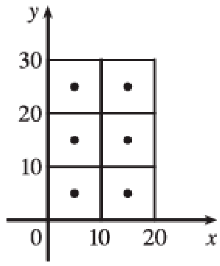
16.1

6. To approximate the volume, let R be the planar region corresponding to the surface of the water in the pool, and place R on coordinate axes so that x and y correspond to the dimensions given. Then we define $f(x, y)$ to be the depth of the water at (x, y), so the volume of water in the pool is the volume of the solid that lies above the rectangle $R = [0, 20] \times [0, 30]$ and below the graph of $f(x, y)$. We can estimate this volume using the Midpoint Rule with $m = 2$ and $n = 3$, so $\Delta A = 100$. Each subrectangle with its midpoint is shown in the figure. Then



$$V \approx \sum_{i=1}^{2}\sum_{j=1}^{3} f(\overline{x}_i, \overline{y}_j)\,\Delta A = \Delta A[f(5,5) + f(5,15) + f(5,25) + f(15,5) + f(15,15) + f(15,25)]$$

$$= 100(3 + 7 + 10 + 3 + 5 + 8) = 3600$$

Thus, we estimate that the pool contains 3600 cubic feet of water.

Alternatively, we can approximate the volume with a Riemann sum where $m = 4$, $n = 6$ and the sample points are taken to be, for example, the upper right corner of each subrectangle. Then $\Delta A = 25$ and

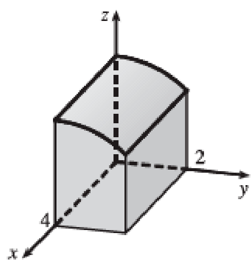
$$V \approx \sum_{i=1}^{4}\sum_{j=1}^{6} f(x_i, y_j)\,\Delta A$$

$$= 25[3 + 4 + 7 + 8 + 10 + 8 + 4 + 6 + 8 + 10 + 12 + 10 + 3 + 4 + 5 + 6 + 8 + 7 + 2 + 2 + 2 + 3 + 4 + 4]$$

$$= 25(140) = 3500$$

So we estimate that the pool contains 3500 ft^3 of water.

14. Here $z = \sqrt{9 - y^2}$, so $z^2 + y^2 = 9$, $z \geq 0$. Thus the integral represents the volume of the top half of the part of the circular cylinder $z^2 + y^2 = 9$ that lies above the rectangle $[0, 4] \times [0, 2]$.



16.2

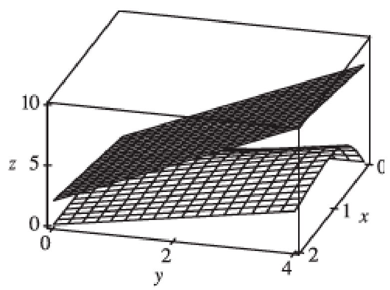
6. $\int_{\pi/6}^{\pi/2}\int_{-1}^{5} \cos y\,dx\,dy = \int_{-1}^{5} dx \int_{\pi/6}^{\pi/2} \cos y\,dy$ [by Equation 5]

$$= \big[x\big]_{-1}^{5}\big[\sin y\big]_{\pi/6}^{\pi/2} = [5 - (-1)](\sin \tfrac{\pi}{2} - \sin \tfrac{\pi}{6}) = 6(1 - \tfrac{1}{2}) = 3$$

26. $V = \iint_R (4 + x^2 - y^2)\,dA = \int_{-1}^{1}\int_0^2 (4 + x^2 - y^2)\,dy\,dx = \int_{-1}^{1} \left[4y + x^2y - \tfrac{1}{3}y^3\right]_{y=0}^{y=2} dx$

$$= \int_{-1}^{1} \left(2x^2 + \tfrac{16}{3}\right) dx = \left[\tfrac{2}{3}x^3 + \tfrac{16}{3}x\right]_{-1}^{1} = \tfrac{2}{3} + \tfrac{16}{3} + \tfrac{2}{3} + \tfrac{16}{3} = 12$$

32.



The solid lies below the plane $z = x + 2y$ and above the surface $z = \dfrac{2xy}{x^2 + 1}$ for $0 \leq x \leq 2$, $0 \leq y \leq 4$. The volume of the solid is the difference in volumes between the solid that lies under $z = x + 2y$ over the rectangle $R = [0, 2] \times [0, 4]$ and the solid that lies under $z = \dfrac{2xy}{x^2 + 1}$ over R.

$$V = \int_0^2\int_0^4 (x + 2y)\,dy\,dx - \int_0^2\int_0^4 \frac{2xy}{x^2 + 1}\,dy\,dx = \int_0^2 \left[xy + y^2\right]_{y=0}^{y=4} dx - \int_0^2 \frac{2x}{x^2 + 1}\,dx \int_0^4 y\,dy$$

$$= \int_0^2 [(4x + 16) - (0 + 0)]\,dx - \left[\ln|x^2 + 1|\right]_0^2 \left[\tfrac{1}{2}y^2\right]_0^4$$

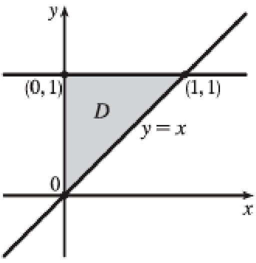
$$= \left[2x^2 + 16x\right]_0^2 - (\ln 5 - \ln 1)(8 - 0) = (8 + 32 - 0) - 8\ln 5 = 40 - 8\ln 5$$

16.3

10. $\iint_D x^3\,dA = \int_1^e \int_0^{\ln x} x^3\,dy\,dx = \int_1^e \left[x^3 y\right]_{y=0}^{y=\ln x} dx = \int_1^e x^3 \ln x\,dx$ $\left[\begin{array}{c}\text{integrate by parts}\\ \text{with } u = \ln x,\, dv = x^3\,dx\end{array}\right]$

$= \left[\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4\right]_1^e = \frac{1}{4}e^4 - \frac{1}{16}e^4 - 0 + \frac{1}{16} = \frac{3}{16}e^4 + \frac{1}{16}$

22.

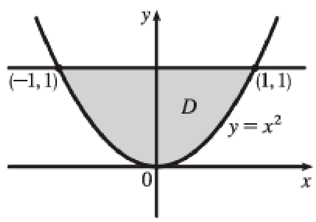


$V = \int_0^1 \int_x^1 (x^2 + 3y^2)\,dy\,dx$

$= \int_0^1 \left[x^2 y + y^3\right]_{y=x}^{y=1} dx = \int_0^1 (x^2 + 1 - 2x^3)\,dx$

$= \left[\frac{1}{3}x^3 + x - \frac{1}{2}x^4\right]_0^1 = \frac{5}{6}$

32. The two planes intersect in the line $y = 1$, $z = 3$, so the region of integration is the plane region enclosed by the parabola $y = x^2$ and the line $y = 1$. We have $2 + y \geq 3y$ for $0 \leq y \leq 1$, so the solid region is bounded above by $z = 2 + y$ and bounded below by $z = 3y$.



$$V = \int_{-1}^1 \int_{x^2}^1 (2 + y)\,dy\,dx - \int_{-1}^1 \int_{x^2}^1 (3y)\,dy\,dx = \int_{-1}^1 \int_{x^2}^1 (2 + y - 3y)\,dy\,dx = \int_{-1}^1 \int_{x^2}^1 (2 - 2y)\,dy\,dx$$

$$= \int_{-1}^1 \left[2y - y^2\right]_{y=x^2}^{y=1} dx = \int_{-1}^1 (1 - 2x^2 + x^4)\,dx = x - \frac{2}{3}x^3 + \frac{1}{5}x^5\Big]_{-1}^1 = \frac{16}{15}$$

16.4

2. The region R is more easily described by rectangular coordinates: $R = \left\{(x, y) \mid -1 \leq x \leq 1, 0 \leq y \leq 1 - x^2\right\}$.

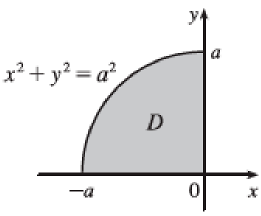
Thus $\iint_R f(x, y)\,dA = \int_{-1}^1 \int_0^{1-x^2} f(x, y)\,dy\,dx$.

22. The sphere $x^2 + y^2 + z^2 = 16$ intersects the xy-plane in the circle $x^2 + y^2 = 16$, so

$$V = 2 \iint_{4 \leq x^2 + y^2 \leq 16} \sqrt{16 - x^2 - y^2}\,dA \quad \text{[by symmetry]} \quad = 2 \int_0^{2\pi} \int_2^4 \sqrt{16 - r^2}\,r\,dr\,d\theta = 2 \int_0^{2\pi} d\theta \int_2^4 r(16 - r^2)^{1/2} dr$$

$$= 2\left[\theta\right]_0^{2\pi} \left[-\frac{1}{3}(16 - r^2)^{3/2}\right]_2^4 = -\frac{2}{3}(2\pi)(0 - 12^{3/2}) = \frac{4\pi}{3}\left(12\sqrt{12}\right) = 32\sqrt{3}\,\pi$$

30.



$\int_{\pi/2}^{\pi} \int_0^a (r\cos\theta)^2\,(r\sin\theta)\,r\,dr\,d\theta = \int_{\pi/2}^{\pi} \int_0^a r^4 \cos^2\theta \sin\theta\,dr\,d\theta$

$= \int_{\pi/2}^{\pi} \cos^2\theta \sin\theta\,d\theta \int_0^a r^4\,dr$

$= \left[-\frac{1}{3}\cos^3\theta\right]_{\pi/2}^{\pi} \left[\frac{1}{5}r^5\right]_0^a$

$= -\frac{1}{3}\left(\cos^3\pi - \cos^3\frac{\pi}{2}\right)\left(\frac{1}{5}a^5\right) = \frac{1}{15}a^5$

16.5

4. $m = \iint_D \rho(x, y)\,dA = \int_0^a \int_0^b cxy\,dy\,dx = c\int_0^a x\,dx \int_0^b y\,dy = c\left[\frac{1}{2}x^2\right]_0^a \left[\frac{1}{2}y^2\right]_0^b = \frac{1}{4}a^2b^2c$,

$M_y = \iint_D x\rho(x, y)\,dA = \int_0^a \int_0^b cx^2y\,dy\,dx = c\int_0^a x^2\,dx \int_0^b y\,dy = c\left[\frac{1}{3}x^3\right]_0^a \left[\frac{1}{2}y^2\right]_0^b = \frac{1}{6}a^3b^2c$, and

$M_x = \iint_D y\rho(x, y)\,dA = \int_0^a \int_0^b cxy^2\,dy\,dx = c\int_0^a x\,dx \int_0^b y^2\,dy = c\left[\frac{1}{2}x^2\right]_0^a \left[\frac{1}{3}y^3\right]_0^b = \frac{1}{6}a^2b^3c$.

Hence, $(\overline{x}, \overline{y}) = \left(\dfrac{M_y}{m}, \dfrac{M_x}{m}\right) = \left(\frac{2}{3}a, \frac{2}{3}b\right)$.

24. Here we assume $b > 0$, $h > 0$ but note that we arrive at the same results if $b < 0$ or $h < 0$. We have

$D = \{(x, y) \mid 0 \le x \le b, 0 \le y \le h - \frac{h}{b}x\}$, so

$$I_x = \int_0^b \int_0^{h-hx/b} y^2 \rho \, dy \, dx = \rho \int_0^b \left[\tfrac{1}{3}y^3\right]_{y=0}^{y=h-hx/b} dx = \tfrac{1}{3}\rho \int_0^b \left(h - \tfrac{h}{b}x\right)^3 dx$$

$$= \tfrac{1}{3}\rho \left[-\tfrac{b}{h}\left(\tfrac{1}{4}\right)\left(h - \tfrac{h}{b}x\right)^4\right]_0^b = -\tfrac{b}{12h}\rho(0 - h^4) = \tfrac{1}{12}\rho b h^3,$$

$$I_y = \int_0^b \int_0^{h-hx/b} x^2 \rho \, dy \, dx = \rho \int_0^b x^2 \left(h - \tfrac{h}{b}x\right) dx = \rho \int_0^b \left(hx^2 - \tfrac{h}{b}x^3\right) dx$$

$$= \rho \left[\tfrac{h}{3}x^3 - \tfrac{h}{4b}x^4\right]_0^b = \rho\left(\tfrac{hb^3}{3} - \tfrac{hb^3}{4}\right) = \tfrac{1}{12}\rho b^3 h,$$

and $m = \int_0^b \int_0^{h-hx/b} \rho \, dy \, dx = \rho \int_0^b \left(h - \frac{h}{b}x\right) dx = \rho\left[hx - \frac{h}{2b}x^2\right]_0^b = \frac{1}{2}\rho bh$. Hence $\overline{\overline{x}}^2 = \dfrac{I_y}{m} = \dfrac{\frac{1}{12}\rho b^3 h}{\frac{1}{2}\rho bh} = \dfrac{b^2}{6} \Rightarrow$

$\overline{\overline{x}} = \dfrac{b}{\sqrt{6}}$ and $\overline{\overline{y}}^2 = \dfrac{I_x}{m} = \dfrac{\frac{1}{12}\rho b h^3}{\frac{1}{2}\rho bh} = \dfrac{h^2}{6} \Rightarrow \overline{\overline{y}} = \dfrac{h}{\sqrt{6}}$.

30. (a) The lifetime of each bulb has exponential density function

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{1000}e^{-t/1000} & \text{if } t \ge 0 \end{cases}$$

If X and Y are the lifetimes of the individual bulbs, then X and Y are independent, so the joint density function is the product of the individual density functions:

$$f(x, y) = \begin{cases} 10^{-6}e^{-(x+y)/1000} & \text{if } x \ge 0, y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

The probability that both of the bulbs fail within 1000 hours is

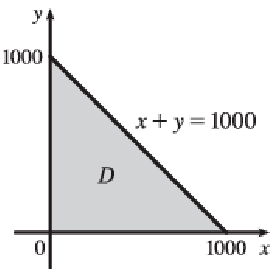
$$P(X \le 1000, Y \le 1000) = \int_{-\infty}^{1000}\int_{-\infty}^{1000} f(x, y)\, dy\, dx = \int_0^{1000}\int_0^{1000} 10^{-6}e^{-(x+y)/1000}\, dy\, dx$$

$$= 10^{-6}\int_0^{1000} e^{-x/1000}\, dx \int_0^{1000} e^{-y/1000}\, dy$$

$$= 10^{-6}\left[-1000e^{-x/1000}\right]_0^{1000}\left[-1000e^{-y/1000}\right]_0^{1000}$$

$$= \left(e^{-1} - 1\right)^2 \approx 0.3996$$

(b) Now we are asked for the probability that the combined lifetimes of both bulbs is 1000 hours or less. Thus we want to find $P(X + Y \le 1000)$, or equivalently $P((X, Y) \in D)$ where D is the triangular region shown in the figure. Then



$$P(X + Y \le 1000) = \iint_D f(x, y)\, dA$$

$$= \int_0^{1000}\int_0^{1000-x} 10^{-6}e^{-(x+y)/1000}\, dy\, dx$$

$$= 10^{-6}\int_0^{1000}\left[-1000e^{-(x+y)/1000}\right]_{y=0}^{y=1000-x} dx = -10^{-3}\int_0^{1000}\left(e^{-1} - e^{-x/1000}\right) dx$$

$$= -10^{-3}\left[e^{-1}x + 1000e^{-x/1000}\right]_0^{1000} = 1 - 2e^{-1} \approx 0.2642$$

24. (a) Divide B into 8 cubes of size $\Delta V = 8$. With $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, the Midpoint Rule gives

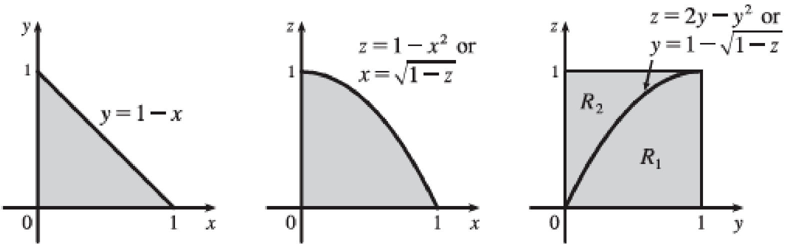
$$\iiint_B \sqrt{x^2 + y^2 + z^2}\, dV \approx \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} f(\overline{x}_i, \overline{y}_j, \overline{z}_k)\, \Delta V$$

$$= 8[f(1,1,1) + f(1,1,3) + f(1,3,1) + f(1,3,3) + f(3,1,1) + f(3,1,3) + f(3,3,1) + f(3,3,3)]$$

$$\approx 239.64$$

(b) Using a CAS we have $\iiint_B \sqrt{x^2 + y^2 + z^2}\, dV = \int_0^4 \int_0^4 \int_0^4 \sqrt{x^2 + y^2 + z^2}\, dz\, dy\, dx \approx 245.91$. This differs from the estimate in part (a) by about 2.5%.

34.



The projections of E onto the xy- and xz-planes are as in the first two diagrams and so

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z)\, dy\, dz\, dx = \int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} f(x, y, z)\, dy\, dx\, dz$$

$$= \int_0^1 \int_0^{1-y} \int_0^{1-x^2} f(x, y, z)\, dz\, dx\, dy = \int_0^1 \int_0^{1-x} \int_0^{1-x^2} f(x, y, z)\, dz\, dy\, dx$$

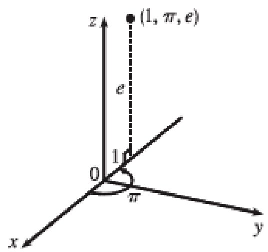
Now the surface $z = 1 - x^2$ intersects the plane $y = 1 - x$ in a curve whose projection in the yz-plane is $z = 1 - (1 - y)^2$ or $z = 2y - y^2$. So we must split up the projection of E on the yz-plane into two regions as in the third diagram. For (y, z) in R_1, $0 \le x \le 1 - y$ and for (y, z) in R_2, $0 \le x \le \sqrt{1 - z}$, and so the given integral is also equal to

$$\int_0^1 \int_0^{1-\sqrt{1-z}} \int_0^{\sqrt{1-z}} f(x, y, z)\, dx\, dy\, dz + \int_0^1 \int_{1-\sqrt{1-z}}^{1} \int_0^{1-y} f(x, y, z)\, dx\, dy\, dz$$

$$= \int_0^1 \int_0^{2y-y^2} \int_0^{1-y} f(x, y, z)\, dx\, dz\, dy + \int_0^1 \int_{2y-y^2}^{1} \int_0^{\sqrt{1-z}} f(x, y, z)\, dx\, dz\, dy.$$

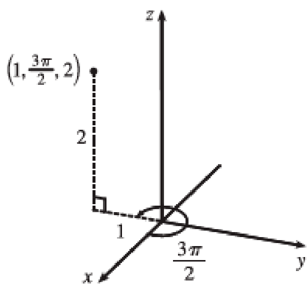
16.7

2. (a)



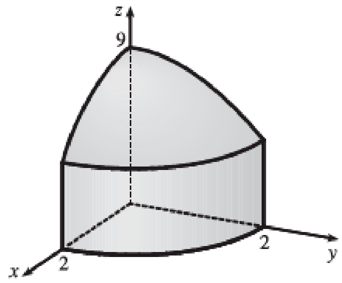
$x = 1\cos\pi = -1$, $y = 1\sin\pi = 0$, and $z = e$, so the point is $(-1, 0, e)$ in rectangular coordinates.

(b)



$x = 1\cos\frac{3\pi}{2} = 0$, $y = 1\sin\frac{3\pi}{2} = -1$, $z = 2$, so the point is $(0, -1, 2)$ in rectangular coordinates.

6. Since $r = 5$, $x^2 + y^2 = 25$ and the surface is a circular cylinder with radius 5 and axis the z-axis.

16. 

The region of integration is given in cylindrical coordinates by $E = \{(r, \theta, z) \mid 0 \le \theta \le \pi/2, 0 \le r \le 2, 0 \le z \le 9 - r^2\}$. This represents the solid region in the first octant enclosed by the circular cylinder $r = 2$, bounded above by $z = 9 - r^2$, a circular paraboloid, and bounded below by the xy-plane.

$$\int_0^{\pi/2} \int_0^2 \int_0^{9-r^2} r\,dz\,dr\,d\theta = \int_0^{\pi/2} \int_0^2 \left[rz\right]_{z=0}^{z=9-r^2} dr\,d\theta$$
$$= \int_0^{\pi/2} \int_0^2 r(9 - r^2)\,dr\,d\theta = \int_0^{\pi/2} d\theta \int_0^2 (9r - r^3)\,dr$$
$$= \left[\theta\right]_0^{\pi/2} \left[\tfrac{9}{2}r^2 - \tfrac{1}{4}r^4\right]_0^2 = \tfrac{\pi}{2}(18 - 4) = 7\pi$$

22. In cylindrical coordinates E is the solid region within the cylinder $r = 1$ bounded above and below by the sphere $r^2 + z^2 = 4$, so $E = \left\{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 1, -\sqrt{4 - r^2} \le z \le \sqrt{4 - r^2}\right\}$. Thus the volume is

$$\iiint_E dV = \int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi} \int_0^1 2r\sqrt{4 - r^2}\,dr\,d\theta$$
$$= \int_0^{2\pi} d\theta \int_0^1 2r\sqrt{4 - r^2}\,dr = 2\pi\left[-\tfrac{2}{3}(4 - r^2)^{3/2}\right]_0^1 = \tfrac{4}{3}\pi(8 - 3^{3/2})$$

16.8

4. (a) $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0 + 3 + 1} = 2$, $\cos\phi = \dfrac{z}{\rho} = \dfrac{1}{2} \quad\Rightarrow\quad \phi = \dfrac{\pi}{3}$, and $\cos\theta = \dfrac{x}{\rho\sin\phi} = \dfrac{0}{2\sin(\pi/3)} = 0 \quad\Rightarrow$

$\theta = \dfrac{\pi}{2}$ [since $y > 0$]. Thus spherical coordinates are $\left(2, \dfrac{\pi}{2}, \dfrac{\pi}{3}\right)$.

(b) $\rho = \sqrt{1 + 1 + 6} = 2\sqrt{2}$, $\cos\phi = \dfrac{\sqrt{6}}{2\sqrt{2}} = \dfrac{\sqrt{3}}{2} \quad\Rightarrow\quad \phi = \dfrac{\pi}{6}$, and $\cos\theta = \dfrac{-1}{2\sqrt{2}\sin(\pi/6)} = -\dfrac{1}{\sqrt{2}} \quad\Rightarrow$

$\theta = \dfrac{3\pi}{4}$ [since $y > 0$]. Thus spherical coordinates are $\left(2\sqrt{2}, \dfrac{3\pi}{4}, \dfrac{\pi}{6}\right)$.

10. (a) $x^2 - 2x + y^2 + z^2 = 0 \quad\Leftrightarrow\quad (x^2 + y^2 + z^2) - 2x = 0 \quad\Leftrightarrow\quad \rho^2 - 2(\rho\sin\phi\cos\theta) = 0$ or $\rho = 2\sin\phi\cos\theta$.

(b) $x + 2y + 3z = 1 \quad\Leftrightarrow\quad \rho\sin\phi\cos\theta + 2\rho\sin\phi\sin\theta + 3\rho\cos\phi = 1$ or $\rho = 1/(\sin\phi\cos\theta + 2\sin\phi\sin\theta + 3\cos\phi)$.

28. If we center the ball at the origin, then the ball is given by $B = \{(\rho, \theta, \phi) \mid 0 \le \rho \le a, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$ and the distance from any point (x, y, z) in the ball to the center $(0, 0, 0)$ is $\sqrt{x^2 + y^2 + z^2} = \rho$. Thus the average distance is

$$\frac{1}{V(B)} \iiint_B \rho\,dV = \frac{1}{\frac{4}{3}\pi a^3} \int_0^{\pi} \int_0^{2\pi} \int_0^a \rho \cdot \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi = \frac{3}{4\pi a^3} \int_0^{\pi} \sin\phi\,d\phi \int_0^{2\pi} d\theta \int_0^a \rho^3\,d\rho$$
$$= \frac{3}{4\pi a^3}\left[-\cos\phi\right]_0^{\pi} \left[\theta\right]_0^{2\pi} \left[\tfrac{1}{4}\rho^4\right]_0^a = \frac{3}{4\pi a^3}(2)(2\pi)\left(\tfrac{1}{4}a^4\right) = \tfrac{3}{4}a$$

16.9

8. S_1 is the line segment $v = 0$, $0 \leq u \leq 1$, so $x = v = 0$ and $y = u(1 + v^2) = u$. Since $0 \leq u \leq 1$, the image is the line segment $x = 0$, $0 \leq y \leq 1$. S_2 is the segment $u = 1$, $0 \leq v \leq 1$, so $x = v$ and $y = u(1 + v^2) = 1 + x^2$. Thus the image is the portion of the parabola $y = 1 + x^2$ for $0 \leq x \leq 1$. S_3 is the segment $v = 1$, $0 \leq u \leq 1$, so $x = 1$ and $y = 2u$. The image is the segment $x = 1$, $0 \leq y \leq 2$. S_4 is described by $u = 0$, $0 \leq v \leq 1$, so $0 \leq x = v \leq 1$ and $y = u(1 + v^2) = 0$. The image is the line segment $y = 0$, $0 \leq x \leq 1$. Thus, the image of S is the region R bounded by the parabola $y = 1 + x^2$, the x-axis, and the lines $x = 0$, $x = 1$.