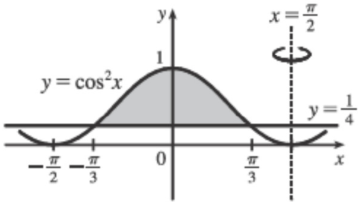
13. A shell has radius $\frac{\pi}{2} - x$, circumference $2\pi\left(\frac{\pi}{2} - x\right)$, and height $\cos^2 x - \frac{1}{4}$.



$y = \cos^2 x$ intersects $y = \frac{1}{4}$ when $\cos^2 x = \frac{1}{4} \quad \Leftrightarrow$

$\cos x = \pm\frac{1}{2} \quad [\,|x| \le \pi/2\,] \quad \Leftrightarrow \quad x = \pm\frac{\pi}{3}$.

$$V = \int_{-\pi/3}^{\pi/3} 2\pi\left(\frac{\pi}{2} - x\right)\left(\cos^2 x - \frac{1}{4}\right) dx$$

15. (a) A cross-section is a washer with inner radius x^2 and outer radius x.

$$V = \int_0^1 \pi\left[(x)^2 - (x^2)^2\right] dx = \int_0^1 \pi(x^2 - x^4)\,dx = \pi\left[\tfrac{1}{3}x^3 - \tfrac{1}{5}x^5\right]_0^1 = \pi\left[\tfrac{1}{3} - \tfrac{1}{5}\right] = \tfrac{2}{15}\pi$$

(b) A cross-section is a washer with inner radius y and outer radius $\sqrt{y}$.

$$V = \int_0^1 \pi\left[\left(\sqrt{y}\right)^2 - y^2\right] dy = \int_0^1 \pi(y - y^2)\,dy = \pi\left[\tfrac{1}{2}y^2 - \tfrac{1}{3}y^3\right]_0^1 = \pi\left[\tfrac{1}{2} - \tfrac{1}{3}\right] = \tfrac{\pi}{6}$$

(c) A cross-section is a washer with inner radius $2 - x$ and outer radius $2 - x^2$.

$$V = \int_0^1 \pi\left[(2 - x^2)^2 - (2 - x)^2\right] dx = \int_0^1 \pi(x^4 - 5x^2 + 4x)\,dx = \pi\left[\tfrac{1}{5}x^5 - \tfrac{5}{3}x^3 + 2x^2\right]_0^1 = \pi\left[\tfrac{1}{5} - \tfrac{5}{3} + 2\right] = \tfrac{8}{15}\pi$$

16. (a) $A = \int_0^1 (2x - x^2 - x^3)\,dx = \left[x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = 1 - \frac{1}{3} - \frac{1}{4} = \frac{5}{12}$

(b) A cross-section is a washer with inner radius x^3 and outer radius $2x - x^2$, so its area is $\pi(2x - x^2)^2 - \pi(x^3)^2$.

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi[(2x - x^2)^2 - (x^3)^2]\,dx = \int_0^1 \pi(4x^2 - 4x^3 + x^4 - x^6)\,dx$$
$$= \pi\left[\tfrac{4}{3}x^3 - x^4 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7\right]_0^1 = \pi\left(\tfrac{4}{3} - 1 + \tfrac{1}{5} - \tfrac{1}{7}\right) = \tfrac{41}{105}\pi$$

(c) Using the method of cylindrical shells,

$$V = \int_0^1 2\pi x(2x - x^2 - x^3)\,dx = \int_0^1 2\pi(2x^2 - x^3 - x^4)\,dx = 2\pi\left[\tfrac{2}{3}x^3 - \tfrac{1}{4}x^4 - \tfrac{1}{5}x^5\right]_0^1 = 2\pi\left(\tfrac{2}{3} - \tfrac{1}{4} - \tfrac{1}{5}\right) = \tfrac{13}{30}\pi.$$

17. (a) Using the Midpoint Rule on $[0, 1]$ with $f(x) = \tan(x^2)$ and $n = 4$, we estimate

$$A = \int_0^1 \tan(x^2)\,dx \approx \tfrac{1}{4}\left[\tan\left(\left(\tfrac{1}{8}\right)^2\right) + \tan\left(\left(\tfrac{3}{8}\right)^2\right) + \tan\left(\left(\tfrac{5}{8}\right)^2\right) + \tan\left(\left(\tfrac{7}{8}\right)^2\right)\right] \approx \tfrac{1}{4}(1.53) \approx 0.38$$

(b) Using the Midpoint Rule on $[0, 1]$ with $f(x) = \pi\tan^2(x^2)$ (for disks) and $n = 4$, we estimate

$$V = \int_0^1 f(x)\,dx \approx \tfrac{1}{4}\pi\left[\tan^2\left(\left(\tfrac{1}{8}\right)^2\right) + \tan^2\left(\left(\tfrac{3}{8}\right)^2\right) + \tan^2\left(\left(\tfrac{5}{8}\right)^2\right) + \tan^2\left(\left(\tfrac{7}{8}\right)^2\right)\right] \approx \tfrac{\pi}{4}(1.114) \approx 0.87$$

21. $\int_0^\pi \pi(2 - \sin x)^2\,dx$

The solid is obtained by rotating the region $\mathcal{R} = \{(x, y) \mid 0 \le x \le \pi, 0 \le y \le 2 - \sin x\}$ about the x-axis.

22. $\int_0^4 2\pi(6 - y)(4y - y^2)\,dy$

The solid is obtained by rotating the region $\mathcal{R} = \left\{(x, y) \mid 0 \le x \le 4y - y^2, 0 \le y \le 4\right\}$ about the line $y = 6$.

25. Equilateral triangles with sides measuring $\frac{1}{4}x$ meters have height $\frac{1}{4}x \sin 60° = \frac{\sqrt{3}}{8}x$. Therefore,

$A(x) = \frac{1}{2} \cdot \frac{1}{4}x \cdot \frac{\sqrt{3}}{8}x = \frac{\sqrt{3}}{64}x^2$. $\quad V = \int_0^{20} A(x)\,dx = \frac{\sqrt{3}}{64}\int_0^{20} x^2\,dx = \frac{\sqrt{3}}{64}\left[\frac{1}{3}x^3\right]_0^{20} = \frac{8000\sqrt{3}}{64 \cdot 3} = \frac{125\sqrt{3}}{3}\ \text{m}^3$.

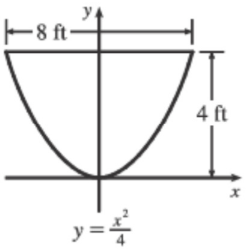
27. $f(x) = kx \quad \Rightarrow \quad 30\ \text{N} = k(15 - 12)\ \text{cm} \quad \Rightarrow \quad k = 10\ \text{N/cm} = 1000\ \text{N/m}$. $\quad 20\ \text{cm} - 12\ \text{cm} = 0.08\ \text{m} \quad \Rightarrow$

$W = \int_0^{0.08} kx\,dx = 1000\int_0^{0.08} x\,dx = 500\left[x^2\right]_0^{0.08} = 500(0.08)^2 = 3.2\ \text{N·m} = 3.2\ \text{J}$.

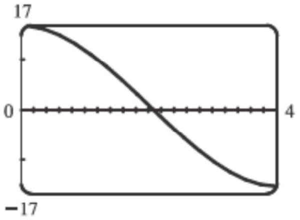
29. (a) The parabola has equation $y = ax^2$ with vertex at the origin and passing through $(4, 4)$. $4 = a \cdot 4^2 \quad \Rightarrow \quad a = \frac{1}{4} \quad \Rightarrow \quad y = \frac{1}{4}x^2 \quad \Rightarrow \quad x^2 = 4y \quad \Rightarrow$ $x = 2\sqrt{y}$. Each circular disk has radius $2\sqrt{y}$ and is moved $4 - y$ ft.

$$W = \int_0^4 \pi \left(2\sqrt{y}\right)^2 62.5(4 - y)\,dy = 250\pi \int_0^4 y(4 - y)\,dy$$

$$= 250\pi \left[2y^2 - \tfrac{1}{3}y^3\right]_0^4 = 250\pi\left(32 - \tfrac{64}{3}\right) = \tfrac{8000\pi}{3} \approx 8378 \text{ ft-lb}$$



(b) In part (a) we knew the final water level (0) but not the amount of work done. Here we use the same equation, except with the work fixed, and the lower limit of integration (that is, the final water level—call it h) unknown: $W = 4000 \quad \Leftrightarrow$ $250\pi\left[2y^2 - \frac{1}{3}y^3\right]_h^4 = 4000 \quad \Leftrightarrow \quad \frac{16}{\pi} = \left[\left(32 - \frac{64}{3}\right) - \left(2h^2 - \frac{1}{3}h^3\right)\right] \quad \Leftrightarrow$ $h^3 - 6h^2 + 32 - \frac{48}{\pi} = 0$. We graph the function $f(h) = h^3 - 6h^2 + 32 - \frac{48}{\pi}$ on the interval $[0, 4]$ to see where it is 0. From the graph, $f(h) = 0$ for $h \approx 2.1$. So the depth of water remaining is about 2.1 ft.



32. (a) $\mathcal{R}_1$ is the region below the graph of $y = x^2$ and above the x-axis between $x = 0$ and $x = b$, and $\mathcal{R}_2$ is the region to the left of the graph of $x = \sqrt{y}$ and to the right of the y-axis between $y = 0$ and $y = b^2$. So the area of $\mathcal{R}_1$ is $A_1 = \int_0^b x^2\,dx = \left[\frac{1}{3}x^3\right]_0^b = \frac{1}{3}b^3$, and the area of $\mathcal{R}_2$ is $A_2 = \int_0^{b^2} \sqrt{y}\,dy = \left[\frac{2}{3}y^{3/2}\right]_0^{b^2} = \frac{2}{3}b^3$. So there is no solution to $A_1 = A_2$ for $b \neq 0$.

(b) Using disks, we calculate the volume of rotation of $\mathcal{R}_1$ about the x-axis to be $V_{1,x} = \pi \int_0^b (x^2)^2\,dx = \frac{1}{5}\pi b^5$. Using cylindrical shells, we calculate the volume of rotation of $\mathcal{R}_1$ about the y-axis to be $V_{1,y} = 2\pi \int_0^b x(x^2)\,dx = 2\pi\left[\frac{1}{4}x^4\right]_0^b = \frac{1}{2}\pi b^4$. So $V_{1,x} = V_{1,y} \quad \Leftrightarrow \quad \frac{1}{5}\pi b^5 = \frac{1}{2}\pi b^4 \quad \Leftrightarrow \quad 2b = 5 \quad \Leftrightarrow \quad b = \frac{5}{2}$. So the volumes of rotation about the x- and y-axes are the same for $b = \frac{5}{2}$.

(c) We use cylindrical shells to calculate the volume of rotation of $\mathcal{R}_2$ about the x-axis:

$\mathcal{R}_{2,x} = 2\pi \displaystyle\int_0^{b^2} y\left(\sqrt{y}\right)dy = 2\pi\left[\tfrac{2}{5}y^{5/2}\right]_0^{b^2} = \tfrac{4}{5}\pi b^5$. We already know the volume of rotation of $\mathcal{R}_1$ about the x-axis from part (b), and $\mathcal{R}_{1,x} = \mathcal{R}_{2,x} \quad \Leftrightarrow \quad \frac{1}{5}\pi b^5 = \frac{4}{5}\pi b^5$, which has no solution for $b \neq 0$.

(d) We use disks to calculate the volume of rotation of $\mathcal{R}_2$ about the y-axis: $\mathcal{R}_{2,y} = \pi \int_0^{b^2} \left(\sqrt{y}\right)^2\,dy = \pi\left[\frac{1}{2}y^2\right]_0^{b^2} = \frac{1}{2}\pi b^4$. We know the volume of rotation of $\mathcal{R}_1$ about the y-axis from part (b), and $\mathcal{R}_{1,y} = \mathcal{R}_{2,y} \quad \Leftrightarrow \quad \frac{1}{2}\pi b^4 = \frac{1}{2}\pi b^4$. But this equation is true for all b, so the volumes of rotation about the y-axis are equal for all values of b.