12.1

2. (a) From Definition 1, a convergent sequence is a sequence for which $\lim_{n\to\infty} a_n$ exists. Examples: $\{1/n\}$, $\{1/2^n\}$

(b) A divergent sequence is a sequence for which $\lim_{n\to\infty} a_n$ *does not* exist. Examples: $\{n\}$, $\{\sin n\}$

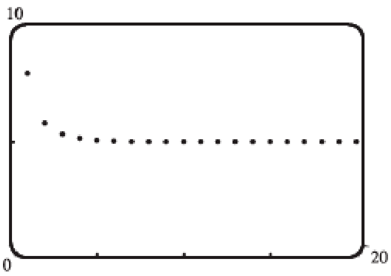
8. $a_1 = 4$, $a_{n+1} = \dfrac{a_n}{a_n - 1}$. Each term is defined in terms of the preceding term.

$a_2 = \dfrac{a_1}{a_1 - 1} = \dfrac{4}{4-1} = \dfrac{4}{3}$. $a_3 = \dfrac{a_2}{a_2 - 1} = \dfrac{4/3}{\frac{4}{3} - 1} = \dfrac{4/3}{1/3} = 4$. Since $a_3 = a_1$, we can see that the terms of the sequence will alternately equal 4 and $4/3$, so the sequence is $\{4, \frac{4}{3}, 4, \frac{4}{3}, 4, \ldots\}$.

28. $a_n = \cos(2/n)$. As $n \to \infty$, $2/n \to 0$, so $\cos(2/n) \to \cos 0 = 1$ because cos is continuous. Converges

36. $a_n = \ln(n+1) - \ln n = \ln\left(\dfrac{n+1}{n}\right) = \ln\left(1 + \dfrac{1}{n}\right) \to \ln(1) = 0$ as $n \to \infty$ because ln is continuous. Converges

50.



From the graph, it appears that the sequence converges to 5.

$$5 = \sqrt[n]{5^n} \le \sqrt[n]{3^n + 5^n} \le \sqrt[n]{5^n + 5^n} = \sqrt[n]{2}\,\sqrt[n]{5^n}$$
$$= \sqrt[n]{2} \cdot 5 \to 5 \text{ as } n \to \infty \quad \left[\lim_{n\to\infty} 2^{1/n} = 2^0 = 1\right]$$

Hence, $a_n \to 5$ by the Squeeze Theorem.

Alternate solution: Let $y = (3^x + 5^x)^{1/x}$. Then

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(3^x + 5^x)}{x} \overset{H}{=} \lim_{x\to\infty} \frac{3^x \ln 3 + 5^x \ln 5}{3^x + 5^x} = \lim_{x\to\infty} \frac{\left(\frac{3}{5}\right)^x \ln 3 + \ln 5}{\left(\frac{3}{5}\right)^x + 1} = \ln 5,$$

so $\lim_{x\to\infty} y = e^{\ln 5} = 5$, and so $\left\{\sqrt[n]{3^n + 5^n}\right\}$ converges to 5.

62. $a_n = \dfrac{2n-3}{3n+4}$ defines an increasing sequence since for $f(x) = \dfrac{2x-3}{3x+4}$,

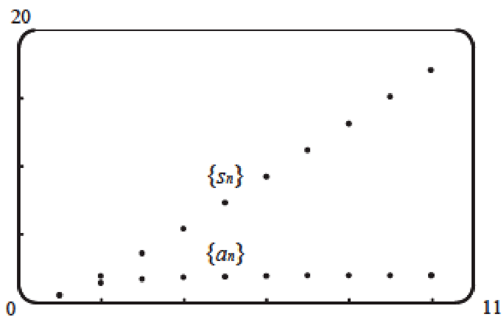
$f'(x) = \dfrac{(3x+4)(2) - (2x-3)(3)}{(3x+4)^2} = \dfrac{17}{(3x+4)^2} > 0$. The sequence is bounded since $a_n \ge a_1 = -\frac{1}{7}$ for $n \ge 1$,

and $a_n < \dfrac{2n-3}{3n} < \dfrac{2n}{3n} = \dfrac{2}{3}$ for $n \ge 1$.

12.2

4.

n	s_n
1	0.50000
2	1.90000
3	3.60000
4	5.42353
5	7.30814
6	9.22706
7	11.16706
8	13.12091
9	15.08432
10	17.05462



The series $\sum_{n=1}^{\infty} \dfrac{2n^2 - 1}{n^2 + 1}$ diverges, since its terms do not approach 0.

14. $1 + 0.4 + 0.16 + 0.064 + \cdots$ is a geometric series with ratio $r = 0.4 = \frac{2}{5}$. Since $|r| = \frac{2}{5} < 1$, the series converges to $\frac{a}{1-r} = \frac{1}{1-2/5} = \frac{5}{3}$.

42. $0.\overline{73} = \frac{73}{10^2} + \frac{73}{10^4} + \cdots = \frac{73/10^2}{1-1/10^2} = \frac{73/100}{99/100} = \frac{73}{99}$

58. (a) Initially, the ball falls a distance H, then rebounds a distance rH, falls rH, rebounds r^2H, falls r^2H, etc. The total distance it travels is

$$H + 2rH + 2r^2H + 2r^3H + \cdots = H(1 + 2r + 2r^2 + 2r^3 + \cdots) = H[1 + 2r(1 + r + r^2 + \cdots)]$$
$$= H\left[1 + 2r\left(\frac{1}{1-r}\right)\right] = H\left(\frac{1+r}{1-r}\right) \text{ meters}$$

(b) From Example 3 in Section 2.1, we know that a ball falls $\frac{1}{2}gt^2$ meters in t seconds, where g is the gravitational acceleration. Thus, a ball falls h meters in $t = \sqrt{2h/g}$ seconds. The total travel time in seconds is

$$\sqrt{\frac{2H}{g}} + 2\sqrt{\frac{2H}{g}r} + 2\sqrt{\frac{2H}{g}r^2} + 2\sqrt{\frac{2H}{g}r^3} + \cdots = \sqrt{\frac{2H}{g}}\left[1 + 2\sqrt{r} + 2\sqrt{r}^2 + 2\sqrt{r}^3 + \cdots\right]$$
$$= \sqrt{\frac{2H}{g}}\left(1 + 2\sqrt{r}\left[1 + \sqrt{r} + \sqrt{r}^2 + \cdots\right]\right)$$
$$= \sqrt{\frac{2H}{g}}\left[1 + 2\sqrt{r}\left(\frac{1}{1-\sqrt{r}}\right)\right] = \sqrt{\frac{2H}{g}}\frac{1+\sqrt{r}}{1-\sqrt{r}}$$

(c) It will help to make a chart of the time for each descent and each rebound of the ball, together with the velocity just before and just after each bounce. Recall that the time in seconds needed to fall h meters is $\sqrt{2h/g}$. The ball hits the ground with velocity $-g\sqrt{2h/g} = -\sqrt{2hg}$ (taking the upward direction to be positive) and rebounds with velocity $kg\sqrt{2h/g} = k\sqrt{2hg}$, taking time $k\sqrt{2h/g}$ to reach the top of its bounce, where its velocity is 0. At that point, its height is k^2h. All these results follow from the formulas for vertical motion with gravitational acceleration $-g$:
$\frac{d^2y}{dt^2} = -g \quad \Rightarrow \quad v = \frac{dy}{dt} = v_0 - gt \quad \Rightarrow \quad y = y_0 + v_0t - \frac{1}{2}gt^2$.

number of descent	time of descent	speed before bounce	speed after bounce	time of ascent	peak height
1	$\sqrt{2H/g}$	$\sqrt{2Hg}$	$k\sqrt{2Hg}$	$k\sqrt{2H/g}$	k^2H
2	$\sqrt{2k^2H/g}$	$\sqrt{2k^2Hg}$	$k\sqrt{2k^2Hg}$	$k\sqrt{2k^2H/g}$	k^4H
3	$\sqrt{2k^4H/g}$	$\sqrt{2k^4Hg}$	$k\sqrt{2k^4Hg}$	$k\sqrt{2k^4H/g}$	k^6H
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

The total travel time in seconds is

$$\sqrt{\frac{2H}{g}} + k\sqrt{\frac{2H}{g}} + k\sqrt{\frac{2H}{g}} + k^2\sqrt{\frac{2H}{g}} + k^2\sqrt{\frac{2H}{g}} + \cdots = \sqrt{\frac{2H}{g}}(1 + 2k + 2k^2 + 2k^3 + \cdots)$$
$$= \sqrt{\frac{2H}{g}}\left[1 + 2k(1 + k + k^2 + \cdots)\right]$$
$$= \sqrt{\frac{2H}{g}}\left[1 + 2k\left(\frac{1}{1-k}\right)\right] = \sqrt{\frac{2H}{g}}\frac{1+k}{1-k}$$

Another method: We could use part (b). At the top of the bounce, the height is $k^2h = rh$, so $\sqrt{r} = k$ and the result follows from part (b).

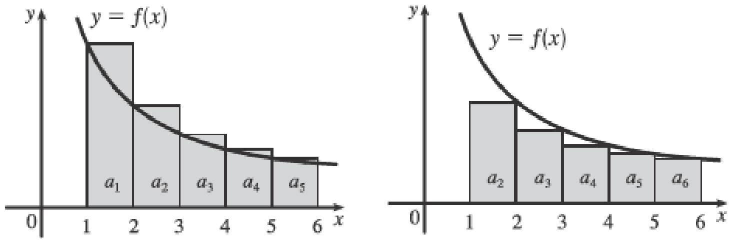
64. $|CD| = b\sin\theta$, $|DE| = |CD|\sin\theta = b\sin^2\theta$, $|EF| = |DE|\sin\theta = b\sin^3\theta$, ... Therefore,

$|CD| + |DE| + |EF| + |FG| + \cdots = b\sum_{n=1}^{\infty} \sin^n\theta = b\left(\dfrac{\sin\theta}{1-\sin\theta}\right)$ since this is a geometric series with $r = \sin\theta$

and $|\sin\theta| < 1$ $\left[\text{because } 0 < \theta < \frac{\pi}{2}\right]$.

12.3

2. From the first figure, we see that $\int_1^6 f(x)\,dx < \sum_{i=1}^{5} a_i$. From the second figure, we see that $\sum_{i=2}^{6} a_i < \int_1^6 f(x)\,dx$. Thus, we have $\sum_{i=2}^{6} a_i < \int_1^6 f(x)\,dx < \sum_{i=1}^{5} a_i$.



12.4

4. $\dfrac{n^3}{n^4-1} > \dfrac{n^3}{n^4} = \dfrac{1}{n}$ for all $n \geq 2$, so $\sum_{n=2}^{\infty} \dfrac{n^3}{n^4-1}$ diverges by comparison with $\sum_{n=2}^{\infty} \dfrac{1}{n}$, which diverges because it is a p-series with $p = 1 \leq 1$ (the harmonic series).

18. Use the Limit Comparison Test with $a_n = \dfrac{1}{2n+3}$ and $b_n = \dfrac{1}{n}$: $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n}{2n+3} = \lim\limits_{n\to\infty} \dfrac{1}{2+(3/n)} = \dfrac{1}{2} > 0$.

Since the harmonic series $\sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges, so does $\sum_{n=1}^{\infty} \dfrac{1}{2n+3}$.

24. If $a_n = \dfrac{n^2-5n}{n^3+n+1}$ and $b_n = \dfrac{1}{n}$, then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n^3-5n^2}{n^3+n+1} = \lim\limits_{n\to\infty} \dfrac{1-5/n}{1+1/n^2+1/n^3} = 1 > 0$,

so $\sum_{n=1}^{\infty} \dfrac{n^2-5n}{n^3+n+1}$ diverges by the Limit Comparison Test with the divergent harmonic series $\sum_{n=1}^{\infty} \dfrac{1}{n}$.

(Note that $a_n > 0$ for $n \geq 6$.)

28. $\dfrac{e^{1/n}}{n} > \dfrac{1}{n}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \dfrac{e^{1/n}}{n}$ diverges by comparison with the harmonic series $\sum_{n=1}^{\infty} \dfrac{1}{n}$.

34. $\sum_{n=1}^{10} \dfrac{\sin^2 n}{n^3} = \dfrac{\sin^2 1}{1} + \dfrac{\sin^2 2}{8} + \dfrac{\sin^2 3}{27} + \cdots + \dfrac{\sin^2 10}{1000} \approx 0.83253$. Now $\dfrac{\sin^2 n}{n^3} \leq \dfrac{1}{n^3}$, so the error is

$$R_{10} \leq T_{10} \leq \int_{10}^{\infty} \frac{1}{x^3}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2x^2}\right]_{10}^{t} = \lim_{t\to\infty}\left(-\frac{1}{2t^2} + \frac{1}{200}\right) = \frac{1}{200} = 0.005.$$

12.5

12. $b_n = \dfrac{e^{1/n}}{n} > 0$ for $n \geq 1$. $\{b_n\}$ is decreasing since $\left(\dfrac{e^{1/x}}{x}\right)' = \dfrac{x \cdot e^{1/x}(-1/x^2) - e^{1/x} \cdot 1}{x^2} = \dfrac{-e^{1/x}(1+x)}{x^3} < 0$ for

$x > 0$. Also, $\lim\limits_{n\to\infty} b_n = 0$ since $\lim\limits_{n\to\infty} e^{1/n} = 1$. Thus, the series $\sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{e^{1/n}}{n}$ converges by the Alternating Series Test.

24. The series $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n\,5^n}$ satisfies (i) of the Alternating Series Test because $\dfrac{1}{(n+1)5^{n+1}} < \dfrac{1}{n\,5^n}$ and (ii) $\lim\limits_{n\to\infty} \dfrac{1}{n\,5^n} = 0$, so

the series is convergent. Now $b_4 = \dfrac{1}{4 \cdot 5^4} = 0.0004 > 0.0001$ and $b_5 = \dfrac{1}{5 \cdot 5^5} = 0.000064 < 0.0001$, so by the Alternating Series Estimation Theorem, $n = 4$. (That is, since the 5th term is less than the desired error, we need to add the first 4 terms to get the sum to the desired accuracy.)

32. If $p > 0$, $\dfrac{1}{(n+1)^p} \le \dfrac{1}{n^p}$ ($\{1/n^p\}$ is decreasing) and $\lim\limits_{n\to\infty} \dfrac{1}{n^p} = 0$, so the series converges by the Alternating Series Test.

If $p \le 0$, $\lim\limits_{n\to\infty} \dfrac{(-1)^{n-1}}{n^p}$ does not exist, so the series diverges by the Test for Divergence. Thus, $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n^p}$ converges $\Leftrightarrow$ $p > 0$.

12.6

2. The series $\sum\limits_{n=1}^{\infty} \dfrac{n^2}{2^n}$ has positive terms and $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \left[\dfrac{(n+1)^2}{2^{n+1}} \cdot \dfrac{2^n}{n^2}\right] = \lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^2 \cdot \dfrac{1}{2} = \dfrac{1}{2} < 1$, so the series is absolutely convergent by the Ratio Test.

12. $\left|\dfrac{\sin 4n}{4^n}\right| \le \dfrac{1}{4^n}$, so $\sum\limits_{n=1}^{\infty} \left|\dfrac{\sin 4n}{4^n}\right|$ converges by comparison with the convergent geometric series $\sum\limits_{n=1}^{\infty} \dfrac{1}{4^n}$ $\left[|r| = \frac{1}{4} < 1\right]$. Thus, $\sum\limits_{n=1}^{\infty} \dfrac{\sin 4n}{4^n}$ is absolutely convergent.

38. (a) $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{[4(n+1)]!\,[1103 + 26{,}390(n+1)]}{[(n+1)!]^4\, 396^{4(n+1)}} \cdot \dfrac{(n!)^4\, 396^{4n}}{(4n)!\,(1103 + 26{,}390n)}\right|$

$$= \lim_{n\to\infty} \frac{(4n+4)(4n+3)(4n+2)(4n+1)(26{,}390n + 27{,}493)}{(n+1)^4\, 396^4\, (26{,}390n + 1103)} = \frac{4^4}{396^4} = \frac{1}{99^4} < 1,$$

so by the Ratio Test, the series $\sum\limits_{n=0}^{\infty} \dfrac{(4n)!\,(1103 + 26{,}390n)}{(n!)^4\, 396^{4n}}$ converges.

(b) $\dfrac{1}{\pi} = \dfrac{2\sqrt{2}}{9801} \sum\limits_{n=0}^{\infty} \dfrac{(4n)!\,(1103 + 26{,}390n)}{(n!)^4\, 396^{4n}}$

With the first term ($n = 0$), $\dfrac{1}{\pi} \approx \dfrac{2\sqrt{2}}{9801} \cdot \dfrac{1103}{1}$ $\Rightarrow$ $\pi \approx 3.141\,592\,73$, so we get 6 correct decimal places of π, which is $3.141\,592\,653\,589\,793\,238$ to 18 decimal places.

With the second term ($n = 1$), $\dfrac{1}{\pi} \approx \dfrac{2\sqrt{2}}{9801}\left(\dfrac{1103}{1} + \dfrac{4!\,(1103 + 26{,}390)}{396^4}\right)$ $\Rightarrow$ $\pi \approx 3.141\,592\,653\,589\,793\,878$, so we get 15 correct decimal places of π.

12.7

2. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \sqrt[n]{\left|\dfrac{(2n+1)^n}{n^{2n}}\right|} = \lim\limits_{n\to\infty} \dfrac{2n+1}{n^2} = \lim\limits_{n\to\infty} \left(\dfrac{2}{n} + \dfrac{1}{n^2}\right) = 0 < 1$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{(2n+1)^n}{n^{2n}}$ converges by the Root Test.

12. The series $\sum\limits_{n=1}^{\infty} \sin n$ diverges by the Test for Divergence since $\lim\limits_{n\to\infty} \sin n$ does not exist.

20. $\lim\limits_{k\to\infty} \left|\dfrac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty} \left|\dfrac{k+6}{5^{k+1}} \cdot \dfrac{5^k}{k+5}\right| = \dfrac{1}{5} \lim\limits_{k\to\infty} \dfrac{k+6}{k+5} = \dfrac{1}{5} < 1$, so the series $\sum\limits_{k=1}^{\infty} \dfrac{k+5}{5^k}$ converges by the Ratio Test.

26. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \left(\dfrac{n^2+2n+2}{5^{n+1}} \cdot \dfrac{5^n}{n^2+1}\right) = \lim\limits_{n\to\infty} \left(\dfrac{1 + 2/n + 2/n^2}{1 + 1/n^2} \cdot \dfrac{1}{5}\right) = \dfrac{1}{5} < 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{n^2+1}{5^n}$ converges by the Ratio Test.

12.8

2. (a) Given the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, the radius of convergence is:

(i) 0 if the series converges only when $x = a$

(ii) ∞ if the series converges for all x, or

(iii) a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

In most cases, R can be found by using the Ratio Test.

(b) The interval of convergence of a power series is the interval that consists of all values of x for which the series converges. Corresponding to the cases in part (a), the interval of convergence is: (i) the single point $\{a\}$, (ii) all real numbers; that is, the real number line $(-\infty, \infty)$, or (iii) an interval with endpoints $a-R$ and $a+R$ which can contain neither, either, or both of the endpoints. In this case, we must test the series for convergence at each endpoint to determine the interval of convergence.

10. If $a_n = \dfrac{10^n\, x^n}{n^3}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{10^{n+1}\, x^{n+1}}{(n+1)^3}\cdot\frac{n^3}{10^n\, x^n}\right| = \lim_{n\to\infty}\left|\frac{10x\, n^3}{(n+1)^3}\right| = \lim_{n\to\infty}\frac{10\,|x|}{(1+1/n)^3} = \frac{10\,|x|}{1^3} = 10\,|x|$$

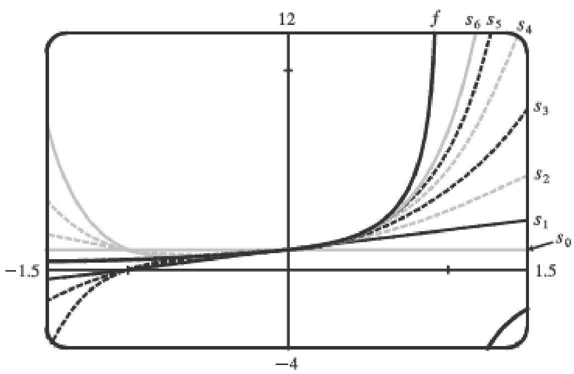
By the Ratio Test, the series $\sum_{n=1}^{\infty}\dfrac{10^n\, x^n}{n^3}$ converges when $10\,|x| < 1 \quad\Leftrightarrow\quad |x| < \frac{1}{10}$, so the radius of convergence is $R = \frac{1}{10}$.

When $x = -\frac{1}{10}$, the series converges by the Alternating Series Test; when $x = \frac{1}{10}$, the series converges because it is a p-series with $p = 3 > 1$. Thus, the interval of convergence is $I = \left[-\frac{1}{10}, \frac{1}{10}\right]$.

24. $a_n = \dfrac{n^2 x^n}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)} = \dfrac{n^2 x^n}{2^n n!} = \dfrac{n x^n}{2^n(n-1)!}$, so

$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)\,|x|^{n+1}}{2^{n+1}n!}\cdot\frac{2^n(n-1)!}{n\,|x|^n} = \lim_{n\to\infty}\frac{n+1}{n^2}\frac{|x|}{2} = 0$. Thus, by the Ratio Test, the series converges for *all* real x and we have $R = \infty$ and $I = (-\infty, \infty)$.

34. The partial sums of the series $\sum_{n=0}^{\infty} x^n$ definitely do not converge to $f(x) = 1/(1-x)$ for $x \geq 1$, since f is undefined at $x = 1$ and negative on $(1, \infty)$, while all the partial sums are positive on this interval. The partial sums also fail to converge to f for $x \leq -1$, since $0 < f(x) < 1$ on this interval, while the partial sums are either larger than 1 or less than 0. The partial sums seem to converge to f on $(-1, 1)$. This graphical evidence is consistent with what we know about geometric series: convergence for $|x| < 1$, divergence for $|x| \geq 1$ (see Examples 1 and 5 in Section 11.2).



12.9

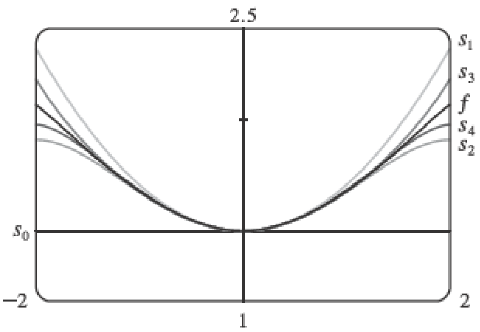
20. $f(x) = \ln(x^2+4) \quad \Rightarrow \quad f'(x) = \dfrac{2x}{x^2+4} = \dfrac{2x}{4}\left(\dfrac{1}{1-(-x^2/4)}\right) = \dfrac{x}{2}\sum\limits_{n=0}^{\infty}\left(-\dfrac{x^2}{4}\right)^n = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{2^{2n+1}}$,

so $f(x) = \displaystyle\int \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2^{2n+1}}\,dx = C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+2}}{2^{2n+1}(2n+2)} = \ln 4 + \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+2}}{(n+1)2^{2n+2}}$

$[f(0) = \ln 4$, so $C = \ln 4]$. The series converges when $\left|-x^2/4\right| < 1 \quad \Leftrightarrow \quad x^2 < 4 \quad \Leftrightarrow \quad |x| < 2$, so $R = 2$. If

$x = \pm 2$, then $f(x) = \ln 4 + \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{1}{n+1}$, which converges by the Alternating Series Test. The partial sums

are $s_0 = \ln 4 \;\; [\approx 1.39]$, $s_1 = s_0 + \dfrac{x^2}{4}$, $s_2 = s_1 - \dfrac{x^4}{2\cdot 2^4}$, $s_3 = s_2 + \dfrac{x^6}{3\cdot 2^6}$, $s_4 = s_3 - \dfrac{x^8}{4\cdot 2^8}, \ldots$.



As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $[-2, 2]$.

28. From Example 6, we know $\ln(1-x) = -\sum\limits_{n=1}^{\infty}\dfrac{x^n}{n}$, so

$\ln(1+x^4) = \ln[1-(-x^4)] = -\sum\limits_{n=1}^{\infty}\dfrac{(-x^4)^n}{n} = \sum\limits_{n=1}^{\infty}(-1)^{n+1}\dfrac{x^{4n}}{n} \quad \Rightarrow$

$\displaystyle\int \ln(1+x^4)\,dx = \int \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{4n}}{n}\,dx = C + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{4n+1}}{n(4n+1)}$. Thus,

$I = \displaystyle\int_0^{0.4}\ln(1+x^4)\,dx = \left[\frac{x^5}{5} - \frac{x^9}{18} + \frac{x^{13}}{39} - \frac{x^{17}}{68} + \cdots\right]_0^{0.4} = \frac{(0.4)^5}{5} - \frac{(0.4)^9}{18} + \frac{(0.4)^{13}}{39} - \frac{(0.4)^{17}}{68} + \cdots$.

The series is alternating, so if we use the first three terms, the error is at most $(0.4)^{17}/68 \approx 2.5\times 10^{-9}$.

So $I \approx (0.4)^5/5 - (0.4)^9/18 + (0.9)^{13}/39 \approx 0.002034$ to six decimal places.

12.10

2. (a) Using Equation 6, a power series expansion of f at 1 must have the form $f(1) + f'(1)(x-1) + \cdots$. Comparing to the given series, $1.6 - 0.8(x-1) + \cdots$, we must have $f'(1) = -0.8$. But from the graph, $f'(1)$ is positive. Hence, the given series is *not* the Taylor series of f centered at 1.

(b) A power series expansion of f at 2 must have the form $f(2) + f'(2)(x-2) + \frac{1}{2}f''(2)(x-2)^2 + \cdots$. Comparing to the given series, $2.8 + 0.5(x-2) + 1.5(x-2)^2 - 0.1(x-2)^3 + \cdots$, we must have $\frac{1}{2}f''(2) = 1.5$; that is, $f''(2)$ is positive. But from the graph, f is concave downward near $x = 2$, so $f''(2)$ must be negative. Hence, the given series is *not* the Taylor series of f centered at 2.

30. $\cos x = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{2n}}{(2n)!} \quad \Rightarrow \quad f(x) = \cos(\pi x/2) = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{(\pi x/2)^{2n}}{(2n)!} = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{\pi^{2n}}{2^{2n}\,(2n)!}x^{2n}$, $R = \infty$.

56. $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{1+x-e^x}=\lim_{x\to 0}\frac{1-\left(1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots\right)}{1+x-\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\cdots\right)}$

$$=\lim_{x\to 0}\frac{\frac{1}{2!}x^2-\frac{1}{4!}x^4+\frac{1}{6!}x^6-\cdots}{-\frac{1}{2!}x^2-\frac{1}{3!}x^3-\frac{1}{4!}x^4-\frac{1}{5!}x^5-\frac{1}{6!}x^6-\cdots}$$

$$=\lim_{x\to 0}\frac{\frac{1}{2!}-\frac{1}{4!}x^2+\frac{1}{6!}x^4-\cdots}{-\frac{1}{2!}-\frac{1}{3!}x-\frac{1}{4!}x^2-\frac{1}{5!}x^3-\frac{1}{6!}x^4-\cdots}=\frac{\frac{1}{2}-0}{-\frac{1}{2}-0}=-1$$

since power series are continuous functions.

60. $\sec x=\dfrac{1}{\cos x}\overset{(16)}{=}\dfrac{1}{1-\frac{1}{2}x^2+\frac{1}{24}x^4-\cdots}$.

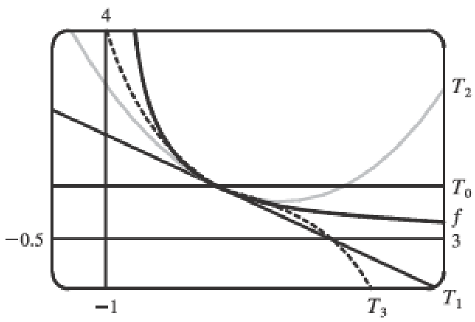
$$\begin{array}{r|l}
 & 1+\frac{1}{2}x^2+\frac{5}{24}x^4+\cdots \\ \hline
1-\frac{1}{2}x^2+\frac{1}{24}x^4-\cdots & 1 \\
 & 1-\frac{1}{2}x^2+\frac{1}{24}x^4-\cdots \\ \hline
 & \quad\frac{1}{2}x^2-\frac{1}{24}x^4+\cdots \\
 & \quad\frac{1}{2}x^2-\ \frac{1}{4}x^4+\cdots \\ \hline
 & \qquad\frac{5}{24}x^4+\cdots \\
 & \qquad\frac{5}{24}x^4+\cdots \\ \hline
 & \qquad\qquad\cdots
\end{array}$$

From the long division above, $\sec x=1+\frac{1}{2}x^2+\frac{5}{24}x^4+\cdots$.

12.11

2. (a)

n	$f^{(n)}(x)$	$f^{(n)}(1)$	$T_n(x)$
0	x^{-1}	1	1
1	$-x^{-2}$	-1	$1-(x-1)=2-x$
2	$2x^{-3}$	2	$1-(x-1)+(x-1)^2=x^2-3x+3$
3	$-6x^{-4}$	-6	$1-(x-1)+(x-1)^2-(x-1)^3=-x^3+4x^2-6x+4$



(b)

x	f	T_0	T_1	T_2	T_3
0.9	$1.\overline{1}$	1	1.1	1.11	1.111
1.3	0.7692	1	0.7	0.79	0.763

(c) As n increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval.

28. $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$. By the Alternating Series Estimation Theorem, the error is less than $\left|-\frac{1}{6!}x^6\right| < 0.005 \quad \Leftrightarrow$ $x^6 < 720(0.005) \quad \Leftrightarrow \quad |x| < (3.6)^{1/6} \approx 1.238$. The curves $y = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$ and $y = \cos x + 0.005$ intersect at $x \approx 1.244$, so the graph confirms our estimate. Since both the cosine function and the given approximation are even functions, we need to check the estimate only for $x > 0$. Thus, the desired range of values for x is $-1.238 < x < 1.238$.