10.1

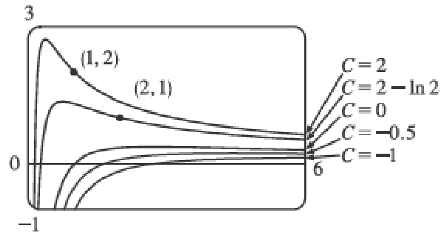
2. $y = \sin x\ \cos x - \cos x \quad \Rightarrow \quad y' = \sin x\,(-\sin x) + \cos x\,(\cos x) - (-\sin x) = \cos^2 x - \sin^2 x + \sin x.$

$$\begin{aligned}\text{LHS} = y' + (\tan x)y &= \cos^2 x - \sin^2 x + \sin x + (\tan x)(\sin x\ \cos x - \cos x)\\ &= \cos^2 x - \sin^2 x + \sin x + \sin^2 x - \sin x = \cos^2 x = \text{ RHS},\end{aligned}$$

so y is a solution of the differential equation. Also, $y(0) = \sin 0\ \cos 0 - \cos 0 = 0 \cdot 1 - 1 = -1$, so the initial condition is satisfied.

6. (a) $y = \dfrac{\ln x + C}{x} \quad \Rightarrow \quad y' = \dfrac{x \cdot (1/x) - (\ln x + C)}{x^2} = \dfrac{1 - \ln x - C}{x^2}.$

$$\begin{aligned}\text{LHS} = x^2 y' + xy &= x^2 \cdot \frac{1 - \ln x - C}{x^2} + x \cdot \frac{\ln x + C}{x}\\ &= 1 - \ln x - C + \ln x + C = 1 = \text{RHS},\end{aligned}$$ so y is a solution of the differential equation.

(b) 

A few notes about the graph of $y = (\ln x + C)/x$:

(1) There is a vertical asymptote of $x = 0$.

(2) There is a horizontal asymptote of $y = 0$.

(3) $y = 0 \quad \Rightarrow \quad \ln x + C = 0 \quad \Rightarrow \quad x = e^{-C}$, so there is an x-intercept at e^{-C}.

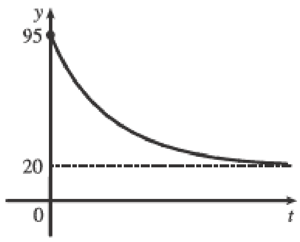
(4) $y' = 0 \quad \Rightarrow \quad \ln x = 1 - C \quad \Rightarrow \quad x = e^{1-C}$, so there is a local maximum at $x = e^{1-C}$.

(c) $y(1) = 2 \quad \Rightarrow \quad 2 = \dfrac{\ln 1 + C}{1} \quad \Rightarrow \quad 2 = C$, so the solution is $y = \dfrac{\ln x + 2}{x}$ [shown in part (b)].

(d) $y(2) = 1 \quad \Rightarrow \quad 1 = \dfrac{\ln 2 + C}{2} \quad \Rightarrow \quad 2 + \ln 2 + C \quad \Rightarrow \quad C = 2 - \ln 2$, so the solution is $y = \dfrac{\ln x + 2 - \ln 2}{x}$ [shown in part (b)].

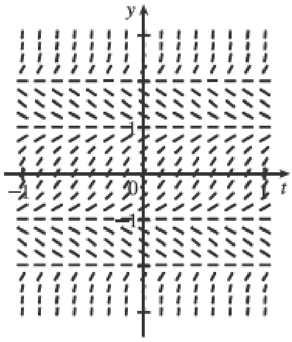
14. (a) The coffee cools most quickly as soon as it is removed from the heat source. The rate of cooling decreases toward 0 since the coffee approaches room temperature.

(b) $\dfrac{dy}{dt} = k(y - R)$, where k is a proportionality constant, y is the temperature of the coffee, and R is the room temperature. The initial condition is $y(0) = 95°\text{C}$. The answer and the model support each other because as y approaches R, dy/dt approaches 0, so the model seems appropriate.

(c) 

10.2

6. $y' = \sin x \sin y = 0$ on the lines $x = 0$ and $y = 0$, and $y' > 0$ for $0 < x < \pi$, $0 < y < \pi$. Direction field II satisfies these conditions.

18. 

Note that when $f(y) = 0$ on the graph in the text, we have $y' = f(y) = 0$; so we get horizontal segments at $y = \pm 1, \pm 2$. We get segments with negative slopes only for $1 < |y| < 2$. All other segments have positive slope. For the limiting behavior of solutions:

- If $y(0) > 2$, then $\lim_{t\to\infty} y = \infty$ and $\lim_{t\to-\infty} y = 2$.
- If $1 < y(0) < 2$, then $\lim_{t\to\infty} y = 1$ and $\lim_{t\to-\infty} y = 2$.
- If $-1 < y(0) < 1$, then $\lim_{t\to\infty} y = 1$ and $\lim_{t\to-\infty} y = -1$.
- If $-2 < y(0) < -1$, then $\lim_{t\to\infty} y = -2$ and $\lim_{t\to-\infty} y = -1$.
- If $y < -2$, then $\lim_{t\to\infty} y = -2$ and $\lim_{t\to-\infty} y = -\infty$.

10.3

12. $\frac{dy}{dx} = \frac{y\cos x}{1+y^2}$, $y(0) = 1$. $(1+y^2)\,dy = y\cos x\,dx \;\Rightarrow\; \frac{1+y^2}{y}dy = \cos x\,dx \;\Rightarrow\; \int\left(\frac{1}{y}+y\right)dy = \int\cos x\,dx \;\Rightarrow$

$\ln|y| + \frac{1}{2}y^2 = \sin x + C$. $y(0) = 1 \;\Rightarrow\; \ln 1 + \frac{1}{2} = \sin 0 + C \;\Rightarrow\; C = \frac{1}{2}$, so $\ln|y| + \frac{1}{2}y^2 = \sin x + \frac{1}{2}$.

We cannot solve explicitly for y.

16. $xy' + y = y^2 \;\Rightarrow\; x\frac{dy}{dx} = y^2 - y \;\Rightarrow\; x\,dy = (y^2-y)\,dx \;\Rightarrow\; \frac{dy}{y^2-y} = \frac{dx}{x} \;\Rightarrow$

$\int\frac{dy}{y(y-1)} = \int\frac{dx}{x}\quad [y \neq 0, 1] \;\Rightarrow\; \int\left(\frac{1}{y-1} - \frac{1}{y}\right)dy = \int\frac{dx}{x} \;\Rightarrow\; \ln|y-1| - \ln|y| = \ln|x| + C \;\Rightarrow$

$\ln\left|\frac{y-1}{y}\right| = \ln\left(e^C|x|\right) \;\Rightarrow\; \left|\frac{y-1}{y}\right| = e^C|x| \;\Rightarrow\; \frac{y-1}{y} = Kx$, where $K = \pm e^C \;\Rightarrow\; 1 - \frac{1}{y} = Kx \;\Rightarrow$

$\frac{1}{y} = 1 - Kx \;\Rightarrow\; y = \frac{1}{1-Kx}$. [The excluded cases, $y = 0$ and $y = 1$, are ruled out by the initial condition $y(1) = -1$.]

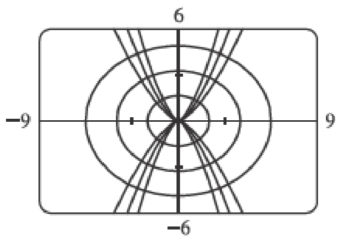
Now $y(1) = -1 \;\Rightarrow\; -1 = \frac{1}{1-K} \;\Rightarrow\; 1 - K = -1 \;\Rightarrow\; K = 2$, so $y = \frac{1}{1-2x}$.

30. The curves $y^2 = kx^3$ form a family of power functions. Differentiating gives $\frac{d}{dx}(y^2) = \frac{d}{dx}(kx^3) \;\Rightarrow\; 2yy' = 3kx^2 \;\Rightarrow$

$y' = \frac{3kx^2}{2y} = \frac{3(y^2/x^3)x^2}{2y} = \frac{3y}{2x}$, the slope of the tangent line at (x, y) on one of the curves. Thus, the orthogonal trajectories must satisfy $y' = -\frac{2x}{3y} \;\Leftrightarrow\; \frac{dy}{dx} = -\frac{2x}{3y} \;\Leftrightarrow$

$3y\,dy = -2x\,dx \;\Leftrightarrow\; \int 3y\,dy = \int -2x\,dx \;\Leftrightarrow\; \frac{3}{2}y^2 = -x^2 + C_1 \;\Leftrightarrow$

$3y^2 = -2x^2 + C_2 \;\Leftrightarrow\; 2x^2 + 3y^2 = C$. This is a family of ellipses.



38. If $S = \frac{dT}{dr}$, then $\frac{dS}{dr} = \frac{d^2T}{dr^2}$. The differential equation $\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} = 0$ can be written as $\frac{dS}{dr} + \frac{2}{r}S = 0$. Thus,

$\frac{dS}{dr} = \frac{-2S}{r} \;\Rightarrow\; \frac{dS}{S} = -\frac{2}{r}\,dr \;\Rightarrow\; \int\frac{1}{S}\,dS = \int -\frac{2}{r}\,dr \;\Rightarrow\; \ln|S| = -2\ln|r| + C$. Assuming $S = dT/dr > 0$

and $r > 0$, we have $S = e^{-2\ln r + C} = e^{\ln r^{-2}}e^C = r^{-2}k\;\; [k = e^C] \;\Rightarrow\; S = \frac{1}{r^2}k \;\Rightarrow\; \frac{dT}{dr} = \frac{1}{r^2}k \;\Rightarrow$

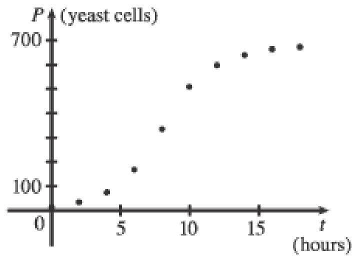
$dT = \frac{1}{r^2}k\,dr \;\Rightarrow\; \int dT = \int\frac{1}{r^2}k\,dr \;\Rightarrow\; T(r) = -\frac{k}{r} + A$.

$T(1) = 15 \;\Rightarrow\; 15 = -k + A$ (1) and $T(2) = 25 \;\Rightarrow\; 25 = -\frac{1}{2}k + A$ (2).

Now solve for k and A: $-2(2) + (1) \;\Rightarrow\; -35 = -A$, so $A = 35$ and $k = 20$, and $T(r) = -20/r + 35$.

10.4

4. (a)



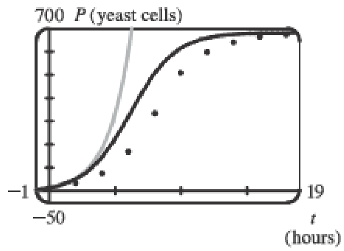
From the graph, we estimate the carrying capacity K for the yeast population to be 680.

(b) An estimate of the initial relative growth rate is $\frac{1}{P_0}\frac{dP}{dt} = \frac{1}{18} \cdot \frac{39-18}{2-0} = \frac{7}{12} = 0.58\overline{3}$.

(c) An exponential model is $P(t) = 18e^{7t/12}$. A logistic model is $P(t) = \dfrac{680}{1 + Ae^{-7t/12}}$, where $A = \frac{680-18}{18} = \frac{331}{9}$.

(d)

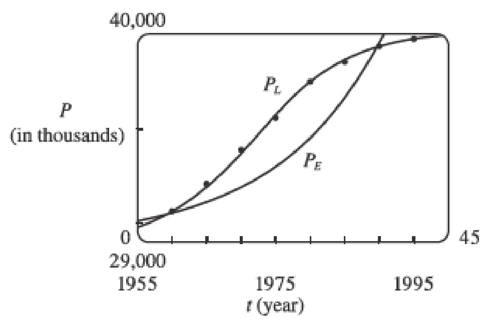
Time in Hours	Observed Values	Exponential Model	Logistic Model
0	18	18	18
2	39	58	55
4	80	186	149
6	171	596	322
8	336	1914	505
10	509	6147	614
12	597	19,739	658
14	640	63,389	673
16	664	203,558	678
18	672	653,679	679



The exponential model is a poor fit for anything beyond the first two observed values. The logistic model varies more for the middle values than it does for the values at either end, but provides a good general fit, as shown in the figure.

(e) $P(7) = \dfrac{680}{1 + \frac{331}{9}e^{-7(7/12)}} \approx 420$ yeast cells

12. Following the hint, we choose $t = 0$ to correspond to 1955 and subtract 29,000 from each of the population figures. We then use a calculator to obtain the models and add 29,000 to get the exponential function $P_E(t) = 1094(1.0668)^t + 29{,}000$ and the logistic function $P_L(t) = \dfrac{11{,}103.3}{1 + 12.34e^{-0.1471t}} + 29{,}000$. P_L is a reasonably accurate model, while P_E is not, since an exponential model would only be used for the first few data points.



10.5

6. $y' = x + 5y \quad\Rightarrow\quad y' - 5y = x$. $\quad I(x) = e^{\int P(x)\,dx} = e^{\int (-5)\,dx} = e^{-5x}$. Multiplying the differential equation by $I(x)$ gives $e^{-5x}y' - 5e^{-5x}y = xe^{-5x} \quad\Rightarrow\quad (e^{-5x}y)' = xe^{-5x} \quad\Rightarrow\quad e^{-5x}y = \int xe^{-5x}\,dx = -\frac{1}{5}xe^{-5x} - \frac{1}{25}e^{-5x} + C$ [by parts] $\quad\Rightarrow\quad y = -\frac{1}{5}x - \frac{1}{25} + Ce^{5x}$.

26. $xy'' + 2y' = 12x^2$ and $u = y' \quad\Rightarrow\quad xu' + 2u = 12x^2 \quad\Rightarrow\quad u' + \dfrac{2}{x}u = 12x$.

$I(x) = e^{\int (2/x)\,dx} = e^{2\ln|x|} = \left(e^{\ln|x|}\right)^2 = |x|^2 = x^2$. Multiplying the last differential equation by x^2 gives

$x^2u' + 2xu = 12x^3 \quad\Rightarrow\quad (x^2u)' = 12x^3 \quad\Rightarrow\quad x^2u = \int 12x^3\,dx = 3x^4 + C \quad\Rightarrow\quad u = 3x^2 + C/x^2 \quad\Rightarrow$

$y' = 3x^2 + C/x^2 \quad\Rightarrow\quad y = x^3 - C/x + D$.

28. (a) $\frac{dI}{dt} + 20I = 40\sin 60t$, so the integrating factor is e^{20t}. Multiplying the differential equation by the integrating factor gives $e^{20t}\frac{dI}{dt} + 20Ie^{20t} = 40e^{20t}\sin 60t \Rightarrow (e^{20t}I)' = 40e^{20t}\sin 60t \Rightarrow$

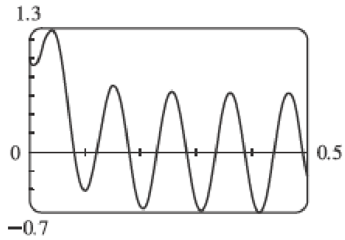
$$I(t) = e^{-20t}\left[\int 40e^{20t}\sin 60t\,dt + C\right] = e^{-20t}\left[40e^{20t}\left(\tfrac{1}{4000}\right)(20\sin 60t - 60\cos 60t)\right] + Ce^{-20t}$$

$$= \frac{\sin 60t - 3\cos 60t}{5} + Ce^{-20t}$$

But $1 = I(0) = -\frac{3}{5} + C$, so $I(t) = \dfrac{\sin 60t - 3\cos 60t + 8e^{-20t}}{5}$.

(b) $I(0.1) = \dfrac{\sin 6 - 3\cos 6 + 8e^{-2}}{5} \approx -0.42$ A

(c)



34. Let $y(t)$ denote the amount of chlorine in the tank at time t (in seconds). $y(0) = (0.05\text{ g/L})(400\text{ L}) = 20$ g. The amount of liquid in the tank at time t is $(400 - 6t)$ L since 4 L of water enters the tank each second and 10 L of liquid leaves the tank each second. Thus, the concentration of chlorine at time t is $\frac{y(t)}{400-6t}\frac{\text{g}}{\text{L}}$. Chlorine doesn't enter the tank, but it leaves at a rate of $\left[\frac{y(t)}{400-6t}\frac{\text{g}}{\text{L}}\right]\left[10\frac{\text{L}}{\text{s}}\right] = \frac{10\,y(t)}{400-6t}\frac{\text{g}}{\text{s}} = \frac{5\,y(t)}{200-3t}\frac{\text{g}}{\text{s}}$. Therefore, $\frac{dy}{dt} = -\frac{5y}{200-3t} \Rightarrow \int\frac{dy}{y} = \int\frac{-5\,dt}{200-3t} \Rightarrow$ $\ln y = \frac{5}{3}\ln(200-3t) + C \Rightarrow y = \exp\left(\frac{5}{3}\ln(200-3t) + C\right) = e^C(200-3t)^{5/3}$. Now $20 = y(0) = e^C \cdot 200^{5/3} \Rightarrow$ $e^C = \frac{20}{200^{5/3}}$, so $y(t) = 20\frac{(200-3t)^{5/3}}{200^{5/3}} = 20(1-0.015t)^{5/3}$ g for $0 \le t \le 66\frac{2}{3}$ s, at which time the tank is empty.

10.6

6.