SOLUTION We use a graphing device to produce the graphs for the cases $a = -2, -1, -0.5, -0.2, 0, 0.5, 1$, and 2 shown in Figure 17. Notice that all of these curves (except the case $a = 0$) have two branches, and both branches approach the vertical asymptote $x = a$ as x approaches a from the left or right.

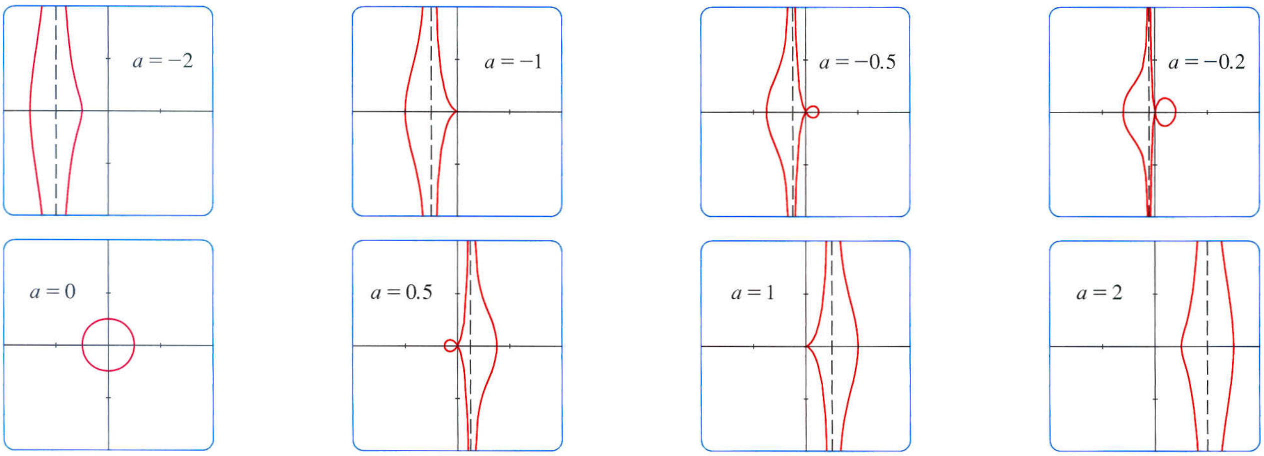


FIGURE 17 Members of the family $x = a + \cos t$, $y = a \tan t + \sin t$, all graphed in the viewing rectangle $[-4, 4]$ by $[-4, 4]$

When $a < -1$, both branches are smooth; but when a reaches -1, the right branch acquires a sharp point, called a *cusp*. For a between -1 and 0 the cusp turns into a loop, which becomes larger as a approaches 0. When $a = 0$, both branches come together and form a circle (see Example 2). For a between 0 and 1, the left branch has a loop, which shrinks to become a cusp when $a = 1$. For $a > 1$, the branches become smooth again, and as a increases further, they become less curved. Notice that the curves with a positive are reflections about the y-axis of the corresponding curves with a negative.

These curves are called **conchoids of Nicomedes** after the ancient Greek scholar Nicomedes. He called them conchoids because the shape of their outer branches resembles that of a conch shell or mussel shell. □

11.1 EXERCISES

1–4 Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as t increases.

1. $x = 1 + \sqrt{t}, \quad y = t^2 - 4t, \quad 0 \leqslant t \leqslant 5$

2. $x = 2\cos t, \quad y = t - \cos t, \quad 0 \leqslant t \leqslant 2\pi$

3. $x = 5\sin t, \quad y = t^2, \quad -\pi \leqslant t \leqslant \pi$

4. $x = e^{-t} + t, \quad y = e^t - t, \quad -2 \leqslant t \leqslant 2$

5–10
(a) Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as t increases.
(b) Eliminate the parameter to find a Cartesian equation of the curve.

5. $x = 3t - 5, \quad y = 2t + 1$

6. $x = 1 + t, \quad y = 5 - 2t, \quad -2 \leqslant t \leqslant 3$

7. $x = t^2 - 2, \quad y = 5 - 2t, \quad -3 \leqslant t \leqslant 4$

8. $x = 1 + 3t, \quad y = 2 - t^2$

9. $x = \sqrt{t}, \quad y = 1 - t$

10. $x = t^2, \quad y = t^3$

11–18
(a) Eliminate the parameter to find a Cartesian equation of the curve.
(b) Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

11. $x = \sin\theta, \quad y = \cos\theta, \quad 0 \leqslant \theta \leqslant \pi$

12. $x = 4\cos\theta, \quad y = 5\sin\theta, \quad -\pi/2 \leqslant \theta \leqslant \pi/2$

13. $x = \sin t, \quad y = \csc t, \quad 0 < t < \pi/2$

14. $x = \sec\theta, \quad y = \tan\theta, \quad -\pi/2 < \theta < \pi/2$

15. $x = e^t, \quad y = e^{-t}$

16. $x = \ln t, \quad y = \sqrt{t}, \quad t \geqslant 1$

17. $x = \sinh t, \quad y = \cosh t$

18. $x = 2\cosh t, \quad y = 5\sinh t$

19–22 Describe the motion of a particle with position (x, y) as t varies in the given interval.

19. $x = \cos \pi t, \quad y = \sin \pi t, \quad 1 \leqslant t \leqslant 2$

20. $x = 2 + \cos t, \quad y = 3 + \sin t, \quad 0 \leqslant t \leqslant 2\pi$

21. $x = 5 \sin t, \quad y = 2 \cos t, \quad -\pi \leqslant t \leqslant 5\pi$

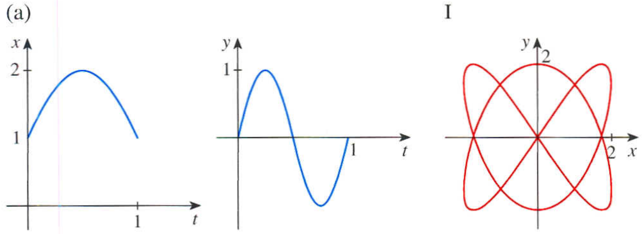
22. $x = \sin t, \quad y = \cos^2 t, \quad -2\pi \leqslant t \leqslant 2\pi$

---

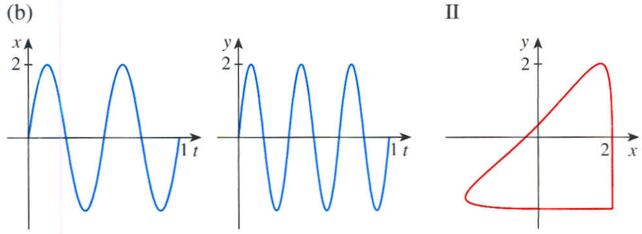
23. Suppose a curve is given by the parametric equations $x = f(t)$, $y = g(t)$, where the range of f is $[1, 4]$ and the range of g is $[2, 3]$. What can you say about the curve?

24. Match the graphs of the parametric equations $x = f(t)$ and $y = g(t)$ in (a)–(d) with the parametric curves labeled I–IV. Give reasons for your choices.

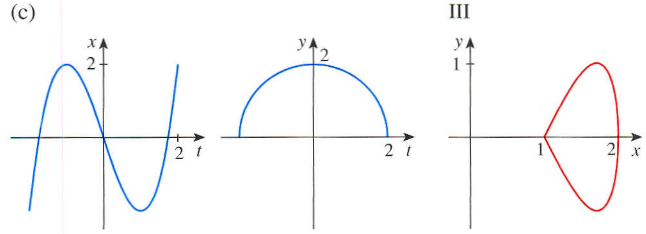
(a) I



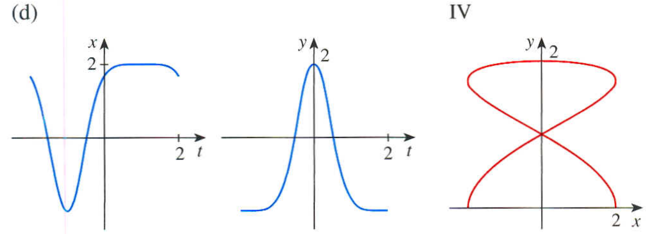
(b) II



(c) III

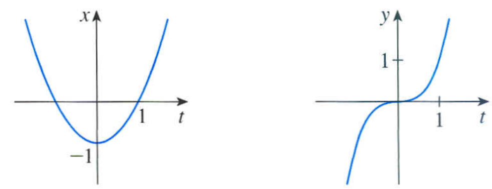


(d) IV

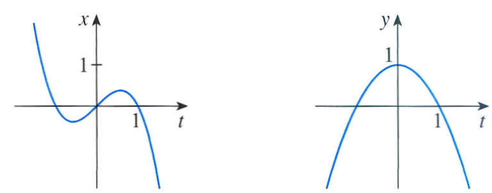


25–27 Use the graphs of $x = f(t)$ and $y = g(t)$ to sketch the parametric curve $x = f(t)$, $y = g(t)$. Indicate with arrows the direction in which the curve is traced as t increases.

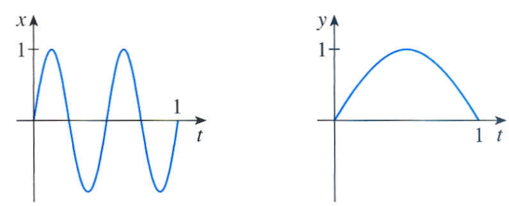
25.



26.



27.



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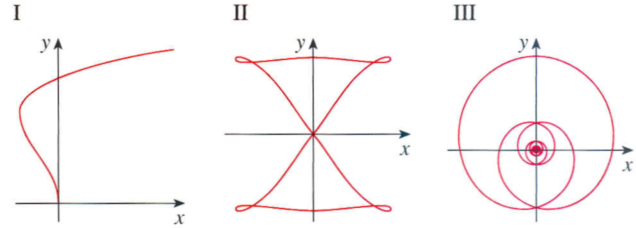
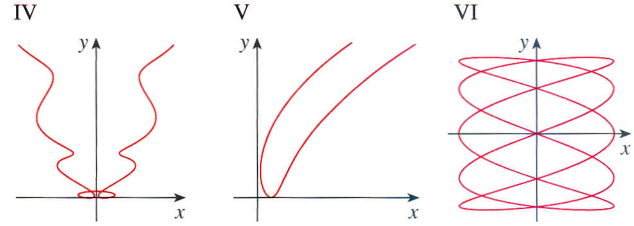
28. Match the parametric equations with the graphs labeled I-VI. Give reasons for your choices. (Do not use a graphing device.)

(a) $x = t^4 - t + 1, \quad y = t^2$
(b) $x = t^2 - 2t, \quad y = \sqrt{t}$
(c) $x = \sin 2t, \quad y = \sin(t + \sin 2t)$
(d) $x = \cos 5t, \quad y = \sin 2t$
(e) $x = t + \sin 4t, \quad y = t^2 + \cos 3t$
(f) $x = \dfrac{\sin 2t}{4 + t^2}, \quad y = \dfrac{\cos 2t}{4 + t^2}$

I II III



IV V VI



29. Graph the curve $x = y - 3y^3 + y^5$.

30. Graph the curves $y = x^5$ and $x = y(y - 1)^2$ and find their points of intersection correct to one decimal place.

31. (a) Show that the parametric equations

$$x = x_1 + (x_2 - x_1)t \qquad y = y_1 + (y_2 - y_1)t$$

where $0 \leqslant t \leqslant 1$, describe the line segment that joins the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

(b) Find parametric equations to represent the line segment from $(-2, 7)$ to $(3, -1)$.

32. Use a graphing device and the result of Exercise 31(a) to draw the triangle with vertices $A(1, 1)$, $B(4, 2)$, and $C(1, 5)$.

33. Find parametric equations for the path of a particle that moves along the circle $x^2 + (y - 1)^2 = 4$ in the manner described.

(a) Once around clockwise, starting at $(2, 1)$
(b) Three times around counterclockwise, starting at $(2, 1)$
(c) Halfway around counterclockwise, starting at $(0, 3)$

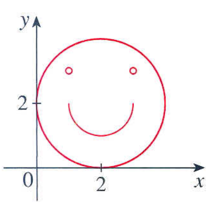
34. (a) Find parametric equations for the ellipse $x^2/a^2 + y^2/b^2 = 1$. [*Hint:* Modify the equations of the circle in Example 2.]

(b) Use these parametric equations to graph the ellipse when $a = 3$ and $b = 1, 2, 4$, and 8.

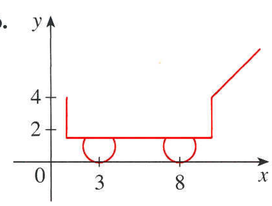
(c) How does the shape of the ellipse change as b varies?

35–36 Use a graphing calculator or computer to reproduce the picture.

35.



36.



37–38 Compare the curves represented by the parametric equations. How do they differ?

37. (a) $x = t^3, \quad y = t^2$ (b) $x = t^6, \quad y = t^4$
(c) $x = e^{-3t}, \quad y = e^{-2t}$

38. (a) $x = t, \quad y = t^{-2}$ (b) $x = \cos t, \quad y = \sec^2 t$
(c) $x = e^t, \quad y = e^{-2t}$

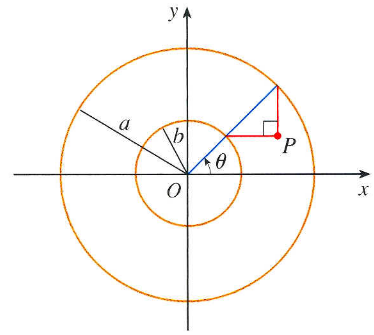
39. Derive Equations 1 for the case $\pi/2 < \theta < \pi$.

40. Let P be a point at a distance d from the center of a circle of radius r. The curve traced out by P as the circle rolls along a straight line is called a **trochoid**. (Think of the motion of a point on a spoke of a bicycle wheel.) The cycloid is the special case of a trochoid with $d = r$. Using the same parameter θ as for the cycloid and, assuming the line is the x-axis and $\theta = 0$ when P is at one of its lowest points, show that parametric equations of the trochoid are

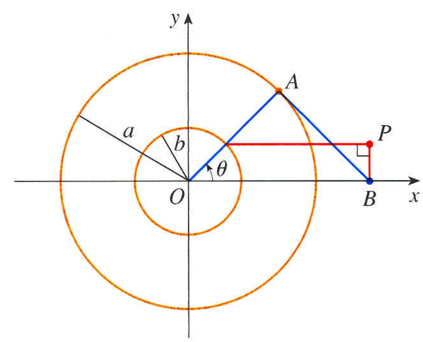
$$x = r\theta - d\sin\theta \qquad y = r - d\cos\theta$$

Sketch the trochoid for the cases $d < r$ and $d > r$.

41. If a and b are fixed numbers, find parametric equations for the curve that consists of all possible positions of the point P in the figure, using the angle θ as the parameter. Then eliminate the parameter and identify the curve.



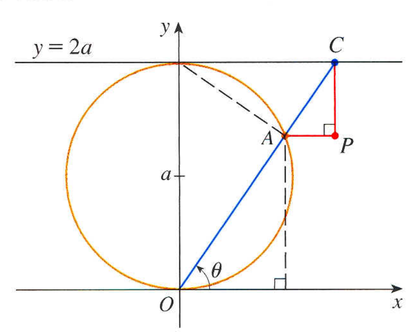
42. If a and b are fixed numbers, find parametric equations for the curve that consists of all possible positions of the point P in the figure, using the angle θ as the parameter. The line segment AB is tangent to the larger circle.



43. A curve, called a **witch of Maria Agnesi**, consists of all possible positions of the point P in the figure. Show that parametric equations for this curve can be written as

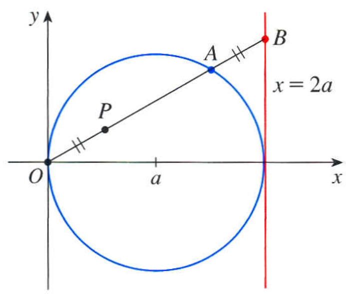
$$x = 2a\cot\theta \qquad y = 2a\sin^2\theta$$

Sketch the curve.



44. (a) Find parametric equations for the set of all points P as shown in the figure such that $|OP| = |AB|$. (This curve is called the **cissoid of Diocles** after the Greek scholar Diocles, who introduced the cissoid as a graphical method for constructing the edge of a cube whose volume is twice that of a given cube.)

(b) Use the geometric description of the curve to draw a rough sketch of the curve by hand. Check your work by using the parametric equations to graph the curve.



45. Suppose that the position of one particle at time t is given by

$$x_1 = 3 \sin t \qquad y_1 = 2 \cos t \qquad 0 \leqslant t \leqslant 2\pi$$

and the position of a second particle is given by

$$x_2 = -3 + \cos t \qquad y_2 = 1 + \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

(a) Graph the paths of both particles. How many points of intersection are there?
(b) Are any of these points of intersection *collision points*? In other words, are the particles ever at the same place at the same time? If so, find the collision points.
(c) Describe what happens if the path of the second particle is given by

$$x_2 = 3 + \cos t \qquad y_2 = 1 + \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

46. If a projectile is fired with an initial velocity of v_0 meters per second at an angle α above the horizontal and air resistance is assumed to be negligible, then its position after t seconds is given by the parametric equations

$$x = (v_0 \cos \alpha)t \qquad y = (v_0 \sin \alpha)t - \tfrac{1}{2}gt^2$$

where g is the acceleration due to gravity (9.8 m/s^2).
(a) If a gun is fired with $\alpha = 30°$ and $v_0 = 500$ m/s, when will the bullet hit the ground? How far from the gun will it hit the ground? What is the maximum height reached by the bullet?
(b) Use a graphing device to check your answers to part (a). Then graph the path of the projectile for several other values of the angle α to see where it hits the ground. Summarize your findings.
(c) Show that the path is parabolic by eliminating the parameter.

47. Investigate the family of curves defined by the parametric equations $x = t^2$, $y = t^3 - ct$. How does the shape change as c increases? Illustrate by graphing several members of the family.

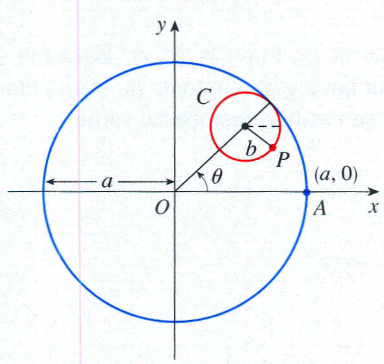
48. The **swallowtail catastrophe curves** are defined by the parametric equations $x = 2ct - 4t^3$, $y = -ct^2 + 3t^4$. Graph several of these curves. What features do the curves have in common? How do they change when c increases?

49. The curves with equations $x = a \sin nt$, $y = b \cos t$ are called **Lissajous figures**. Investigate how these curves vary when a, b, and n vary. (Take n to be a positive integer.)

50. Investigate the family of curves defined by the parametric equations $x = \cos t$, $y = \sin t - \sin ct$, where $c > 0$. Start by letting c be a positive integer and see what happens to the shape as c increases. Then explore some of the possibilities that occur when c is a fraction.

LABORATORY PROJECT

RUNNING CIRCLES AROUND CIRCLES



TEC Look at Module 11.1B to see how hypocycloids and epicycloids are formed by the motion of rolling circles.

In this project we investigate families of curves, called *hypocycloids* and *epicycloids,* that are generated by the motion of a point on a circle that rolls inside or outside another circle.

1. A **hypocycloid** is a curve traced out by a fixed point P on a circle C of radius b as C rolls on the inside of a circle with center O and radius a. Show that if the initial position of P is $(a, 0)$ and the parameter θ is chosen as in the figure, then parametric equations of the hypocycloid are

$$x = (a - b)\cos\theta + b\cos\left(\frac{a-b}{b}\theta\right) \qquad y = (a - b)\sin\theta - b\sin\left(\frac{a-b}{b}\theta\right)$$

2. Use a graphing device (or the interactive graphic in TEC Module 11.1B) to draw the graphs of hypocycloids with a a positive integer and $b = 1$. How does the value of a affect the graph? Show that if we take $a = 4$, then the parametric equations of the hypocycloid reduce to

$$x = 4\cos^3\theta \qquad y = 4\sin^3\theta$$

This curve is called a **hypocycloid of four cusps**, or an **astroid**.

about the x-axis. Therefore, from Formula 7, we get

$$S = \int_0^\pi 2\pi r \sin t \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt$$

$$= 2\pi \int_0^\pi r \sin t \sqrt{r^2(\sin^2 t + \cos^2 t)}\, dt = 2\pi \int_0^\pi r\sin t \cdot r\, dt$$

$$= 2\pi r^2 \int_0^\pi \sin t\, dt = 2\pi r^2(-\cos t)\Big]_0^\pi = 4\pi r^2$$

□

11.2 EXERCISES

1–2 Find dy/dx.

1. $x = t - t^3, \quad y = 2 - 5t$

2. $x = te^t, \quad y = t + e^t$

3–6 Find an equation of the tangent to the curve at the point corresponding to the given value of the parameter.

3. $x = t^2 + t, \quad y = t^2 - t; \quad t = 0$

4. $x = t - t^{-1}, \quad y = 1 + t^2; \quad t = 1$

5. $x = e^{\sqrt{t}}, \quad y = t - \ln t^2; \quad t = 1$

6. $x = t\sin t, \quad y = t\cos t; \quad t = \pi$

7–8 Find an equation of the tangent to the curve at the given point by two methods: (a) without eliminating the parameter and (b) by first eliminating the parameter.

7. $x = 1 + \ln t, \quad y = t^2 + 2; \quad (1, 3)$

8. $x = \tan\theta, \quad y = \sec\theta; \quad (1, \sqrt{2})$

9–10 Find an equation of the tangent(s) to the curve at the given point. Then graph the curve and the tangent(s).

9. $x = 6\sin t, \quad y = t^2 + t; \quad (0, 0)$

10. $x = \cos t + \cos 2t, \quad y = \sin t + \sin 2t; \quad (-1, 1)$

11–16 Find dy/dx and d^2y/dx^2. For which values of t is the curve concave upward?

11. $x = 4 + t^2, \quad y = t^2 + t^3$

12. $x = t^3 - 12t, \quad y = t^2 - 1$

13. $x = t - e^t, \quad y = t + e^{-t}$

14. $x = t + \ln t, \quad y = t - \ln t$

15. $x = 2\sin t, \quad y = 3\cos t, \quad 0 < t < 2\pi$

16. $x = \cos 2t, \quad y = \cos t, \quad 0 < t < \pi$

17–20 Find the points on the curve where the tangent is horizontal or vertical. If you have a graphing device, graph the curve to check your work.

17. $x = 10 - t^2, \quad y = t^3 - 12t$

18. $x = 2t^3 + 3t^2 - 12t, \quad y = 2t^3 + 3t^2 + 1$

19. $x = 2\cos\theta, \quad y = \sin 2\theta$

20. $x = \cos 3\theta, \quad y = 2\sin\theta$

21. Use a graph to estimate the coordinates of the rightmost point on the curve $x = t - t^6$, $y = e^t$. Then use calculus to find the exact coordinates.

22. Use a graph to estimate the coordinates of the lowest point and the leftmost point on the curve $x = t^4 - 2t$, $y = t + t^4$. Then find the exact coordinates.

23–24 Graph the curve in a viewing rectangle that displays all the important aspects of the curve.

23. $x = t^4 - 2t^3 - 2t^2, \quad y = t^3 - t$

24. $x = t^4 + 4t^3 - 8t^2, \quad y = 2t^2 - t$

25. Show that the curve $x = \cos t$, $y = \sin t \cos t$ has two tangents at $(0, 0)$ and find their equations. Sketch the curve.

26. Graph the curve $x = \cos t + 2\cos 2t$, $y = \sin t + 2\sin 2t$ to discover where it crosses itself. Then find equations of both tangents at that point.

27. (a) Find the slope of the tangent line to the trochoid $x = r\theta - d\sin\theta$, $y = r - d\cos\theta$ in terms of θ. (See Exercise 40 in Section 11.1.)
(b) Show that if $d < r$, then the trochoid does not have a vertical tangent.

28. (a) Find the slope of the tangent to the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$ in terms of θ. (Astroids are explored in the Laboratory Project on page 665.)
(b) At what points is the tangent horizontal or vertical?
(c) At what points does the tangent have slope 1 or -1?

29. At what points on the curve $x = 2t^3$, $y = 1 + 4t - t^2$ does the tangent line have slope 1?

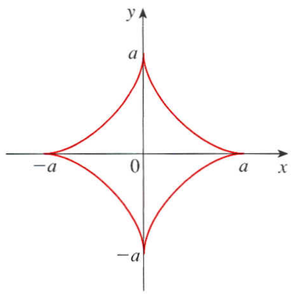
30. Find equations of the tangents to the curve $x = 3t^2 + 1$, $y = 2t^3 + 1$ that pass through the point $(4, 3)$.

31. Use the parametric equations of an ellipse, $x = a\cos\theta$, $y = b\sin\theta$, $0 \leqslant \theta \leqslant 2\pi$, to find the area that it encloses.

32. Find the area enclosed by the curve $x = t^2 - 2t$, $y = \sqrt{t}$ and the y-axis.

33. Find the area enclosed by the x-axis and the curve $x = 1 + e^t$, $y = t - t^2$.

34. Find the area of the region enclosed by the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$. (Astroids are explored in the Laboratory Project on page 665.)



35. Find the area under one arch of the trochoid of Exercise 40 in Section 11.1 for the case $d < r$.

36. Let $\mathcal{R}$ be the region enclosed by the loop of the curve in Example 1.
(a) Find the area of $\mathcal{R}$.
(b) If $\mathcal{R}$ is rotated about the x-axis, find the volume of the resulting solid.
(c) Find the centroid of $\mathcal{R}$.

37–40 Set up an integral that represents the length of the curve. Then use your calculator to find the length correct to four decimal places.

37. $x = t - t^2$, $y = \frac{4}{3}t^{3/2}$, $1 \leqslant t \leqslant 2$

38. $x = 1 + e^t$, $y = t^2$, $-3 \leqslant t \leqslant 3$

39. $x = t + \cos t$, $y = t - \sin t$, $0 \leqslant t \leqslant 2\pi$

40. $x = \ln t$, $y = \sqrt{t + 1}$, $1 \leqslant t \leqslant 5$

41–44 Find the exact length of the curve.

41. $x = 1 + 3t^2$, $y = 4 + 2t^3$, $0 \leqslant t \leqslant 1$

42. $x = e^t + e^{-t}$, $y = 5 - 2t$, $0 \leqslant t \leqslant 3$

43. $x = \dfrac{t}{1 + t}$, $y = \ln(1 + t)$, $0 \leqslant t \leqslant 2$

44. $x = 3\cos t - \cos 3t$, $y = 3\sin t - \sin 3t$, $0 \leqslant t \leqslant \pi$

45–47 Graph the curve and find its length.

45. $x = e^t\cos t$, $y = e^t\sin t$, $0 \leqslant t \leqslant \pi$

46. $x = \cos t + \ln(\tan \frac{1}{2}t)$, $y = \sin t$, $\pi/4 \leqslant t \leqslant 3\pi/4$

47. $x = e^t - t$, $y = 4e^{t/2}$, $-8 \leqslant t \leqslant 3$

48. Find the length of the loop of the curve $x = 3t - t^3$, $y = 3t^2$.

49. Use Simpson's Rule with $n = 6$ to estimate the length of the curve $x = t - e^t$, $y = t + e^t$, $-6 \leqslant t \leqslant 6$.

50. In Exercise 43 in Section 11.1 you were asked to derive the parametric equations $x = 2a\cot\theta$, $y = 2a\sin^2\theta$ for the curve called the witch of Maria Agnesi. Use Simpson's Rule with $n = 4$ to estimate the length of the arc of this curve given by $\pi/4 \leqslant \theta \leqslant \pi/2$.

51–52 Find the distance traveled by a particle with position (x, y) as t varies in the given time interval. Compare with the length of the curve.

51. $x = \sin^2 t$, $y = \cos^2 t$, $0 \leqslant t \leqslant 3\pi$

52. $x = \cos^2 t$, $y = \cos t$, $0 \leqslant t \leqslant 4\pi$

53. Show that the total length of the ellipse $x = a\sin\theta$, $y = b\cos\theta$, $a > b > 0$, is

$$L = 4a\int_0^{\pi/2} \sqrt{1 - e^2\sin^2\theta}\, d\theta$$

where e is the eccentricity of the ellipse $\left(e = c/a, \text{ where } c = \sqrt{a^2 - b^2}\right)$.

54. Find the total length of the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$, where $a > 0$.

CAS **55.** (a) Graph the **epitrochoid** with equations

$$x = 11\cos t - 4\cos(11t/2)$$
$$y = 11\sin t - 4\sin(11t/2)$$

What parameter interval gives the complete curve?
(b) Use your CAS to find the approximate length of this curve.

CAS **56.** A curve called **Cornu's spiral** is defined by the parametric equations

$$x = C(t) = \int_0^t \cos(\pi u^2/2)\, du$$
$$y = S(t) = \int_0^t \sin(\pi u^2/2)\, du$$

where C and S are the Fresnel functions that were introduced in Chapter 5.
(a) Graph this curve. What happens as $t \to \infty$ and as $t \to -\infty$?
(b) Find the length of Cornu's spiral from the origin to the point with parameter value t.

57–58 Set up an integral that represents the area of the surface obtained by rotating the given curve about the x-axis. Then use your calculator to find the surface area correct to four decimal places.

57. $x = 1 + te^t$, $y = (t^2 + 1)e^t$, $0 \leqslant t \leqslant 1$

58. $x = \sin^2 t$, $y = \sin 3t$, $0 \leqslant t \leqslant \pi/3$

59–61 Find the exact area of the surface obtained by rotating the given curve about the x-axis.

59. $x = t^3, \quad y = t^2, \quad 0 \leqslant t \leqslant 1$

60. $x = 3t - t^3, \quad y = 3t^2, \quad 0 \leqslant t \leqslant 1$

61. $x = a \cos^3\theta, \quad y = a \sin^3\theta, \quad 0 \leqslant \theta \leqslant \pi/2$

---

62. Graph the curve

$$x = 2 \cos \theta - \cos 2\theta \qquad y = 2 \sin \theta - \sin 2\theta$$

If this curve is rotated about the x-axis, find the area of the resulting surface. (Use your graph to help find the correct parameter interval.)

63. If the curve

$$x = t + t^3 \qquad y = t - \frac{1}{t^2} \qquad 1 \leqslant t \leqslant 2$$

is rotated about the x-axis, use your calculator to estimate the area of the resulting surface to three decimal places.

64. If the arc of the curve in Exercise 50 is rotated about the x-axis, estimate the area of the resulting surface using Simpson's Rule with $n = 4$.

65–66 Find the surface area generated by rotating the given curve about the y-axis.

65. $x = 3t^2, \quad y = 2t^3, \quad 0 \leqslant t \leqslant 5$

66. $x = e^t - t, \quad y = 4e^{t/2}, \quad 0 \leqslant t \leqslant 1$

---

67. If f' is continuous and $f'(t) \neq 0$ for $a \leqslant t \leqslant b$, show that the parametric curve $x = f(t)$, $y = g(t)$, $a \leqslant t \leqslant b$, can be put in the form $y = F(x)$. [*Hint:* Show that f^{-1} exists.]

68. Use Formula 2 to derive Formula 7 from Formula 9.2.5 for the case in which the curve can be represented in the form $y = F(x)$, $a \leqslant x \leqslant b$.

69. The **curvature** at a point P of a curve is defined as

$$\kappa = \left| \frac{d\phi}{ds} \right|$$

where ϕ is the angle of inclination of the tangent line at P, as shown in the figure. Thus the curvature is the absolute value of the rate of change of ϕ with respect to arc length. It can be regarded as a measure of the rate of change of direction of the curve at P and will be studied in greater detail in Chapter 14.

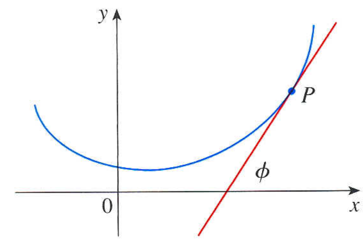
(a) For a parametric curve $x = x(t)$, $y = y(t)$, derive the formula

$$\kappa = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}}$$

where the dots indicate derivatives with respect to t, so $\dot{x} = dx/dt$. [*Hint:* Use $\phi = \tan^{-1}(dy/dx)$ and Formula 2 to find $d\phi/dt$. Then use the Chain Rule to find $d\phi/ds$.]

(b) By regarding a curve $y = f(x)$ as the parametric curve $x = x$, $y = f(x)$, with parameter x, show that the formula in part (a) becomes

$$\kappa = \frac{|d^2y/dx^2|}{[1 + (dy/dx)^2]^{3/2}}$$



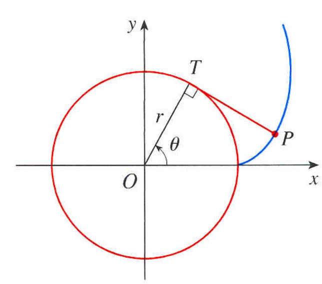
70. (a) Use the formula in Exercise 69(b) to find the curvature of the parabola $y = x^2$ at the point $(1, 1)$.
(b) At what point does this parabola have maximum curvature?

71. Use the formula in Exercise 69(a) to find the curvature of the cycloid $x = \theta - \sin\theta$, $y = 1 - \cos\theta$ at the top of one of its arches.

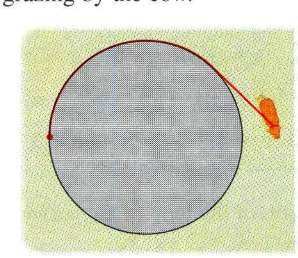
72. (a) Show that the curvature at each point of a straight line is $\kappa = 0$.
(b) Show that the curvature at each point of a circle of radius r is $\kappa = 1/r$.

73. A string is wound around a circle and then unwound while being held taut. The curve traced by the point P at the end of the string is called the **involute** of the circle. If the circle has radius r and center O and the initial position of P is $(r, 0)$, and if the parameter θ is chosen as in the figure, show that parametric equations of the involute are

$$x = r(\cos\theta + \theta \sin\theta) \qquad y = r(\sin\theta - \theta\cos\theta)$$



74. A cow is tied to a silo with radius r by a rope just long enough to reach the opposite side of the silo. Find the area available for grazing by the cow.



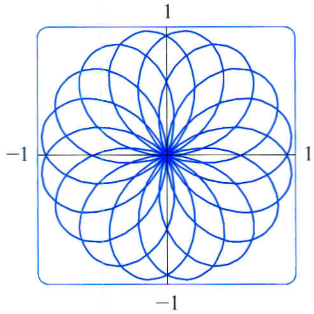


FIGURE 18
$r = \sin(8\theta/5)$

and so we require that $16n\pi/5$ be an even multiple of π. This will first occur when $n = 5$. Therefore we will graph the entire curve if we specify that $0 \leqslant \theta \leqslant 10\pi$. Switching from θ to t, we have the equations

$$x = \sin(8t/5)\cos t \qquad y = \sin(8t/5)\sin t \qquad 0 \leqslant t \leqslant 10\pi$$

and Figure 18 shows the resulting curve. Notice that this rose has 16 loops. □

EXAMPLE 11 Investigate the family of polar curves given by $r = 1 + c \sin\theta$. How does the shape change as c changes? (These curves are called **limaçons**, after a French word for snail, because of the shape of the curves for certain values of c.)

■ In Exercise 55 you are asked to prove analytically what we have discovered from the graphs in Figure 19.

SOLUTION Figure 19 shows computer-drawn graphs for various values of c. For $c > 1$ there is a loop that decreases in size as c decreases. When $c = 1$ the loop disappears and the curve becomes the cardioid that we sketched in Example 7. For c between 1 and $\frac{1}{2}$ the cardioid's cusp is smoothed out and becomes a "dimple." When c decreases from $\frac{1}{2}$ to 0, the limaçon is shaped like an oval. This oval becomes more circular as $c \to 0$, and when $c = 0$ the curve is just the circle $r = 1$.

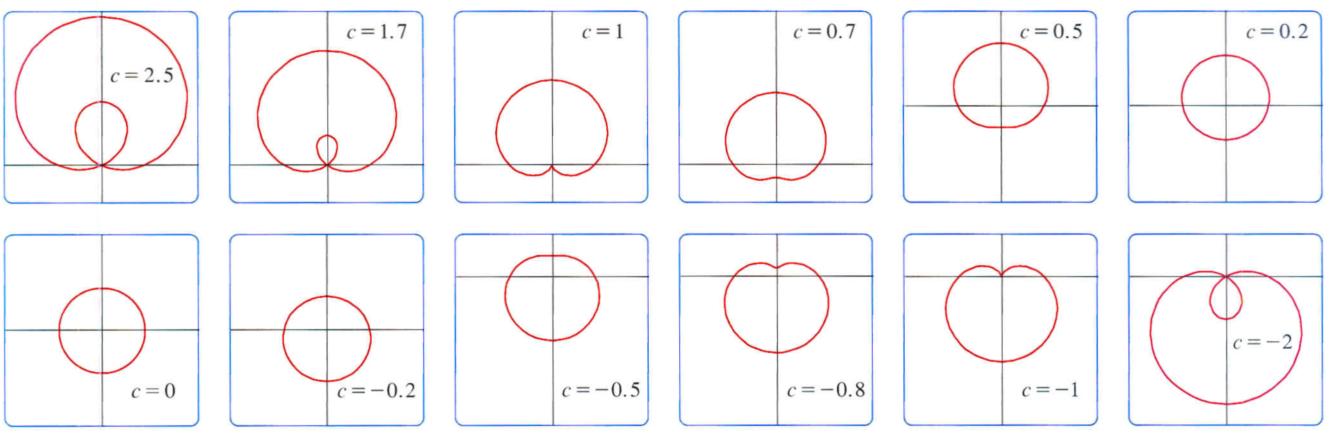


FIGURE 19
Members of the family of limaçons $r = 1 + c\sin\theta$

The remaining parts of Figure 19 show that as c becomes negative, the shapes change in reverse order. In fact, these curves are reflections about the horizontal axis of the corresponding curves with positive c. □

11.3 EXERCISES

1–2 Plot the point whose polar coordinates are given. Then find two other pairs of polar coordinates of this point, one with $r > 0$ and one with $r < 0$.

1. (a) $(2, \pi/3)$ (b) $(1, -3\pi/4)$ (c) $(-1, \pi/2)$

2. (a) $(1, 7\pi/4)$ (b) $(-3, \pi/6)$ (c) $(1, -1)$

3–4 Plot the point whose polar coordinates are given. Then find the Cartesian coordinates of the point.

3. (a) $(1, \pi)$ (b) $(2, -2\pi/3)$ (c) $(-2, 3\pi/4)$

4. (a) $(-\sqrt{2}, 5\pi/4)$ (b) $(1, 5\pi/2)$ (c) $(2, -7\pi/6)$

5–6 The Cartesian coordinates of a point are given.
(i) Find polar coordinates (r, θ) of the point, where $r > 0$ and $0 \leqslant \theta < 2\pi$.
(ii) Find polar coordinates (r, θ) of the point, where $r < 0$ and $0 \leqslant \theta < 2\pi$.

5. (a) $(2, -2)$ (b) $(-1, \sqrt{3})$

6. (a) $(3\sqrt{3}, 3)$ (b) $(1, -2)$

7–12 Sketch the region in the plane consisting of points whose polar coordinates satisfy the given conditions.

7. $1 \leqslant r \leqslant 2$

8. $r \geqslant 0, \quad \pi/3 \leqslant \theta \leqslant 2\pi/3$

9. $0 \leqslant r < 4, \quad -\pi/2 \leqslant \theta < \pi/6$

10. $2 < r \leqslant 5, \quad 3\pi/4 < \theta < 5\pi/4$

11. $2 < r < 3, \quad 5\pi/3 \leqslant \theta \leqslant 7\pi/3$

12. $r \geqslant 1, \quad \pi \leqslant \theta \leqslant 2\pi$

13. Find the distance between the points with polar coordinates $(2, \pi/3)$ and $(4, 2\pi/3)$.

14. Find a formula for the distance between the points with polar coordinates (r_1, θ_1) and (r_2, θ_2).

15–20 Identify the curve by finding a Cartesian equation for the curve.

15. $r = 2$

16. $r \cos\theta = 1$

17. $r = 3 \sin\theta$

18. $r = 2\sin\theta + 2\cos\theta$

19. $r = \csc\theta$

20. $r = \tan\theta \sec\theta$

21–26 Find a polar equation for the curve represented by the given Cartesian equation.

21. $y = 5$

22. $x^2 + y^2 = 9$

23. $x = -y^2$

24. $y = 2x - 1$

25. $x^2 + y^2 = 2cx$

26. $xy = 4$

27–28 For each of the described curves, decide if the curve would be more easily given by a polar equation or a Cartesian equation. Then write an equation for the curve.

27. (a) A line through the origin that makes an angle of $\pi/6$ with the positive x-axis
(b) A vertical line through the point $(3, 3)$

28. (a) A circle with radius 5 and center $(2, 3)$
(b) A circle centered at the origin with radius 4

29–48 Sketch the curve with the given polar equation.

29. $\theta = -\pi/6$

30. $r^2 - 3r + 2 = 0$

31. $r = \sin\theta$

32. $r = -3\cos\theta$

33. $r = 2(1 - \sin\theta), \quad \theta \geqslant 0$

34. $r = 1 - 3\cos\theta$

35. $r = \theta, \quad \theta \geqslant 0$

36. $r = \ln\theta, \quad \theta \geqslant 1$

37. $r = \sin 2\theta$

38. $r = 2\cos 3\theta$

39. $r = 2\cos 4\theta$

40. $r = 3\cos 6\theta$

41. $r = 1 - 2\sin\theta$

42. $r = 2 + \sin\theta$

43. $r^2 = 9\sin 2\theta$

44. $r^2 = \cos 4\theta$

45. $r = 2\cos(3\theta/2)$

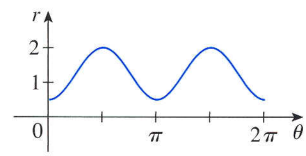
46. $r^2\theta = 1$

47. $r = 1 + 2\cos 2\theta$

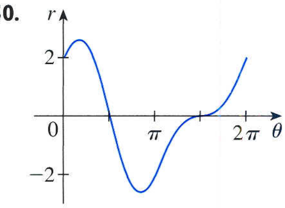
48. $r = 1 + 2\cos(\theta/2)$

49–50 The figure shows the graph of r as a function of θ in Cartesian coordinates. Use it to sketch the corresponding polar curve.

49.



50.



51. Show that the polar curve $r = 4 + 2\sec\theta$ (called a **conchoid**) has the line $x = 2$ as a vertical asymptote by showing that $\lim_{r\to\pm\infty} x = 2$. Use this fact to help sketch the conchoid.

52. Show that the curve $r = 2 - \csc\theta$ (also a conchoid) has the line $y = -1$ as a horizontal asymptote by showing that $\lim_{r\to\pm\infty} y = -1$. Use this fact to help sketch the conchoid.

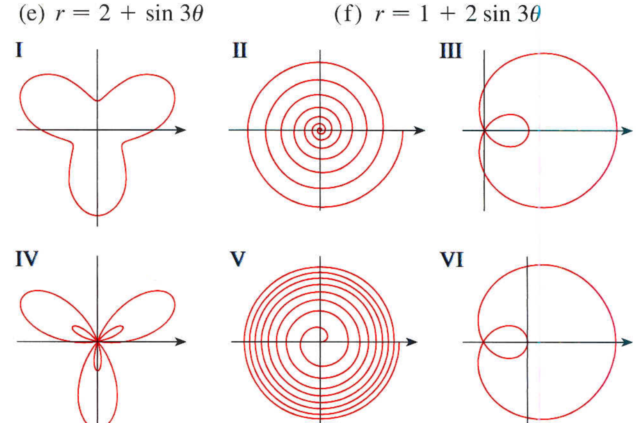
53. Show that the curve $r = \sin\theta\,\tan\theta$ (called a **cissoid of Diocles**) has the line $x = 1$ as a vertical asymptote. Show also that the curve lies entirely within the vertical strip $0 \leqslant x < 1$. Use these facts to help sketch the cissoid.

54. Sketch the curve $(x^2 + y^2)^3 = 4x^2y^2$.

55. (a) In Example 11 the graphs suggest that the limaçon $r = 1 + c\sin\theta$ has an inner loop when $|c| > 1$. Prove that this is true, and find the values of θ that correspond to the inner loop.
(b) From Figure 19 it appears that the limaçon loses its dimple when $c = \frac{1}{2}$. Prove this.

56. Match the polar equations with the graphs labeled I–VI. Give reasons for your choices. (Don't use a graphing device.)

(a) $r = \sqrt{\theta}, \quad 0 \leqslant \theta \leqslant 16\pi$
(b) $r = \theta^2, \quad 0 \leqslant \theta \leqslant 16\pi$
(c) $r = \cos(\theta/3)$
(d) $r = 1 + 2\cos\theta$
(e) $r = 2 + \sin 3\theta$
(f) $r = 1 + 2\sin 3\theta$



57–62 Find the slope of the tangent line to the given polar curve at the point specified by the value of θ.

57. $r = 2\sin\theta, \quad \theta = \pi/6$

58. $r = 2 - \sin\theta, \quad \theta = \pi/3$

59. $r = 1/\theta, \quad \theta = \pi$

60. $r = \cos(\theta/3), \quad \theta = \pi$

61. $r = \cos 2\theta, \quad \theta = \pi/4$

62. $r = 1 + 2\cos\theta, \quad \theta = \pi/3$

63–68 Find the points on the given curve where the tangent line is horizontal or vertical.

63. $r = 3\cos\theta$

64. $r = 1 - \sin\theta$

65. $r = 1 + \cos\theta$

66. $r = e^{\theta}$

67. $r = 2 + \sin\theta$

68. $r^2 = \sin 2\theta$

69. Show that the polar equation $r = a\sin\theta + b\cos\theta$, where $ab \neq 0$, represents a circle, and find its center and radius.

70. Show that the curves $r = a\sin\theta$ and $r = a\cos\theta$ intersect at right angles.

71–76 Use a graphing device to graph the polar curve. Choose the parameter interval to make sure that you produce the entire curve.

71. $r = 1 + 2\sin(\theta/2)$ (nephroid of Freeth)

72. $r = \sqrt{1 - 0.8\sin^2\theta}$ (hippopede)

73. $r = e^{\sin\theta} - 2\cos(4\theta)$ (butterfly curve)

74. $r = \sin^2(4\theta) + \cos(4\theta)$

75. $r = 2 - 5\sin(\theta/6)$

76. $r = \cos(\theta/2) + \cos(\theta/3)$

77. How are the graphs of $r = 1 + \sin(\theta - \pi/6)$ and $r = 1 + \sin(\theta - \pi/3)$ related to the graph of $r = 1 + \sin\theta$? In general, how is the graph of $r = f(\theta - \alpha)$ related to the graph of $r = f(\theta)$?

78. Use a graph to estimate the y-coordinate of the highest points on the curve $r = \sin 2\theta$. Then use calculus to find the exact value.

79. (a) Investigate the family of curves defined by the polar equations $r = \sin n\theta$, where n is a positive integer. How is the number of loops related to n?

(b) What happens if the equation in part (a) is replaced by $r = |\sin n\theta|$?

80. A family of curves is given by the equations $r = 1 + c\sin n\theta$, where c is a real number and n is a positive integer. How does the graph change as n increases? How does it change as c changes? Illustrate by graphing enough members of the family to support your conclusions.

81. A family of curves has polar equations

$$r = \frac{1 - a\cos\theta}{1 + a\cos\theta}$$

Investigate how the graph changes as the number a changes. In particular, you should identify the transitional values of a for which the basic shape of the curve changes.

82. The astronomer Giovanni Cassini (1625–1712) studied the family of curves with polar equations

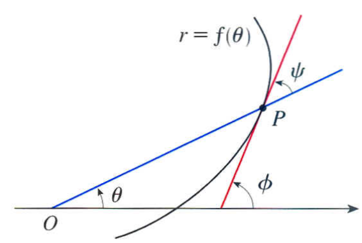
$$r^4 - 2c^2r^2\cos 2\theta + c^4 - a^4 = 0$$

where a and c are positive real numbers. These curves are called the **ovals of Cassini** even though they are oval shaped only for certain values of a and c. (Cassini thought that these curves might represent planetary orbits better than Kepler's ellipses.) Investigate the variety of shapes that these curves may have. In particular, how are a and c related to each other when the curve splits into two parts?

83. Let P be any point (except the origin) on the curve $r = f(\theta)$. If ψ is the angle between the tangent line at P and the radial line OP, show that

$$\tan\psi = \frac{r}{dr/d\theta}$$

[*Hint:* Observe that $\psi = \phi - \theta$ in the figure.]



84. (a) Use Exercise 83 to show that the angle between the tangent line and the radial line is $\psi = \pi/4$ at every point on the curve $r = e^{\theta}$.

(b) Illustrate part (a) by graphing the curve and the tangent lines at the points where $\theta = 0$ and $\pi/2$.

(c) Prove that any polar curve $r = f(\theta)$ with the property that the angle ψ between the radial line and the tangent line is a constant must be of the form $r = Ce^{k\theta}$, where C and k are constants.

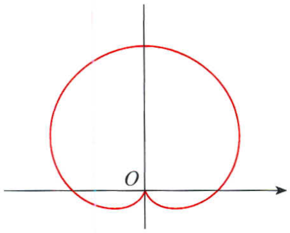


FIGURE 8
$r = 1 + \sin\theta$

Formula 5 gives

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta$$

We could evaluate this integral by multiplying and dividing the integrand by $\sqrt{2 - 2\sin\theta}$, or we could use a computer algebra system. In any event, we find that the length of the cardioid is $L = 8$. □

11.4 EXERCISES

1–4 Find the area of the region that is bounded by the given curve and lies in the specified sector.

1. $r = \theta^2$, $0 \leqslant \theta \leqslant \pi/4$

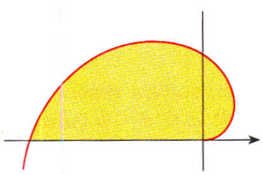
2. $r = e^{\theta/2}$, $\pi \leqslant \theta \leqslant 2\pi$

3. $r = \sin\theta$, $\pi/3 \leqslant \theta \leqslant 2\pi/3$

4. $r = \sqrt{\sin\theta}$, $0 \leqslant \theta \leqslant \pi$

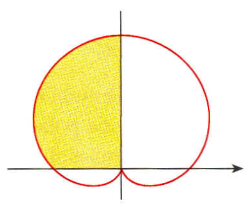
5–8 Find the area of the shaded region.

5.



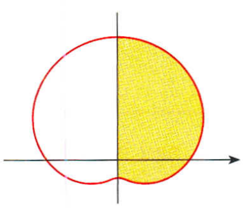
$r = \theta$

6.



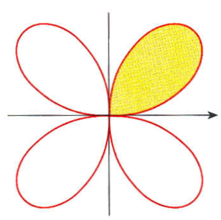
$r = 1 + \sin\theta$

7.



$r = 4 + 3\sin\theta$

8.



$r = \sin 2\theta$

9–14 Sketch the curve and find the area that it encloses.

9. $r = 3\cos\theta$

10. $r = 3(1 + \cos\theta)$

11. $r^2 = 4\cos 2\theta$

12. $r = 2 - \sin\theta$

13. $r = 2\cos 3\theta$

14. $r = 2 + \cos 2\theta$

15–16 Graph the curve and find the area that it encloses.

15. $r = 1 + 2\sin 6\theta$

16. $r = 2\sin\theta + 3\sin 9\theta$

17–21 Find the area of the region enclosed by one loop of the curve.

17. $r = \sin 2\theta$

18. $r = 4\sin 3\theta$

19. $r = 3\cos 5\theta$

20. $r = 2\sin 6\theta$

21. $r = 1 + 2\sin\theta$ (inner loop)

22. Find the area enclosed by the loop of the **strophoid** $r = 2\cos\theta - \sec\theta$.

23–28 Find the area of the region that lies inside the first curve and outside the second curve.

23. $r = 2\cos\theta$, $r = 1$

24. $r = 1 - \sin\theta$, $r = 1$

25. $r = 4\sin\theta$, $r = 2$

26. $r = 3\cos\theta$, $r = 2 - \cos\theta$

27. $r = 3\cos\theta$, $r = 1 + \cos\theta$

28. $r = 3\sin\theta$, $r = 2 - \sin\theta$

29–34 Find the area of the region that lies inside both curves.

29. $r = \sqrt{3}\cos\theta$, $r = \sin\theta$

30. $r = 1 + \cos\theta$, $r = 1 - \cos\theta$

31. $r = \sin 2\theta$, $r = \cos 2\theta$

32. $r = 3 + 2\cos\theta$, $r = 3 + 2\sin\theta$

33. $r^2 = \sin 2\theta$, $r^2 = \cos 2\theta$

34. $r = a\sin\theta$, $r = b\cos\theta$, $a > 0$, $b > 0$

35. Find the area inside the larger loop and outside the smaller loop of the limaçon $r = \frac{1}{2} + \cos\theta$.

36. Find the area between a large loop and the enclosed small loop of the curve $r = 1 + 2\cos 3\theta$.

37–42 Find all points of intersection of the given curves.

37. $r = 1 + \sin\theta$, $r = 3\sin\theta$

38. $r = 1 - \cos\theta$, $r = 1 + \sin\theta$

39. $r = 2\sin 2\theta$, $r = 1$

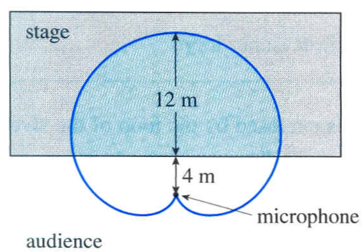
40. $r = \cos 3\theta$, $r = \sin 3\theta$

41. $r = \sin\theta$, $r = \sin 2\theta$

42. $r^2 = \sin 2\theta$, $r^2 = \cos 2\theta$

43. The points of intersection of the cardioid $r = 1 + \sin\theta$ and the spiral loop $r = 2\theta$, $-\pi/2 \leqslant \theta \leqslant \pi/2$, can't be found exactly. Use a graphing device to find the approximate values of θ at which they intersect. Then use these values to estimate the area that lies inside both curves.

44. When recording live performances, sound engineers often use a microphone with a cardioid pickup pattern because it suppresses noise from the audience. Suppose the microphone is placed 4 m from the front of the stage (as in the figure) and the boundary of the optimal pickup region is given by the cardioid $r = 8 + 8\sin\theta$, where r is measured in meters and the microphone is at the pole. The musicians want to know the area they will have on stage within the optimal pickup range of the microphone. Answer their question.



45–48 Find the exact length of the polar curve.

45. $r = 3\sin\theta$, $0 \leqslant \theta \leqslant \pi/3$

46. $r = e^{2\theta}$, $0 \leqslant \theta \leqslant 2\pi$

47. $r = \theta^2$, $0 \leqslant \theta \leqslant 2\pi$

48. $r = \theta$, $0 \leqslant \theta \leqslant 2\pi$

49–52 Use a calculator to find the length of the curve correct to four decimal places.

49. $r = 3\sin 2\theta$

50. $r = 4\sin 3\theta$

51. $r = \sin(\theta/2)$

52. $r = 1 + \cos(\theta/3)$

53–54 Graph the curve and find its length.

53. $r = \cos^4(\theta/4)$

54. $r = \cos^2(\theta/2)$

55. (a) Use Formula 11.2.7 to show that the area of the surface generated by rotating the polar curve

$$r = f(\theta) \qquad a \leqslant \theta \leqslant b$$

(where f' is continuous and $0 \leqslant a < b \leqslant \pi$) about the polar axis is

$$S = \int_a^b 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

(b) Use the formula in part (a) to find the surface area generated by rotating the lemniscate $r^2 = \cos 2\theta$ about the polar axis.

56. (a) Find a formula for the area of the surface generated by rotating the polar curve $r = f(\theta)$, $a \leqslant \theta \leqslant b$ (where f' is continuous and $0 \leqslant a < b \leqslant \pi$), about the line $\theta = \pi/2$.
(b) Find the surface area generated by rotating the lemniscate $r^2 = \cos 2\theta$ about the line $\theta = \pi/2$.

11.5 CONIC SECTIONS

In this section we give geometric definitions of parabolas, ellipses, and hyperbolas and derive their standard equations. They are called **conic sections**, or **conics**, because they result from intersecting a cone with a plane as shown in Figure 1.

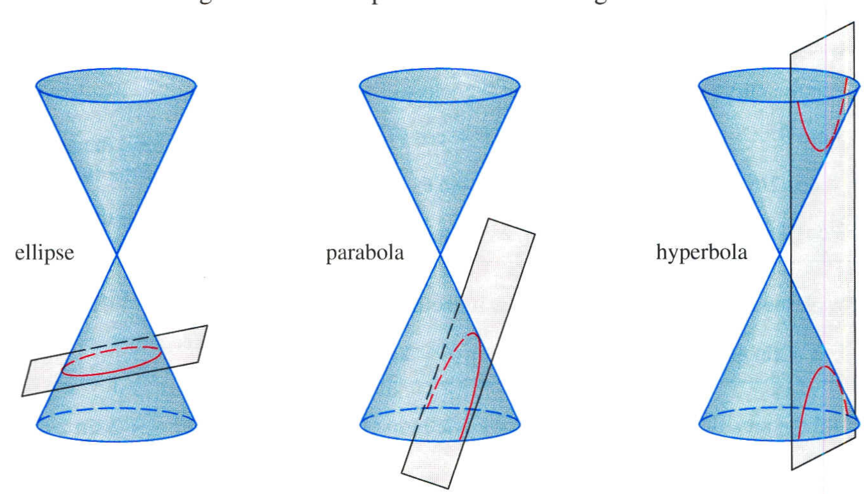


FIGURE 1
Conics

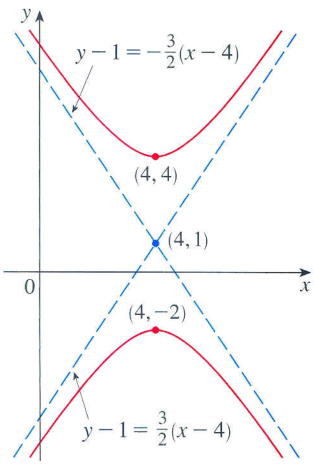


FIGURE 15
$9x^2-4y^2-72x+8y+176=0$

EXAMPLE 7 Sketch the conic

$$9x^2 - 4y^2 - 72x + 8y + 176 = 0$$

and find its foci.

SOLUTION We complete the squares as follows:

$$4(y^2 - 2y) - 9(x^2 - 8x) = 176$$
$$4(y^2 - 2y + 1) - 9(x^2 - 8x + 16) = 176 + 4 - 144$$
$$4(y - 1)^2 - 9(x - 4)^2 = 36$$
$$\frac{(y-1)^2}{9} - \frac{(x-4)^2}{4} = 1$$

This is in the form (8) except that x and y are replaced by $x - 4$ and $y - 1$. Thus $a^2 = 9$, $b^2 = 4$, and $c^2 = 13$. The hyperbola is shifted four units to the right and one unit upward. The foci are $(4, 1 + \sqrt{13})$ and $(4, 1 - \sqrt{13})$ and the vertices are $(4, 4)$ and $(4, -2)$. The asymptotes are $y - 1 = \pm\frac{3}{2}(x - 4)$. The hyperbola is sketched in Figure 15. □

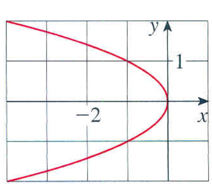
11.5 EXERCISES

1–8 Find the vertex, focus, and directrix of the parabola and sketch its graph.

1. $x = 2y^2$
2. $4y + x^2 = 0$
3. $4x^2 = -y$
4. $y^2 = 12x$
5. $(x + 2)^2 = 8(y - 3)$
6. $x - 1 = (y + 5)^2$
7. $y^2 + 2y + 12x + 25 = 0$
8. $y + 12x - 2x^2 = 16$

9–10 Find an equation of the parabola. Then find the focus and directrix.

9.



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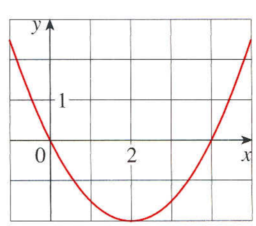
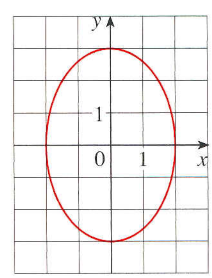


11–16 Find the vertices and foci of the ellipse and sketch its graph.

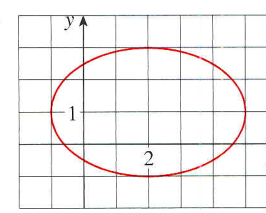
11. $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$
12. $\dfrac{x^2}{64} + \dfrac{y^2}{100} = 1$
13. $25x^2 + 9y^2 = 225$
14. $4x^2 + 25y^2 = 25$
15. $9x^2 - 18x + 4y^2 = 27$
16. $x^2 + 3y^2 + 2x - 12y + 10 = 0$

17–18 Find an equation of the ellipse. Then find its foci.

17.



18.



19–24 Find the vertices, foci, and asymptotes of the hyperbola and sketch its graph.

19. $\dfrac{x^2}{144} - \dfrac{y^2}{25} = 1$
20. $\dfrac{y^2}{16} - \dfrac{x^2}{36} = 1$
21. $y^2 - x^2 = 4$
22. $9x^2 - 4y^2 = 36$
23. $4x^2 - y^2 - 24x - 4y + 28 = 0$
24. $y^2 - 4x^2 - 2y + 16x = 31$

25–30 Identify the type of conic section whose equation is given and find the vertices and foci.

25. $x^2 = y + 1$
26. $x^2 = y^2 + 1$
27. $x^2 = 4y - 2y^2$
28. $y^2 - 8y = 6x - 16$
29. $y^2 + 2y = 4x^2 + 3$
30. $4x^2 + 4x + y^2 = 0$

31–48 Find an equation for the conic that satisfies the given conditions.

31. Parabola, vertex $(0, 0)$, focus $(0, -2)$

32. Parabola, vertex $(1, 0)$, directrix $x = -5$

33. Parabola, focus $(-4, 0)$, directrix $x = 2$

34. Parabola, focus $(3, 6)$, vertex $(3, 2)$

35. Parabola, vertex $(0, 0)$, axis the x-axis, passing through $(1, -4)$

36. Parabola, vertical axis, passing through $(-2, 3)$, $(0, 3)$, and $(1, 9)$

37. Ellipse, foci $(\pm 2, 0)$, vertices $(\pm 5, 0)$

38. Ellipse, foci $(0, \pm 5)$, vertices $(0, \pm 13)$

39. Ellipse, foci $(0, 2)$, $(0, 6)$, vertices $(0, 0)$, $(0, 8)$

40. Ellipse, foci $(0, -1)$, $(8, -1)$, vertex $(9, -1)$

41. Ellipse, center $(-1, 4)$, vertex $(-1, 0)$, focus $(-1, 6)$

42. Ellipse, foci $(\pm 4, 0)$, passing through $(-4, 1.8)$

43. Hyperbola, vertices $(\pm 3, 0)$, foci $(\pm 5, 0)$

44. Hyperbola, vertices $(0, \pm 2)$, foci $(0, \pm 5)$

45. Hyperbola, vertices $(-3, -4)$, $(-3, 6)$, foci $(-3, -7)$, $(-3, 9)$

46. Hyperbola, vertices $(-1, 2)$, $(7, 2)$, foci $(-2, 2)$, $(8, 2)$

47. Hyperbola, vertices $(\pm 3, 0)$, asymptotes $y = \pm 2x$

48. Hyperbola, foci $(2, 0)$, $(2, 8)$, asymptotes $y = 3 + \frac{1}{2}x$ and $y = 5 - \frac{1}{2}x$

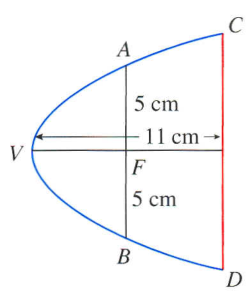
---

49. The point in a lunar orbit nearest the surface of the moon is called *perilune* and the point farthest from the surface is called *apolune*. The *Apollo 11* spacecraft was placed in an elliptical lunar orbit with perilune altitude 110 km and apolune altitude 314 km (above the moon). Find an equation of this ellipse if the radius of the moon is 1728 km and the center of the moon is at one focus.

50. A cross-section of a parabolic reflector is shown in the figure. The bulb is located at the focus and the opening at the focus is 10 cm.

(a) Find an equation of the parabola.

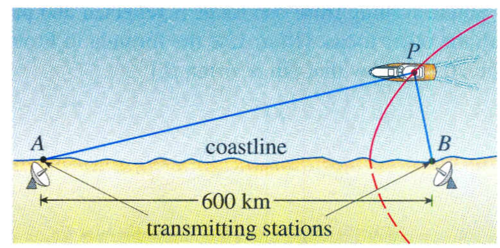
(b) Find the diameter of the opening $|CD|$, 11 cm from the vertex.



51. In the LORAN (LOng RAnge Navigation) radio navigation system, two radio stations located at A and B transmit simultaneous signals to a ship or an aircraft located at P. The onboard computer converts the time difference in receiving these signals into a distance difference $|PA| - |PB|$, and this, according to the definition of a hyperbola, locates the ship or aircraft on one branch of a hyperbola (see the figure). Suppose that station B is located 600 km due east of station A on a coastline. A ship received the signal from B 1200 microseconds (μs) before it received the signal from A.

(a) Assuming that radio signals travel at a speed of 300 m/μs, find an equation of the hyperbola on which the ship lies.

(b) If the ship is due north of B, how far off the coastline is the ship?



52. Use the definition of a hyperbola to derive Equation 6 for a hyperbola with foci $(\pm c, 0)$ and vertices $(\pm a, 0)$.

53. Show that the function defined by the upper branch of the hyperbola $y^2/a^2 - x^2/b^2 = 1$ is concave upward.

54. Find an equation for the ellipse with foci $(1, 1)$ and $(-1, -1)$ and major axis of length 4.

55. Determine the type of curve represented by the equation

$$\frac{x^2}{k} + \frac{y^2}{k - 16} = 1$$

in each of the following cases: (a) $k > 16$, (b) $0 < k < 16$, and (c) $k < 0$.

(d) Show that all the curves in parts (a) and (b) have the same foci, no matter what the value of k is.

56. (a) Show that the equation of the tangent line to the parabola $y^2 = 4px$ at the point (x_0, y_0) can be written as

$$y_0 y = 2p(x + x_0)$$

(b) What is the x-intercept of this tangent line? Use this fact to draw the tangent line.

57. Show that the tangent lines to the parabola $x^2 = 4py$ drawn from any point on the directrix are perpendicular.

58. Show that if an ellipse and a hyperbola have the same foci, then their tangent lines at each point of intersection are perpendicular.

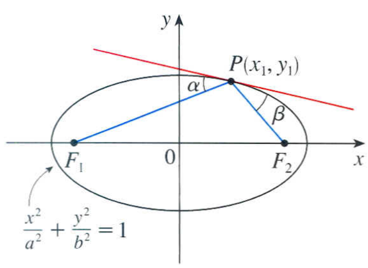
59. Use Simpson's Rule with $n = 10$ to estimate the length of the ellipse $x^2 + 4y^2 = 4$.

60. The planet Pluto travels in an elliptical orbit around the sun (at one focus). The length of the major axis is 1.18×10^{10} km and the length of the minor axis is 1.14×10^{10} km. Use Simpson's Rule with $n = 10$ to estimate the distance traveled by the planet during one complete orbit around the sun.

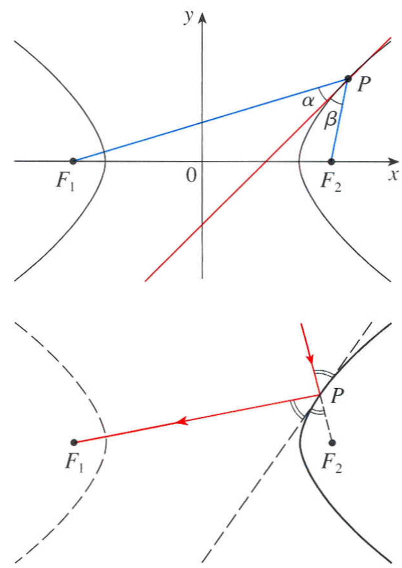
61. Find the area of the region enclosed by the hyperbola $x^2/a^2 - y^2/b^2 = 1$ and the vertical line through a focus.

62. (a) If an ellipse is rotated about its major axis, find the volume of the resulting solid.
(b) If it is rotated about its minor axis, find the resulting volume.

63. Let $P_1(x_1, y_1)$ be a point on the ellipse $x^2/a^2 + y^2/b^2 = 1$ with foci F_1 and F_2 and let α and β be the angles between the lines PF_1, PF_2 and the ellipse as shown in the figure. Prove that $\alpha = \beta$. This explains how whispering galleries and lithotripsy work. Sound coming from one focus is reflected and passes through the other focus. [*Hint:* Use the formula in Problem 15 on page 202 to show that $\tan \alpha = \tan \beta$.]



64. Let $P(x_1, y_1)$ be a point on the hyperbola $x^2/a^2 - y^2/b^2 = 1$ with foci F_1 and F_2 and let α and β be the angles between the lines PF_1, PF_2 and the hyperbola as shown in the figure. Prove that $\alpha = \beta$. (This is the reflection property of the hyperbola. It shows that light aimed at a focus F_2 of a hyperbolic mirror is reflected toward the other focus F_1.)



11.6 CONIC SECTIONS IN POLAR COORDINATES

In the preceding section we defined the parabola in terms of a focus and directrix, but we defined the ellipse and hyperbola in terms of two foci. In this section we give a more unified treatment of all three types of conic sections in terms of a focus and directrix. Furthermore, if we place the focus at the origin, then a conic section has a simple polar equation, which provides a convenient description of the motion of planets, satellites, and comets.

1 THEOREM Let F be a fixed point (called the **focus**) and l be a fixed line (called the **directrix**) in a plane. Let e be a fixed positive number (called the **eccentricity**). The set of all points P in the plane such that

$$\frac{|PF|}{|Pl|} = e$$

(that is, the ratio of the distance from F to the distance from l is the constant e) is a conic section. The conic is

(a) an ellipse if $e < 1$

(b) a parabola if $e = 1$

(c) a hyperbola if $e > 1$

11.6 EXERCISES

1–8 Write a polar equation of a conic with the focus at the origin and the given data.

1. Hyperbola, eccentricity $\frac{7}{4}$, directrix $y = 6$

2. Parabola, directrix $x = 4$

3. Ellipse, eccentricity $\frac{3}{4}$, directrix $x = -5$

4. Hyperbola, eccentricity 2, directrix $y = -2$

5. Parabola, vertex $(4, 3\pi/2)$

6. Ellipse, eccentricity 0.8, vertex $(1, \pi/2)$

7. Ellipse, eccentricity $\frac{1}{2}$, directrix $r = 4 \sec\theta$

8. Hyperbola, eccentricity 3, directrix $r = -6 \csc\theta$

9–16 (a) Find the eccentricity, (b) identify the conic, (c) give an equation of the directrix, and (d) sketch the conic.

9. $r = \dfrac{1}{1 + \sin\theta}$

10. $r = \dfrac{12}{3 - 10\cos\theta}$

11. $r = \dfrac{12}{4 - \sin\theta}$

12. $r = \dfrac{3}{2 + 2\cos\theta}$

13. $r = \dfrac{9}{6 + 2\cos\theta}$

14. $r = \dfrac{8}{4 + 5\sin\theta}$

15. $r = \dfrac{3}{4 - 8\cos\theta}$

16. $r = \dfrac{10}{5 - 6\sin\theta}$

17. (a) Find the eccentricity and directrix of the conic $r = 1/(1 - 2\sin\theta)$ and graph the conic and its directrix.
(b) If this conic is rotated counterclockwise about the origin through an angle $3\pi/4$, write the resulting equation and graph its curve.

18. Graph the conic $r = 4/(5 + 6\cos\theta)$ and its directrix. Also graph the conic obtained by rotating this curve about the origin through an angle $\pi/3$.

19. Graph the conics $r = e/(1 - e\cos\theta)$ with $e = 0.4, 0.6, 0.8$, and 1.0 on a common screen. How does the value of e affect the shape of the curve?

20. (a) Graph the conics $r = ed/(1 + e\sin\theta)$ for $e = 1$ and various values of d. How does the value of d affect the shape of the conic?
(b) Graph these conics for $d = 1$ and various values of e. How does the value of e affect the shape of the conic?

21. Show that a conic with focus at the origin, eccentricity e, and directrix $x = -d$ has polar equation

$$r = \frac{ed}{1 - e\cos\theta}$$

22. Show that a conic with focus at the origin, eccentricity e, and directrix $y = d$ has polar equation

$$r = \frac{ed}{1 + e\sin\theta}$$

23. Show that a conic with focus at the origin, eccentricity e, and directrix $y = -d$ has polar equation

$$r = \frac{ed}{1 - e\sin\theta}$$

24. Show that the parabolas $r = c/(1 + \cos\theta)$ and $r = d/(1 - \cos\theta)$ intersect at right angles.

25. The orbit of Mars around the sun is an ellipse with eccentricity 0.093 and semimajor axis 2.28×10^8 km. Find a polar equation for the orbit.

26. Jupiter's orbit has eccentricity 0.048 and the length of the major axis is 1.56×10^9 km. Find a polar equation for the orbit.

27. The orbit of Halley's comet, last seen in 1986 and due to return in 2062, is an ellipse with eccentricity 0.97 and one focus at the sun. The length of its major axis is 36.18 AU. [An astronomical unit (AU) is the mean distance between the earth and the sun, about 1.5×10^8 km.] Find a polar equation for the orbit of Halley's comet. What is the maximum distance from the comet to the sun?

28. The Hale-Bopp comet, discovered in 1995, has an elliptical orbit with eccentricity 0.9951 and the length of the major axis is 356.5 AU. Find a polar equation for the orbit of this comet. How close to the sun does it come?

29. The planet Mercury travels in an elliptical orbit with eccentricity 0.206. Its minimum distance from the sun is 4.6×10^7 km. Find its maximum distance from the sun.

30. The distance from the planet Pluto to the sun is 4.43×10^9 km at perihelion and 7.37×10^9 km at aphelion. Find the eccentricity of Pluto's orbit.

31. Using the data from Exercise 29, find the distance traveled by the planet Mercury during one complete orbit around the sun. (If your calculator or computer algebra system evaluates definite integrals, use it. Otherwise, use Simpson's Rule.)

11 REVIEW

CONCEPT CHECK

1. (a) What is a parametric curve?
   (b) How do you sketch a parametric curve?
2. (a) How do you find the slope of a tangent to a parametric curve?
   (b) How do you find the area under a parametric curve?
3. Write an expression for each of the following:
   (a) The length of a parametric curve
   (b) The area of the surface obtained by rotating a parametric curve about the x-axis
4. (a) Use a diagram to explain the meaning of the polar coordinates (r, θ) of a point.
   (b) Write equations that express the Cartesian coordinates (x, y) of a point in terms of the polar coordinates.
   (c) What equations would you use to find the polar coordinates of a point if you knew the Cartesian coordinates?
5. (a) How do you find the slope of a tangent line to a polar curve?
   (b) How do you find the area of a region bounded by a polar curve?
   (c) How do you find the length of a polar curve?
6. (a) Give a geometric definition of a parabola.
   (b) Write an equation of a parabola with focus $(0, p)$ and directrix $y = -p$. What if the focus is $(p, 0)$ and the directrix is $x = -p$?
7. (a) Give a definition of an ellipse in terms of foci.
   (b) Write an equation for the ellipse with foci $(\pm c, 0)$ and vertices $(\pm a, 0)$.
8. (a) Give a definition of a hyperbola in terms of foci.
   (b) Write an equation for the hyperbola with foci $(\pm c, 0)$ and vertices $(\pm a, 0)$.
   (c) Write equations for the asymptotes of the hyperbola in part (b).
9. (a) What is the eccentricity of a conic section?
   (b) What can you say about the eccentricity if the conic section is an ellipse? A hyperbola? A parabola?
   (c) Write a polar equation for a conic section with eccentricity e and directrix $x = d$. What if the directrix is $x = -d$? $y = d$? $y = -d$?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If the parametric curve $x = f(t)$, $y = g(t)$ satisfies $g'(1) = 0$, then it has a horizontal tangent when $t = 1$.
2. If $x = f(t)$ and $y = g(t)$ are twice differentiable, then
$$\frac{d^2y}{dx^2} = \frac{d^2y/dt^2}{d^2x/dt^2}$$
3. The length of the curve $x = f(t)$, $y = g(t)$, $a \leqslant t \leqslant b$, is $\int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$.
4. If a point is represented by (x, y) in Cartesian coordinates (where $x \neq 0$) and (r, θ) in polar coordinates, then $\theta = \tan^{-1}(y/x)$.
5. The polar curves $r = 1 - \sin 2\theta$ and $r = \sin 2\theta - 1$ have the same graph.
6. The equations $r = 2$, $x^2 + y^2 = 4$, and $x = 2 \sin 3t$, $y = 2 \cos 3t$ $(0 \leqslant t \leqslant 2\pi)$ all have the same graph.
7. The parametric equations $x = t^2$, $y = t^4$ have the same graph as $x = t^3$, $y = t^6$.
8. The graph of $y^2 = 2y + 3x$ is a parabola.
9. A tangent line to a parabola intersects the parabola only once.
10. A hyperbola never intersects its directrix.

EXERCISES

1–4 Sketch the parametric curve and eliminate the parameter to find the Cartesian equation of the curve.

1. $x = t^2 + 4t, \quad y = 2 - t, \quad -4 \leq t \leq 1$

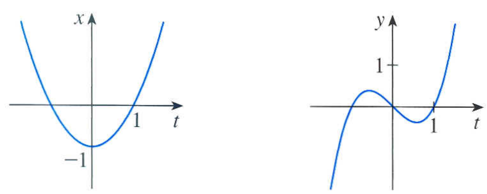
2. $x = 1 + e^{2t}, \quad y = e^t$

3. $x = \cos\theta, \quad y = \sec\theta, \quad 0 \leq \theta < \pi/2$

4. $x = 2\cos\theta, \quad y = 1 + \sin\theta$

5. Write three different sets of parametric equations for the curve $y = \sqrt{x}$.

6. Use the graphs of $x = f(t)$ and $y = g(t)$ to sketch the parametric curve $x = f(t)$, $y = g(t)$. Indicate with arrows the direction in which the curve is traced as t increases.



7. (a) Plot the point with polar coordinates $(4, 2\pi/3)$. Then find its Cartesian coordinates.

(b) The Cartesian coordinates of a point are $(-3, 3)$. Find two sets of polar coordinates for the point.

8. Sketch the region consisting of points whose polar coordinates satisfy $1 \leq r < 2$ and $\pi/6 \leq \theta \leq 5\pi/6$.

9–16 Sketch the polar curve.

9. $r = 1 - \cos\theta$

10. $r = \sin 4\theta$

11. $r = \cos 3\theta$

12. $r = 3 + \cos 3\theta$

13. $r = 1 + \cos 2\theta$

14. $r = 2\cos(\theta/2)$

15. $r = \dfrac{3}{1 + 2\sin\theta}$

16. $r = \dfrac{3}{2 - 2\cos\theta}$

17–18 Find a polar equation for the curve represented by the given Cartesian equation.

17. $x + y = 2$

18. $x^2 + y^2 = 2$

19. The curve with polar equation $r = (\sin\theta)/\theta$ is called a **cochleoid**. Use a graph of r as a function of θ in Cartesian coordinates to sketch the cochleoid by hand. Then graph it with a machine to check your sketch.

20. Graph the ellipse $r = 2/(4 - 3\cos\theta)$ and its directrix. Also graph the ellipse obtained by rotation about the origin through an angle $2\pi/3$.

21–24 Find the slope of the tangent line to the given curve at the point corresponding to the specified value of the parameter.

21. $x = \ln t, \quad y = 1 + t^2; \quad t = 1$

22. $x = t^3 + 6t + 1, \quad y = 2t - t^2; \quad t = -1$

23. $r = e^{-\theta}; \quad \theta = \pi$

24. $r = 3 + \cos 3\theta; \quad \theta = \pi/2$

25–26 Find dy/dx and d^2y/dx^2.

25. $x = t + \sin t, \quad y = t - \cos t$

26. $x = 1 + t^2, \quad y = t - t^3$

27. Use a graph to estimate the coordinates of the lowest point on the curve $x = t^3 - 3t$, $y = t^2 + t + 1$. Then use calculus to find the exact coordinates.

28. Find the area enclosed by the loop of the curve in Exercise 27.

29. At what points does the curve

$$x = 2a\cos t - a\cos 2t \qquad y = 2a\sin t - a\sin 2t$$

have vertical or horizontal tangents? Use this information to help sketch the curve.

30. Find the area enclosed by the curve in Exercise 29.

31. Find the area enclosed by the curve $r^2 = 9\cos 5\theta$.

32. Find the area enclosed by the inner loop of the curve $r = 1 - 3\sin\theta$.

33. Find the points of intersection of the curves $r = 2$ and $r = 4\cos\theta$.

34. Find the points of intersection of the curves $r = \cot\theta$ and $r = 2\cos\theta$.

35. Find the area of the region that lies inside both of the circles $r = 2\sin\theta$ and $r = \sin\theta + \cos\theta$.

36. Find the area of the region that lies inside the curve $r = 2 + \cos 2\theta$ but outside the curve $r = 2 + \sin\theta$.

37–40 Find the length of the curve.

37. $x = 3t^2, \quad y = 2t^3, \quad 0 \leq t \leq 2$

38. $x = 2 + 3t, \quad y = \cosh 3t, \quad 0 \leq t \leq 1$

39. $r = 1/\theta, \quad \pi \leq \theta \leq 2\pi$

40. $r = \sin^3(\theta/3), \quad 0 \leq \theta \leq \pi$

41–42 Find the area of the surface obtained by rotating the given curve about the x-axis.

41. $x = 4\sqrt{t}, \quad y = \dfrac{t^3}{3} + \dfrac{1}{2t^2}, \quad 1 \leqslant t \leqslant 4$

42. $x = 2 + 3t, \quad y = \cosh 3t, \quad 0 \leqslant t \leqslant 1$

43. The curves defined by the parametric equations

$$x = \frac{t^2 - c}{t^2 + 1} \qquad y = \frac{t(t^2 - c)}{t^2 + 1}$$

are called **strophoids** (from a Greek word meaning "to turn or twist"). Investigate how these curves vary as c varies.

44. A family of curves has polar equations $r^a = |\sin 2\theta|$ where a is a positive number. Investigate how the curves change as a changes.

45–48 Find the foci and vertices and sketch the graph.

45. $\dfrac{x^2}{9} + \dfrac{y^2}{8} = 1$

46. $4x^2 - y^2 = 16$

47. $6y^2 + x - 36y + 55 = 0$

48. $25x^2 + 4y^2 + 50x - 16y = 59$

49. Find an equation of the ellipse with foci $(\pm 4, 0)$ and vertices $(\pm 5, 0)$.

50. Find an equation of the parabola with focus $(2, 1)$ and directrix $x = -4$.

51. Find an equation of the hyperbola with foci $(0, \pm 4)$ and asymptotes $y = \pm 3x$.

52. Find an equation of the ellipse with foci $(3, \pm 2)$ and major axis with length 8.

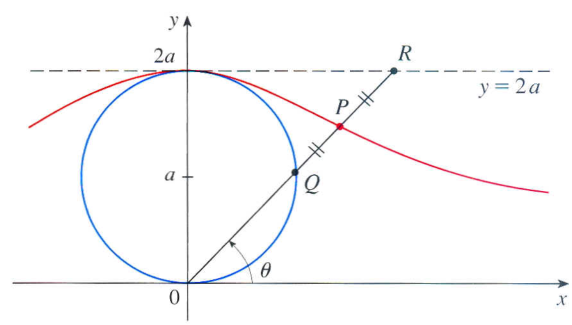
53. Find an equation for the ellipse that shares a vertex and a focus with the parabola $x^2 + y = 100$ and that has its other focus at the origin.

54. Show that if m is any real number, then there are exactly two lines of slope m that are tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ and their equations are $y = mx \pm \sqrt{a^2m^2 + b^2}$.

55. Find a polar equation for the ellipse with focus at the origin, eccentricity $\frac{1}{3}$, and directrix with equation $r = 4 \sec \theta$.

56. Show that the angles between the polar axis and the asymptotes of the hyperbola $r = ed/(1 - e \cos \theta)$, $e > 1$, are given by $\cos^{-1}(\pm 1/e)$.

57. In the figure the circle of radius a is stationary, and for every θ, the point P is the midpoint of the segment QR. The curve traced out by P for $0 < \theta < \pi$ is called the **longbow curve**. Find parametric equations for this curve.



1. A curve is defined by the parametric equations

$$x = \int_1^t \frac{\cos u}{u}\,du \qquad y = \int_1^t \frac{\sin u}{u}\,du$$

Find the length of the arc of the curve from the origin to the nearest point where there is a vertical tangent line.

2. (a) Find the highest and lowest points on the curve $x^4 + y^4 = x^2 + y^2$.

(b) Sketch the curve. (Notice that it is symmetric with respect to both axes and both of the lines $y = \pm x$, so it suffices to consider $y \geqslant x \geqslant 0$ initially.)

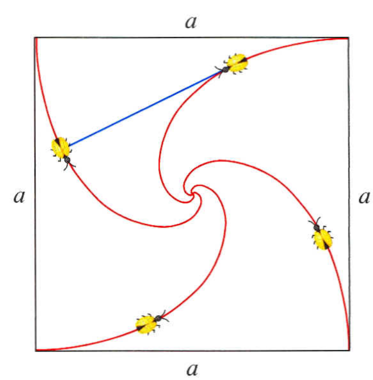
CAS (c) Use polar coordinates and a computer algebra system to find the area enclosed by the curve.

3. What is the smallest viewing rectangle that contains every member of the family of polar curves $r = 1 + c \sin \theta$, where $0 \leqslant c \leqslant 1$? Illustrate your answer by graphing several members of the family in this viewing rectangle.

4. Four bugs are placed at the four corners of a square with side length a. The bugs crawl counterclockwise at the same speed and each bug crawls directly toward the next bug at all times. They approach the center of the square along spiral paths.

(a) Find the polar equation of a bug's path assuming the pole is at the center of the square. (Use the fact that the line joining one bug to the next is tangent to the bug's path.)

(b) Find the distance traveled by a bug by the time it meets the other bugs at the center.



5. A curve called the **folium of Descartes** is defined by the parametric equations

$$x = \frac{3t}{1 + t^3} \qquad y = \frac{3t^2}{1 + t^3}$$

(a) Show that if (a, b) lies on the curve, then so does (b, a); that is, the curve is symmetric with respect to the line $y = x$. Where does the curve intersect this line?

(b) Find the points on the curve where the tangent lines are horizontal or vertical.

(c) Show that the line $y = -x - 1$ is a slant asymptote.

(d) Sketch the curve.

(e) Show that a Cartesian equation of this curve is $x^3 + y^3 = 3xy$.

(f) Show that the polar equation can be written in the form

$$r = \frac{3 \sec \theta \tan \theta}{1 + \tan^3 \theta}$$

(g) Find the area enclosed by the loop of this curve.

CAS (h) Show that the area of the loop is the same as the area that lies between the asymptote and the infinite branches of the curve. (Use a computer algebra system to evaluate the integral.)

6. A circle C of radius $2r$ has its center at the origin. A circle of radius r rolls without slipping in the counterclockwise direction around C. A point P is located on a fixed radius of the rolling circle at a distance b from its center, $0 < b < r$. [See parts (i) and (ii) of the figure.] Let L be the line from the center of C to the center of the rolling circle and let θ be the angle that L makes with the positive x-axis.

(a) Using θ as a parameter, show that parametric equations of the path traced out by P are

$$x = b\cos 3\theta + 3r\cos\theta \qquad y = b\sin 3\theta + 3r\sin\theta$$

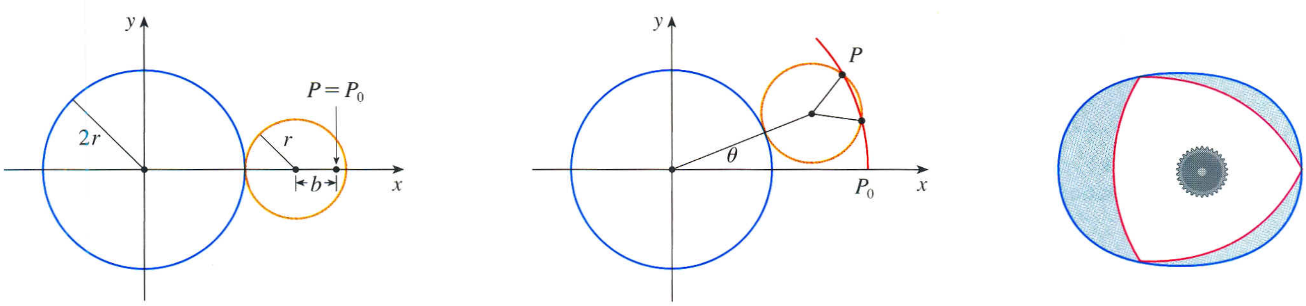
Note: If $b = 0$, the path is a circle of radius $3r$; if $b = r$, the path is an *epicycloid.* The path traced out by P for $0 < b < r$ is called an *epitrochoid.*

(b) Graph the curve for various values of b between 0 and r.

(c) Show that an equilateral triangle can be inscribed in the epitrochoid and that its centroid is on the circle of radius b centered at the origin.

Note: This is the principle of the Wankel rotary engine. When the equilateral triangle rotates with its vertices on the epitrochoid, its centroid sweeps out a circle whose center is at the center of the curve.

(d) In most rotary engines the sides of the equilateral triangles are replaced by arcs of circles centered at the opposite vertices as in part (iii) of the figure. (Then the diameter of the rotor is constant.) Show that the rotor will fit in the epitrochoid if $b \leqslant \frac{3}{2}(2 - \sqrt{3})r$.



(i) (ii) (iii)