15.1

2. (a) From Table 3, $f(95, 70) = 124$, which means that when the actual temperature is $95°\text{F}$ and the relative humidity is 70%, the perceived air temperature is approximately $124°\text{F}$.

(b) Looking at the row corresponding to $T = 90$, we see that $f(90, h) = 100$ when $h = 60$.

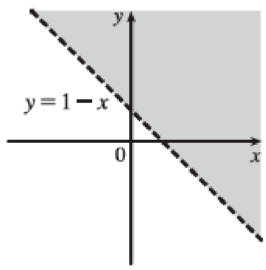
(c) Looking at the column corresponding to $h = 50$, we see that $f(T, 50) = 88$ when $T = 85$.

(d) $I = f(80, h)$ means that T is fixed at 80 and h is allowed to vary, resulting in a function of h that gives the humidex values for different relative humidities when the actual temperature is $80°\text{F}$. Similarly, $I = f(100, h)$ is a function of one variable that gives the humidex values for different relative humidities when the actual temperature is $100°\text{F}$. Looking at the rows of the table corresponding to $T = 80$ and $T = 100$, we see that $f(80, h)$ increases at a relatively constant rate of approximately $1°\text{F}$ per 10% relative humidity, while $f(100, h)$ increases more quickly (at first with an average rate of change of $5°\text{F}$ per 10% relative humidity) and at an increasing rate (approximately $12°\text{F}$ per 10% relative humidity for larger values of h).

6. (a) $f(1, 1) = \ln(1 + 1 - 1) = \ln 1 = 0$

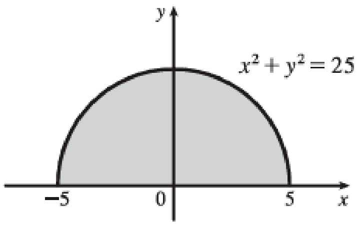
(b) $f(e, 1) = \ln(e + 1 - 1) = \ln e = 1$

(c) $\ln(x + y - 1)$ is defined only when $x + y - 1 > 0$, that is, $y > 1 - x$. So the domain of f is $\{(x, y) \mid y > 1 - x\}$.

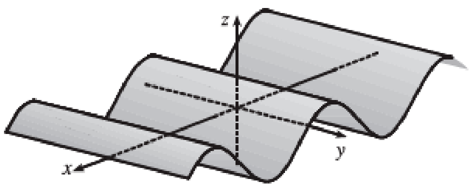


(d) Since $\ln(x + y - 1)$ can be any real number, the range is $\mathbb{R}$.

16. $\sqrt{y} + \sqrt{25 - x^2 - y^2}$ is defined only when $y \geq 0$ and $25 - x^2 - y^2 \geq 0 \quad \Leftrightarrow \quad x^2 + y^2 \leq 25$. So the domain of f is $\{(x, y) \mid x^2 + y^2 \leq 25,\ y \geq 0\}$, a half disk of radius 5.



24. $z = \cos x$, a "wave."



30. All six graphs have different traces in the planes $x = 0$ and $y = 0$, so we investigate these for each function.

(a) $f(x, y) = |x| + |y|$. The trace in $x = 0$ is $z = |y|$, and in $y = 0$ is $z = |x|$, so it must be graph VI.

(b) $f(x, y) = |xy|$. The trace in $x = 0$ is $z = 0$, and in $y = 0$ is $z = 0$, so it must be graph V.

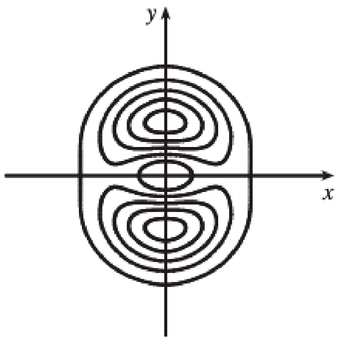
(c) $f(x, y) = \dfrac{1}{1 + x^2 + y^2}$. The trace in $x = 0$ is $z = \dfrac{1}{1 + y^2}$, and in $y = 0$ is $z = \dfrac{1}{1 + x^2}$. In addition, we can see that f is close to 0 for large values of x and y, so this is graph I.

(d) $f(x, y) = (x^2 - y^2)^2$. The trace in $x = 0$ is $z = y^4$, and in $y = 0$ is $z = x^4$. Both graph II and graph IV seem plausible; notice the trace in $z = 0$ is $0 = (x^2 - y^2)^2 \Rightarrow y = \pm x$, so it must be graph IV.

(e) $f(x, y) = (x - y)^2$. The trace in $x = 0$ is $z = y^2$, and in $y = 0$ is $z = x^2$. Both graph II and graph IV seem plausible; notice the trace in $z = 0$ is $0 = (x - y)^2 \Rightarrow y = x$, so it must be graph II.

(f) $f(x, y) = \sin(|x| + |y|)$. The trace in $x = 0$ is $z = \sin|y|$, and in $y = 0$ is $z = \sin|x|$. In addition, notice that the oscillating nature of the graph is characteristic of trigonometric functions. So this is graph III.

34.



15.2

2. (a) The outdoor temperature as a function of longitude, latitude, and time is continuous. Small changes in longitude, latitude, or time can produce only small changes in temperature, as the temperature doesn't jump abruptly from one value to another.

(b) Elevation is not necessarily continuous. If we think of a cliff with a sudden drop-off, a very small change in longitude or latitude can produce a comparatively large change in elevation, without all the intermediate values being attained. Elevation *can* jump from one value to another.

(c) The cost of a taxi ride is usually discontinuous. The cost normally increases in jumps, so small changes in distance traveled or time can produce a jump in cost. A graph of the function would show breaks in the surface.

4. We make a table of values of $f(x,y) = \dfrac{2xy}{x^2+2y^2}$ for a set of (x,y) points near the origin.

$x \backslash y$	−0.3	−0.2	−0.1	0	0.1	0.2	0.3
−0.3	0.667	0.706	0.545	0.000	−0.545	−0.706	−0.667
−0.2	0.545	0.667	0.667	0.000	−0.667	−0.667	−0.545
−0.1	0.316	0.444	0.667	0.000	−0.667	−0.444	−0.316
0	0.000	0.000	0.000		0.000	0.000	0.000
0.1	−0.316	−0.444	−0.667	0.000	0.667	0.444	0.316
0.2	−0.545	−0.667	−0.667	0.000	0.667	0.667	0.545
0.3	−0.667	−0.706	−0.545	0.000	0.545	0.706	0.667

It appears from the table that the values of $f(x,y)$ are not approaching a single value as (x,y) approaches the origin. For verification, if we first approach $(0,0)$ along the x-axis, we have $f(x,0) = 0$, so $f(x,y) \to 0$. But if we approach $(0,0)$ along the line $y = x$, $f(x,x) = \dfrac{2x^2}{x^2+2x^2} = \dfrac{2}{3}$ $(x \neq 0)$, so $f(x,y) \to \frac{2}{3}$. Since f approaches different values along different paths to the origin, this limit does not exist.

8. $\dfrac{1+y^2}{x^2+xy}$ is a rational function and hence continuous on its domain, which includes $(1,0)$. $\ln t$ is a continuous function for $t > 0$, so the composition $f(x,y) = \ln\left(\dfrac{1+y^2}{x^2+xy}\right)$ is continuous wherever $\dfrac{1+y^2}{x^2+xy} > 0$. In particular, f is continuous at $(1,0)$ and so $\displaystyle\lim_{(x,y)\to(1,0)} f(x,y) = f(1,0) = \ln\left(\frac{1+0^2}{1^2+1\cdot 0}\right) = \ln\frac{1}{1} = 0$.

16. We can use the Squeeze Theorem to show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2\sin^2 y}{x^2+2y^2} = 0$:

$0 \le \dfrac{x^2\sin^2 y}{x^2+2y^2} \le \sin^2 y$ since $\dfrac{x^2}{x^2+2y^2} \le 1$, and $\sin^2 y \to 0$ as $(x,y) \to (0,0)$, so $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2\sin^2 y}{x^2+2y^2} = 0$.

20. $f(x,y,z) = \dfrac{x^2+2y^2+3z^2}{x^2+y^2+z^2}$. Then $f(x,0,0) = \dfrac{x^2+0+0}{x^2+0+0} = 1$ for $x \neq 0$, so $f(x,y,z) \to 1$ as $(x,y,z) \to (0,0,0)$ along the x-axis. But $f(0,y,0) = \dfrac{0+2y^2+0}{0+y^2+0} = 2$ for $y \neq 0$, so $f(x,y,z) \to 2$ as $(x,y,z) \to (0,0,0)$ along the y-axis. Thus, the limit doesn't exist.

36. $f(x,y,z) = \sqrt{x+y+z} = h(g(x,y,z))$ where $g(x,y,z) = x+y+z$, continuous everywhere, and $h(t) = \sqrt{t}$ is continuous on its domain $\{t \mid t \ge 0\}$. Thus f is continuous on its domain $\{(x,y,z) \mid x+y+z \ge 0\}$, so f is continuous on and above the plane $z = -x - y$.

15.3

20. $z = \tan xy \quad \Rightarrow \quad \dfrac{\partial z}{\partial x} = (\sec^2 xy)(y) = y\sec^2 xy$, $\dfrac{\partial z}{\partial y} = (\sec^2 xy)(x) = x\sec^2 xy$

46. $yz = \ln(x+z) \quad \Rightarrow \quad \dfrac{\partial}{\partial x}(yz) = \dfrac{\partial}{\partial x}(\ln(x+z)) \quad \Rightarrow \quad y\dfrac{\partial z}{\partial x} = \dfrac{1}{x+z}\left(1+\dfrac{\partial z}{\partial x}\right) \quad \Leftrightarrow \quad \left(y - \dfrac{1}{x+z}\right)\dfrac{\partial z}{\partial x} = \dfrac{1}{x+z}$,

so $\dfrac{\partial z}{\partial x} = \dfrac{1/(x+z)}{y - 1/(x+z)} = \dfrac{1}{y(x+z)-1}$.

$\dfrac{\partial}{\partial y}(yz) = \dfrac{\partial}{\partial y}(\ln(x+z)) \quad \Rightarrow \quad y\dfrac{\partial z}{\partial y} + z\cdot 1 = \dfrac{1}{x+z}\left(0+\dfrac{\partial z}{\partial y}\right) \quad \Leftrightarrow \quad \left(y - \dfrac{1}{x+z}\right)\dfrac{\partial z}{\partial y} = -z$,

so $\dfrac{\partial z}{\partial y} = \dfrac{-z}{y-1/(x+z)} = \dfrac{z(x+z)}{1-y(x+z)}$.

62. $f(x,t) = x^2 e^{-ct} \quad \Rightarrow \quad f_t = x^2(-ce^{-ct}),\ f_{tt} = x^2(c^2 e^{-ct}),\ f_{ttt} = x^2(-c^3 e^{-ct}) = -c^3 x^2 e^{-ct}$ and

$f_{tx} = 2x(-ce^{-ct}),\ f_{txx} = 2(-ce^{-ct}) = -2ce^{-ct}$.

15.4

4. $z = f(x,y) = y \ln x \quad \Rightarrow \quad f_x(x,y) = y/x,\ f_y(x,y) = \ln x$, so $f_x(1,4) = 4,\ f_y(1,4) = 0$, and an equation of the tangent plane is $z - 0 = f_x(1,4)(x-1) + f_y(1,4)(y-4) \quad \Rightarrow \quad z = 4(x-1) + 0(y-4)$ or $z = 4x - 4$.

16. $f(x,y) = \sin(2x+3y)$. The partial derivatives are $f_x(x,y) = 2\cos(2x+3y)$ and $f_y(x,y) = 3\cos(2x+3y)$, so $f_x(-3,2) = 2$ and $f_y(-3,2) = 3$. Both f_x and f_y are continuous functions, so f is differentiable at $(-3,2)$, and the linearization of f at $(-3,2)$ is

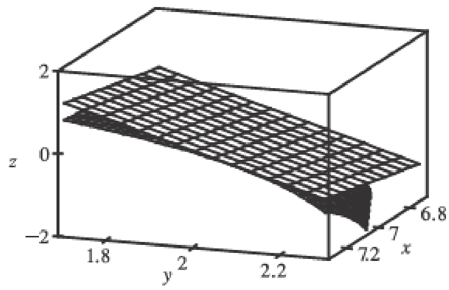
$$L(x,y) = f(-3,2) + f_x(-3,2)(x+3) + f_y(-3,2)(y-2) = 0 + 2(x+3) + 3(y-2) = 2x + 3y.$$

20. $f(x,y) = \ln(x-3y) \quad \Rightarrow \quad f_x(x,y) = \dfrac{1}{x-3y}$ and $f_y(x,y) = -\dfrac{3}{x-3y}$, so $f_x(7,2) = 1$ and $f_y(7,2) = -3$.

Then the linear approximation of f at $(7,2)$ is given by

$$\begin{aligned} f(x,y) &\approx f(7,2) + f_x(7,2)(x-7) + f_y(7,2)(y-2) \\ &= 0 + 1(x-7) - 3(y-2) = x - 3y - 1 \end{aligned}$$

Thus $f(6.9, 2.06) \approx 6.9 - 3(2.06) - 1 = -0.28$. The graph shows that our approximated value is slightly greater than the actual value.



22. From the table, $f(40,20) = 28$. To estimate $f_v(40,20)$ and $f_t(40,20)$ we follow the procedure used in Exercise 15.3.4 [ET 14.3.4]. Since $f_v(40,20) = \lim\limits_{h\to 0} \dfrac{f(40+h,20) - f(40,20)}{h}$, we approximate this quantity with $h = \pm 10$ and use the values given in the table:

$$f_v(40,20) \approx \frac{f(50,20) - f(40,20)}{10} = \frac{40-28}{10} = 1.2, \qquad f_v(40,20) \approx \frac{f(30,20) - f(40,20)}{-10} = \frac{17-28}{-10} = 1.1$$

Averaging these values gives $f_v(40,20) \approx 1.15$. Similarly, $f_t(40,20) = \lim\limits_{h\to 0} \dfrac{f(40,20+h) - f(40,20)}{h}$, so we use $h = 10$ and $h = -5$:

$$f_t(40,20) \approx \frac{f(40,30) - f(40,20)}{10} = \frac{31-28}{10} = 0.3, \qquad f_t(40,20) \approx \frac{f(40,15) - f(40,20)}{-5} = \frac{25-28}{-5} = 0.6$$

Averaging these values gives $f_t(40,15) \approx 0.45$. The linear approximation, then, is

$$f(v,t) \approx f(40,20) + f_v(40,20)(v-40) + f_t(40,20)(t-20) \approx 28 + 1.15(v-40) + 0.45(t-20)$$

When $v = 43$ and $t = 24$, we estimate $f(43,24) \approx 28 + 1.15(43-40) + 0.45(24-20) = 33.25$, so we would expect the wave heights to be approximately 33.25 ft.

15.5

30. $\sin x + \cos y = \sin x \cos y$, so let $F(x,y) = \sin x + \cos y - \sin x \cos y = 0$. Then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\cos x - \cos x \cos y}{-\sin y + \sin x \sin y} = \frac{\cos x(\cos y - 1)}{\sin y(\sin x - 1)}.$$

42. Let x and y be the respective distances of car A and car B from the intersection and let z be the distance between the two cars. Then $dx/dt = -90$, $dy/dt = -80$ and $z^2 = x^2 + y^2$. When $x = 0.3$ and $y = 0.4$, $z = \sqrt{0.25} = 0.5$ and $2z\,(dz/dt) = 2x\,(dx/dt) + 2y\,(dy/dt)$ or $dz/dt = 0.6(-90) + 0.8(-80) = -118$ km/h.

15.6

6. $f(x,y) = x\sin(xy) \quad \Rightarrow \quad f_x(x,y) = x\cos(xy)\cdot y + \sin(xy) = xy\cos(xy) + \sin(xy)$ and $f_y(x,y) = x\cos(xy)\cdot x = x^2\cos(xy)$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = \frac{\pi}{3}$, then from Equation 6,

$D_{\mathbf{u}}\, f(2,0) = f_x(2,0)\cos\frac{\pi}{3} + f_y(2,0)\sin\frac{\pi}{3} = 0 + 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$.

8. $f(x,y) = y^2/x$

(a) $\nabla f(x,y) = \dfrac{\partial f}{\partial x}\mathbf{i} + \dfrac{\partial f}{\partial y}\mathbf{j} = y^2(-x^{-2})\mathbf{i} + (2y/x)\,\mathbf{j} = -\dfrac{y^2}{x^2}\mathbf{i} + \dfrac{2y}{x}\mathbf{j}$

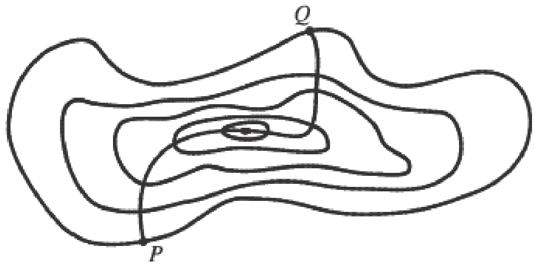
(b) $\nabla f(1,2) = -4\,\mathbf{i} + 4\,\mathbf{j}$

(c) By Equation 9, $D_{\mathbf{u}}\, f(1,2) = \nabla f(1,2)\cdot\mathbf{u} = (-4\,\mathbf{i} + 4\,\mathbf{j})\cdot\frac{1}{3}\,(2\,\mathbf{i} + \sqrt{5}\,\mathbf{j}) = \frac{1}{3}\,(-8 + 4\sqrt{5}) = \frac{4}{3}\,(\sqrt{5} - 2)$.

30. The fisherman is traveling in the direction $\langle -80, -60\rangle$. A unit vector in this direction is $\mathbf{u} = \frac{1}{100}\langle -80, -60\rangle = \left\langle -\frac{4}{5}, -\frac{3}{5}\right\rangle$, and if the depth of the lake is given by $f(x,y) = 200 + 0.02x^2 - 0.001y^3$, then $\nabla f(x,y) = \left\langle 0.04x, -0.003y^2\right\rangle$.

$D_{\mathbf{u}}\, f(80,60) = \nabla f(80,60)\cdot\mathbf{u} = \langle 3.2, -10.8\rangle\cdot\left\langle -\frac{4}{5}, -\frac{3}{5}\right\rangle = 3.92$. Since $D_{\mathbf{u}}\, f(80,60)$ is positive, the depth of the lake is increasing near $(80,60)$ in the direction toward the buoy.

36. The curve of steepest ascent is perpendicular to all of the contour lines.



15.7

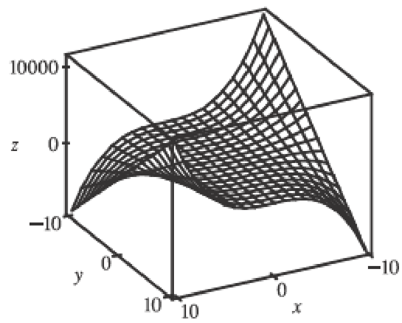
6. $f(x,y) = x^3y + 12x^2 - 8y \quad \Rightarrow \quad f_x = 3x^2y + 24x$,

$f_y = x^3 - 8,\ f_{xx} = 6xy + 24,\ f_{xy} = 3x^2,\ f_{yy} = 0$.

Then $f_y = 0$ implies $x = 2$, and substitution into $f_x = 0$ gives

$12y + 48 = 0 \quad \Rightarrow \quad y = -4$. Thus, the only critical point is $(2,-4)$.

$D(2,-4) = (-24)(0) - 12^2 = -144 < 0$, so $(2,-4)$ is a saddle point.



44. Let x, y, z, be the positive numbers. Then $x + y + z = 12$ and we want to minimize $x^2 + y^2 + z^2 = x^2 + y^2 + (12 - x - y)^2 = f(x,y)$ for $0 < x, y < 12$. $f_x = 2x + 2(12 - x - y)(-1) = 4x + 2y - 24$, $f_y = 2y + 2(12 - x - y)(-1) = 2x + 4y - 24$, $f_{xx} = 4$, $f_{xy} = 2$, $f_{yy} = 4$. Then $f_x = 0$ implies $4x + 2y = 24$ or $y = 12 - 2x$ and substituting into $f_y = 0$ gives $2x + 4(12 - 2x) = 24 \quad \Rightarrow \quad 6x = 24 \quad \Rightarrow \quad x = 4$ and then $y = 4$, so the only critical point is $(4,4)$. $D(4,4) = 16 - 4 > 0$ and $f_{xx}(4,4) = 4 > 0$, so $f(4,4)$ is a local minimum. $f(4,4)$ is also the absolute minimum [compare to the values of f as $x, y \to 0$ or 12] so the numbers are $x = y = z = 4$.

50. The cost equals $5xy + 2(xz + yz)$ and $xyz = V$, so $C(x,y) = 5xy + 2V(x+y)/(xy) = 5xy + 2V(x^{-1} + y^{-1})$. Then $C_x = 5y - 2Vx^{-2}$, $C_y = 5x - 2\,Vy^{-2}$, $f_x = 0$ implies $y = 2\,V/(5x^2)$, $f_y = 0$ implies $x = \sqrt[3]{\frac{2}{5}V} = y$. Thus the dimensions of the aquarium which minimize the cost are $x = y = \sqrt[3]{\frac{2}{5}V}$ units, $z = V^{1/3}\left(\frac{5}{2}\right)^{2/3}$.

15.8

10. $f(x,y,z) = x^2y^2z^2,\ g(x,y,z) = x^2+y^2+z^2 = 1 \quad\Rightarrow\quad \nabla f = \langle 2xy^2z^2, 2yx^2z^2, 2zx^2y^2\rangle, \lambda\nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z\rangle$.

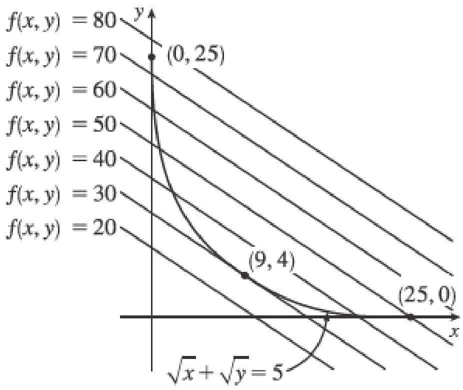
Then $\nabla f = \lambda\nabla g$ implies **(1)** $\lambda = y^2z^2 = x^2z^2 = x^2y^2$ and $\lambda \neq 0$, or **(2)** $\lambda = 0$ and one or two (but not three) of the coordinates are 0. If **(1)** then $x^2 = y^2 = z^2 = \frac{1}{3}$. The minimum value of f on the sphere occurs in case **(2)** with a value of 0 and the maximum value is $\frac{1}{27}$ which arises from all the points from **(1)**, that is, the points $\left(\pm\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(\pm\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(\pm\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

18. $f(x,y) = 2x^2+3y^2-4x-5 \quad\Rightarrow\quad \nabla f = \langle 4x-4, 6y\rangle = \langle 0,0\rangle \quad\Rightarrow\quad x = 1, y = 0$. Thus $(1,0)$ is the only critical point of f, and it lies in the region $x^2+y^2 < 16$. On the boundary, $g(x,y) = x^2+y^2 = 16 \quad\Rightarrow\quad \lambda\nabla g = \langle 2\lambda x, 2\lambda y\rangle$, so $6y = 2\lambda y \quad\Rightarrow\quad$ either $y = 0$ or $\lambda = 3$. If $y = 0$, then $x = \pm 4$; if $\lambda = 3$, then $4x - 4 = 2\lambda x \quad\Rightarrow\quad x = -2$ and $y = \pm 2\sqrt{3}$. Now $f(1,0) = -7$, $f(4,0) = 11$, $f(-4,0) = 43$, and $f\left(-2, \pm 2\sqrt{3}\right) = 47$. Thus the maximum value of $f(x,y)$ on the disk $x^2+y^2 \leq 16$ is $f\left(-2, \pm 2\sqrt{3}\right) = 47$, and the minimum value is $f(1,0) = -7$.

20. (a) $f(x,y) = 2x+3y, g(x,y) = \sqrt{x}+\sqrt{y} = 5 \quad\Rightarrow\quad \nabla f = \langle 2,3\rangle = \lambda\nabla g = \lambda\left\langle \dfrac{1}{2\sqrt{x}}, \dfrac{1}{2\sqrt{y}}\right\rangle$. Then

$2 = \dfrac{\lambda}{2\sqrt{x}}$ and $3 = \dfrac{\lambda}{2\sqrt{y}}$ so $4\sqrt{x} = \lambda = 6\sqrt{y} \quad\Rightarrow\quad \sqrt{y} = \frac{2}{3}\sqrt{x}$. With $\sqrt{x}+\sqrt{y} = 5$ we have $\sqrt{x}+\frac{2}{3}\sqrt{x} = 5 \quad\Rightarrow\quad \sqrt{x} = 3 \quad\Rightarrow\quad x = 9$. Substituting into $\sqrt{y} = \frac{2}{3}\sqrt{x}$ gives $\sqrt{y} = 2$ or $y = 4$. Thus the only possible extreme value subject to the constraint is $f(9,4) = 30$. (The question remains whether this is indeed the maximum of f.)

(b) $f(25,0) = 50$ which is larger than the result of part (a).

(c) 

We can see from the level curves of f that the maximum occurs at the left endpoint $(0,25)$ of the constraint curve g. The maximum value is $f(0,25) = 75$.

(d) Here ∇g does not exist if $x = 0$ or $y = 0$, so the method will not locate any associated points. Also, the method of Lagrange multipliers identifies points where the level curves of f share a common tangent line with the constraint curve g. This normally does not occur at an endpoint, although an absolute maximum or minimum may occur there.

(e) Here $f(9,4)$ is the absolute *minimum* of f subject to g.

36. $f(x,y,z) = xyz,\ g(x,y,z) = xy+yz+xz = 32 \quad\Rightarrow\quad \nabla f = \langle yz, xz, xy\rangle = \lambda\nabla g = \langle \lambda(y+z), \lambda(x+z), \lambda(x+y)\rangle$. Then $\lambda(y+z) = yz$ **(1)**, $\lambda(x+z) = xz$ **(2)**, and $\lambda(x+y) = xy$ **(3)**. And **(1)** minus **(2)** implies $\lambda(y-x) = z(y-x)$ so $x = y$ or $\lambda = z$. If $\lambda = z$, then **(1)** implies $z(y+z) = yz$ or $z = 0$ which is false. Thus $x = y$. Similarly **(2)** minus **(3)** implies $\lambda(z-y) = x(z-y)$ so $y = z$ or $\lambda = x$. As above, $\lambda \neq x$, so $x = y = z$ and $3x^2 = 32$ or $x = y = z = \frac{8}{\sqrt{6}}$ cm.