Rightarrow\; -kt = \ln\frac{1}{2} = -\ln 2 \;\Rightarrow\; t = (\ln 2)/k$

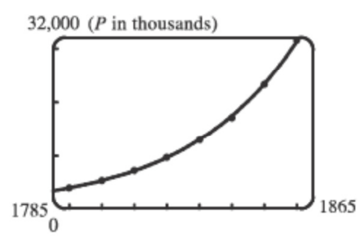
7.4

2. $f(x) = \ln(x^2 + 10) \;\Rightarrow\; f'(x) = \dfrac{1}{x^2 + 10}\dfrac{d}{dx}(x^2 + 10) = \dfrac{2x}{x^2 + 10}$

16. $y = \ln(x^4 \sin^2 x) = \ln x^4 + \ln(\sin x)^2 = 4\ln x + 2\ln\sin x \;\Rightarrow\; y' = 4 \cdot \dfrac{1}{x} + 2 \cdot \dfrac{1}{\sin x} \cdot \cos x = \dfrac{4}{x} + 2\cot x$

38. $y = \ln(x^3 - 7) \;\Rightarrow\; y' = \dfrac{1}{x^3 - 7} \cdot 3x^2 \;\Rightarrow\; y'(2) = \dfrac{12}{8 - 7} = 12$, so an equation of a tangent line at $(2, 0)$ is $y - 0 = 12(x - 2)$ or $y = 12x - 24$.

68. (a) $P = ab^t$ with $a = 4.502714 \times 10^{-20}$ and $b = 1.029953851$, where P is measured in thousands of people. The fit appears to be very good.



(b) **For 1800:** $m_1 = \dfrac{5308 - 3929}{1800 - 1790} = 137.9$, $m_2 = \dfrac{7240 - 5308}{1810 - 1800} = 193.2$.

So $P'(1800) \approx (m_1 + m_2)/2 = 165.55$ thousand people/year.

For 1850: $m_1 = \dfrac{23{,}192 - 17{,}063}{1850 - 1840} = 612.9$, $m_2 = \dfrac{31{,}443 - 23{,}192}{1860 - 1850} = 825.1$.

So $P'(1850) \approx (m_1 + m_2)/2 = 719$ thousand people/year.

(c) Using $P'(t) = ab^t \ln b$ (from Formula 7) with the values of a and b from part (a), we get $P'(1800) \approx 156.85$ and $P'(1850) \approx 686.07$. These estimates are somewhat less than the ones in part (b).

(d) $P(1870) \approx 41{,}946.56$. The difference of 3.4 million people is most likely due to the Civil War (1861–1865).

78. Let $u = e^x + 1$. Then $du = e^x\,dx$, so $\displaystyle\int \frac{e^x}{e^x+1}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln(e^x+1) + C$.

7.5

2. (a) By Theorem 2, $P(t) = P(0)e^{kt} = 60e^{kt}$. In 20 minutes ($\frac{1}{3}$ hour), there are 120 cells, so $P\left(\frac{1}{3}\right) = 60e^{k/3} = 120 \;\Rightarrow$

$e^{k/3} = 2 \;\Rightarrow\; k/3 = \ln 2 \;\Rightarrow\; k = 3\ln 2 = \ln(2^3) = \ln 8$.

(b) $P(t) = 60e^{(\ln 8)t} = 60 \cdot 8^t$

(c) $P(8) = 60 \cdot 8^8 = 60 \cdot 2^{24} = 1{,}006{,}632{,}960$

(d) $dP/dt = kP \;\Rightarrow\; P'(8) = kP(8) = (\ln 8)P(8) \approx 2.093$ billion cells/h

(e) $P(t) = 20{,}000 \;\Rightarrow\; 60 \cdot 8^t = 20{,}000 \;\Rightarrow\; 8^t = 1000/3 \;\Rightarrow\; t\ln 8 = \ln(1000/3) \;\Rightarrow$

$t = \dfrac{\ln(1000/3)}{\ln 8} \approx 2.79$ h

8. (a) The mass remaining after t days is $y(t) = y(0)e^{kt} = 800e^{kt}$. Since the half-life is 5.0 days, $y(5) = 800e^{5k} = 400 \;\Rightarrow$

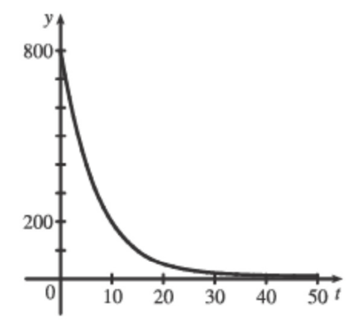
$e^{5k} = \frac{1}{2} \;\Rightarrow\; 5k = \ln\frac{1}{2} \;\Rightarrow\; k = -(\ln 2)/5$, so $y(t) = 800e^{-(\ln 2)t/5} = 800 \cdot 2^{-t/5}$.

(b) $y(30) = 800 \cdot 2^{-30/5} = 12.5$ mg

(c) $800e^{-(\ln 2)t/5} = 1 \;\Leftrightarrow\; -(\ln 2)\frac{t}{5} = \ln\frac{1}{800} = -\ln 800 \;\Leftrightarrow$

$t = 5\,\frac{\ln 800}{\ln 2} \approx 48$ days

(d)



18. (a) Using $A = A_0\left(1 + \frac{r}{n}\right)^{nt}$ with $A_0 = 1000$, $r = 0.08$, and $t = 3$, we have:

(i) Annually: $n = 1$; $\quad A = 1000\left(1 + \frac{0.08}{1}\right)^{1\cdot 3} = \$1259.71$

(ii) Quarterly: $n = 4$; $\quad A = 1000\left(1 + \frac{0.08}{4}\right)^{4\cdot 3} = \$1268.24$

(iii) Monthly: $n = 12$; $\quad A = 1000\left(1 + \frac{0.08}{12}\right)^{12\cdot 3} = \$1270.24$

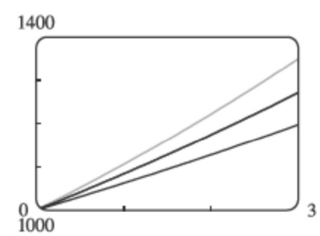
(iv) Weekly: $n = 52$ $\quad A = 1000\left(1 + \frac{0.08}{52}\right)^{52\cdot 3} = \$1271.01$

(v) Daily: $n = 365$; $\quad A = 1000\left(1 + \frac{0.08}{365}\right)^{365\cdot 3} = \$1271.22$

(vi) Hourly: $n = 365 \cdot 24$; $\quad A = 1000\left(1 + \frac{0.08}{365\cdot 24}\right)^{365\cdot 24\cdot 3} = \$1271.25$

(vii) Continuously: $\quad A = 1000e^{(0.08)3} = \$1271.25$

(b)



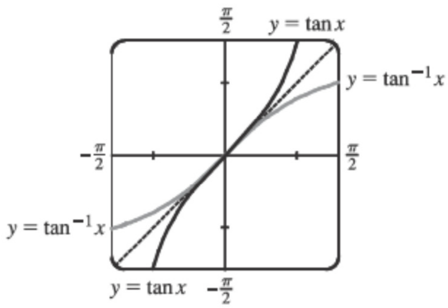
$A_{0.10}(3) = \$1349.86$,

$A_{0.08}(3) = \$1271.25$, and

$A_{0.06}(3) = \$1197.22$.

7.6

16.



The graph of $\tan^{-1} x$ is the reflection of the graph of $\tan x$ about the line $y = x$.

18. (a) Let $a = \sin^{-1} x$ and $b = \cos^{-1} x$. Then $\cos a = \sqrt{1 - \sin^2 a} = \sqrt{1 - x^2}$ since $\cos a \geq 0$ for $-\frac{\pi}{2} \leq a \leq \frac{\pi}{2}$.

Similarly, $\sin b = \sqrt{1 - x^2}$. So

$$\sin(\sin^{-1} x + \cos^{-1} x) = \sin(a + b) = \sin a \cos b + \cos a \sin b = x \cdot x + \sqrt{1 - x^2}\,\sqrt{1 - x^2} = x^2 + (1 - x^2) = 1$$

But $-\frac{\pi}{2} \leq \sin^{-1} x + \cos^{-1} x \leq \frac{3\pi}{2}$, and so $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.

(b) We differentiate $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ with respect to x, and get

$$\frac{1}{\sqrt{1 - x^2}} + \frac{d}{dx}(\cos^{-1} x) = 0 \quad \Rightarrow \quad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}}.$$

24. $f(x) = x \ln(\arctan x) \quad \Rightarrow \quad f'(x) = x \cdot \dfrac{1}{\arctan x} \cdot \dfrac{1}{1 + x^2} + \ln(\arctan x) \cdot 1 = \dfrac{x}{(1 + x^2) \arctan x} + \ln(\arctan x)$

64. Let $u = -\cos x$. Then $du = \sin x\,dx$, so

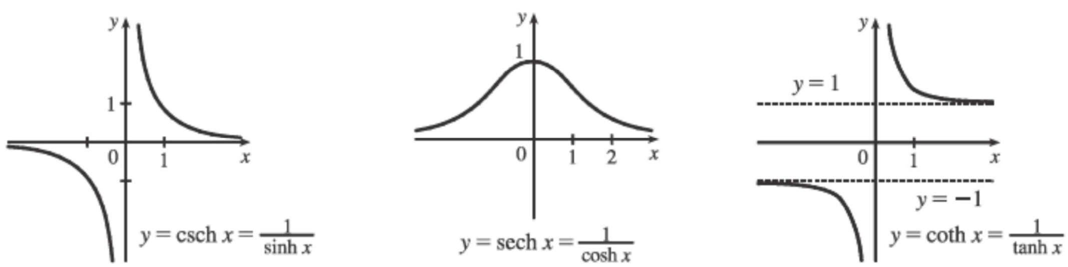
$$\int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x}\,dx = \int_{-1}^0 \frac{1}{1 + u^2}\,du = \left[\tan^{-1} u\right]_{-1}^0 = \tan^{-1} 0 - \tan^{-1}(-1) = 0 - \left(-\tfrac{\pi}{4}\right) = \tfrac{\pi}{4}.$$

7.7

2. (a) $\tanh 0 = \dfrac{(e^0 - e^{-0})/2}{(e^0 + e^{-0})/2} = 0$ (b) $\tanh 1 = \dfrac{e^1 - e^{-1}}{e^1 + e^{-1}} = \dfrac{e^2 - 1}{e^2 + 1} \approx 0.76159$

8. $\cosh(-x) = \frac{1}{2}[e^{-x} + e^{-(-x)}] = \frac{1}{2}(e^{-x} + e^x) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$

22. (a)



32. $g(x) = \cosh(\ln x) \quad \Rightarrow \quad g'(x) = \sinh(\ln x) \cdot (\ln x)' = \frac{1}{x} \sinh(\ln x)$

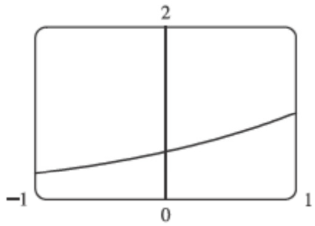
Or: $g(x) = \cosh(\ln x) = \frac{1}{2}(e^{\ln x} + e^{-\ln x}) = \frac{1}{2}(x + x^{-1}) \quad \Rightarrow \quad g'(x) = \frac{1}{2}(1 - x^{-2}) = \frac{1}{2} - 1/(2x^2)$

54. $\displaystyle \lim_{x \to \infty} \frac{\sinh x}{e^x} = \lim_{x \to \infty} \frac{e^x - e^{-x}}{2e^x} = \lim_{x \to \infty} \frac{1 - e^{-2x}}{2} = \frac{1 - 0}{2} = \frac{1}{2}$

58. Let $u = 1 + 4x$. Then $du = 4\,dx$, so $\int \sinh(1 + 4x)\,dx = \frac{1}{4} \int \sinh u\,du = \frac{1}{4} \cosh u + C = \frac{1}{4} \cosh(1 + 4x) + C$.

7.8

6. This limit has the form $\frac{0}{0}$. $\displaystyle \lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \to 2} \frac{(x + 3)(x - 2)}{x - 2} = \lim_{x \to 2}(x + 3) = 2 + 3 = 5$

66. 

From the graph, as $x \to 0$, $y \approx 0.55$. The limit has the form $\frac{0}{0}$.

$$\lim_{x\to 0} \frac{5^x - 4^x}{3^x - 2^x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{5^x \ln 5 - 4^x \ln 4}{3^x \ln 3 - 2^x \ln 2} = \frac{\ln 5 - \ln 4}{\ln 3 - \ln 2} = \frac{\ln \frac{5}{4}}{\ln \frac{3}{2}} \; [\approx 0.55]$$