■ Figure 8 shows the interpretation of the arc length function in Example 4. Figure 9 shows the graph of this arc length function. Why is $s(x)$ negative when x is less than 1?

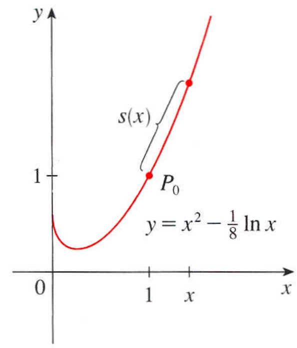


FIGURE 8

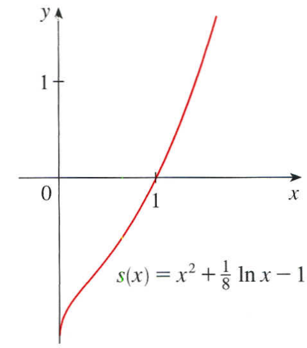


FIGURE 9

9.1 EXERCISES

1. Use the arc length formula (3) to find the length of the curve $y = 2x - 5$, $-1 \leqslant x \leqslant 3$. Check your answer by noting that the curve is a line segment and calculating its length by the distance formula.

2. Use the arc length formula to find the length of the curve $y = \sqrt{2 - x^2}$, $0 \leqslant x \leqslant 1$. Check your answer by noting that the curve is part of a circle.

3–6 Set up, but do not evaluate, an integral for the length of the curve.

3. $y = x^3, \quad 0 \leqslant x \leqslant 1$

4. $y = xe^{-x^2}, \quad 0 \leqslant x \leqslant 1$

5. $x = y + y^3, \quad 1 \leqslant y \leqslant 4$

6. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

7–18 Find the length of the curve.

7. $y = 1 + 6x^{3/2}, \quad 0 \leqslant x \leqslant 1$

8. $y^2 = 4(x + 4)^3, \quad 0 \leqslant x \leqslant 2, \quad y > 0$

9. $y = \dfrac{x^5}{6} + \dfrac{1}{10x^3}, \quad 1 \leqslant x \leqslant 2$

10. $x = \dfrac{y^4}{8} + \dfrac{1}{4y^2}, \quad 1 \leqslant y \leqslant 2$

11. $x = \frac{1}{3}\sqrt{y}\,(y - 3), \quad 1 \leqslant y \leqslant 9$

12. $y = \ln(\cos x), \quad 0 \leqslant x \leqslant \pi/3$

13. $y = \ln(\sec x), \quad 0 \leqslant x \leqslant \pi/4$

14. $y = 3 + \frac{1}{2}\cosh 2x, \quad 0 \leqslant x \leqslant 1$

15. $y = \ln(1 - x^2), \quad 0 \leqslant x \leqslant \frac{1}{2}$

16. $y^2 = 4x, \quad 0 \leqslant y \leqslant 2$

17. $y = e^x, \quad 0 \leqslant x \leqslant 1$

18. $y = \ln\left(\dfrac{e^x + 1}{e^x - 1}\right), \quad a \leqslant x \leqslant b, \quad a > 0$

19–20 Find the length of the arc of the curve from point P to point Q.

19. $y = \frac{1}{2}x^2, \quad P\left(-1, \frac{1}{2}\right), \quad Q\left(1, \frac{1}{2}\right)$

20. $x^2 = (y - 4)^3, \quad P(1, 5), \quad Q(8, 8)$

21–22 Graph the curve and visually estimate its length. Then find its exact length.

21. $y = \frac{2}{3}(x^2 - 1)^{3/2}, \quad 1 \leqslant x \leqslant 3$

22. $y = \dfrac{x^3}{6} + \dfrac{1}{2x}, \quad \frac{1}{2} \leqslant x \leqslant 1$

23–26 Use Simpson's Rule with $n = 10$ to estimate the arc length of the curve. Compare your answer with the value of the integral produced by your calculator.

23. $y = xe^{-x}, \quad 0 \leqslant x \leqslant 5$

24. $x = y + \sqrt{y}, \quad 1 \leqslant y \leqslant 2$

25. $y = \sec x, \quad 0 \leqslant x \leqslant \pi/3$

26. $y = x \ln x, \quad 1 \leqslant x \leqslant 3$

27. (a) Graph the curve $y = x\sqrt[3]{4 - x}$, $0 \leqslant x \leqslant 4$.
(b) Compute the lengths of inscribed polygons with $n = 1$, 2, and 4 sides. (Divide the interval into equal subintervals.) Illustrate by sketching these polygons (as in Figure 6).
(c) Set up an integral for the length of the curve.
(d) Use your calculator to find the length of the curve to four decimal places. Compare with the approximations in part (b).

28. Repeat Exercise 27 for the curve

$$y = x + \sin x \qquad 0 \leqslant x \leqslant 2\pi$$

29. Use either a computer algebra system or a table of integrals to find the *exact* length of the arc of the curve $y = \ln x$ that lies between the points $(1, 0)$ and $(2, \ln 2)$.

30. Use either a computer algebra system or a table of integrals to find the *exact* length of the arc of the curve $y = x^{4/3}$ that lies between the points $(0, 0)$ and $(1, 1)$. If your CAS has trouble evaluating the integral, make a substitution that changes the integral into one that the CAS can evaluate.

31. Sketch the curve with equation $x^{2/3} + y^{2/3} = 1$ and use symmetry to find its length.

32. (a) Sketch the curve $y^3 = x^2$.
(b) Use Formulas 3 and 4 to set up two integrals for the arc length from $(0, 0)$ to $(1, 1)$. Observe that one of these is an improper integral and evaluate both of them.
(c) Find the length of the arc of this curve from $(-1, 1)$ to $(8, 4)$.

33. Find the arc length function for the curve $y = 2x^{3/2}$ with starting point $P_0(1, 2)$.

34. (a) Graph the curve $y = \frac{1}{3}x^3 + 1/(4x)$, $x > 0$.
(b) Find the arc length function for this curve with starting point $P_0(1, \frac{7}{12})$.
(c) Graph the arc length function.

35. Find the arc length function for the curve $y = \sin^{-1}x + \sqrt{1 - x^2}$ with starting point $(0, 1)$.

36. A steady wind blows a kite due west. The kite's height above ground from horizontal position $x = 0$ to $x = 25$ m is given by $y = 50 - 0.1(x - 15)^2$. Find the distance traveled by the kite.

37. A hawk flying at 15 m/s at an altitude of 180 m accidentally drops its prey. The parabolic trajectory of the falling prey is described by the equation

$$y = 180 - \frac{x^2}{45}$$

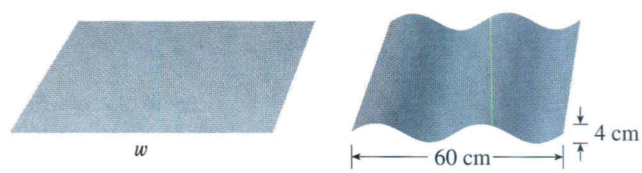
until it hits the ground, where y is its height above the ground and x is the horizontal distance traveled in meters. Calculate the distance traveled by the prey from the time it is dropped until the time it hits the ground. Express your answer correct to the nearest tenth of a meter.

38. The Gateway Arch in St. Louis (see the photo on page 465) was constructed using the equation

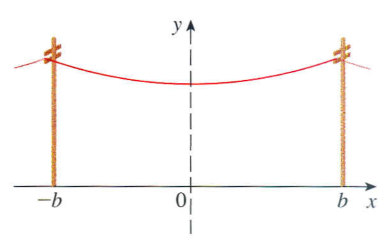
$$y = 211.49 - 20.96 \cosh 0.03291765x$$

for the central curve of the arch, where x and y are measured in meters and $|x| \leqslant 91.20$. Set up an integral for the length of the arch and use your calculator to estimate the length correct to the nearest meter.

39. A manufacturer of corrugated metal roofing wants to produce panels that are 60 cm wide and 4 cm thick by processing flat sheets of metal as shown in the figure. The profile of the roofing takes the shape of a sine wave. Verify that the sine curve has equation $y = 2\sin(\pi x/15)$ and find the width w of a flat metal sheet that is needed to make a 60-cm panel. (Use your calculator to evaluate the integral correct to four significant digits.)



40. (a) The figure shows a telephone wire hanging between two poles at $x = -b$ and $x = b$. It takes the shape of a catenary with equation $y = c + a\cosh(x/a)$. Find the length of the wire.
(b) Suppose two telephone poles are 20 m apart and the length of the wire between the poles is 20.4 m. If the lowest point of the wire must be 9 m above the ground, how high up on each pole should the wire be attached?



41. Find the length of the curve

$$y = \int_1^x \sqrt{t^3 - 1}\,dt \qquad 1 \leqslant x \leqslant 4$$

42. The curves with equations $x^n + y^n = 1$, $n = 4, 6, 8, \ldots$, are called **fat circles**. Graph the curves with $n = 2, 4, 6, 8$, and 10 to see why. Set up an integral for the length L_{2k} of the fat circle with $n = 2k$. Without attempting to evaluate this integral, state the value of $\lim_{k \to \infty} L_{2k}$.

we have

$$S = \int_0^1 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_0^1 e^x \sqrt{1 + e^{2x}}\, dx$$

$$= 2\pi \int_1^e \sqrt{1 + u^2}\, du \qquad \text{(where } u = e^x\text{)}$$

$$= 2\pi \int_{\pi/4}^{\alpha} \sec^3\theta\, d\theta \qquad \text{(where } u = \tan\theta \text{ and } \alpha = \tan^{-1}e\text{)}$$

■ Or use Formula 21 in the Table of Integrals.

$$= 2\pi \cdot \tfrac{1}{2}\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_{\pi/4}^{\alpha} \qquad \text{(by Example 8 in Section 8.2)}$$

$$= \pi\Big[\sec\alpha\tan\alpha + \ln(\sec\alpha + \tan\alpha) - \sqrt{2} - \ln(\sqrt{2} + 1)\Big]$$

Since $\tan\alpha = e$, we have $\sec^2\alpha = 1 + \tan^2\alpha = 1 + e^2$ and

$$S = \pi\Big[e\sqrt{1 + e^2} + \ln\big(e + \sqrt{1 + e^2}\big) - \sqrt{2} - \ln(\sqrt{2} + 1)\Big] \qquad \square$$

9.2 EXERCISES

1–4 Set up, but do not evaluate, an integral for the area of the surface obtained by rotating the curve about (a) the x-axis and (b) the y-axis.

1. $y = x^4, \quad 0 \leqslant x \leqslant 1$

2. $y = xe^{-x}, \quad 1 \leqslant x \leqslant 3$

3. $y = \tan^{-1}x, \quad 0 \leqslant x \leqslant 1$

4. $x = \sqrt{y - y^2}$

5–12 Find the area of the surface obtained by rotating the curve about the x-axis.

5. $y = x^3, \quad 0 \leqslant x \leqslant 2$

6. $9x = y^2 + 18, \quad 2 \leqslant x \leqslant 6$

7. $y = \sqrt{1 + 4x}, \quad 1 \leqslant x \leqslant 5$

8. $y = \cos 2x, \quad 0 \leqslant x \leqslant \pi/6$

9. $y = \cosh x, \quad 0 \leqslant x \leqslant 1$

10. $y = \dfrac{x^3}{6} + \dfrac{1}{2x}, \quad \tfrac{1}{2} \leqslant x \leqslant 1$

11. $x = \tfrac{1}{3}(y^2 + 2)^{3/2}, \quad 1 \leqslant y \leqslant 2$

12. $x = 1 + 2y^2, \quad 1 \leqslant y \leqslant 2$

13–16 The given curve is rotated about the y-axis. Find the area of the resulting surface.

13. $y = \sqrt[3]{x}, \quad 1 \leqslant y \leqslant 2$

14. $y = 1 - x^2, \quad 0 \leqslant x \leqslant 1$

15. $x = \sqrt{a^2 - y^2}, \quad 0 \leqslant y \leqslant a/2$

16. $y = \tfrac{1}{4}x^2 - \tfrac{1}{2}\ln x, \quad 1 \leqslant x \leqslant 2$

17–20 Use Simpson's Rule with $n = 10$ to approximate the area of the surface obtained by rotating the curve about the x-axis. Compare your answer with the value of the integral produced by your calculator.

17. $y = \ln x, \quad 1 \leqslant x \leqslant 3$

18. $y = x + \sqrt{x}, \quad 1 \leqslant x \leqslant 2$

19. $y = \sec x, \quad 0 \leqslant x \leqslant \pi/3$

20. $y = e^{-x^2}, \quad 0 \leqslant x \leqslant 1$

CAS **21–22** Use either a CAS or a table of integrals to find the exact area of the surface obtained by rotating the given curve about the x-axis.

21. $y = 1/x, \quad 1 \leqslant x \leqslant 2$

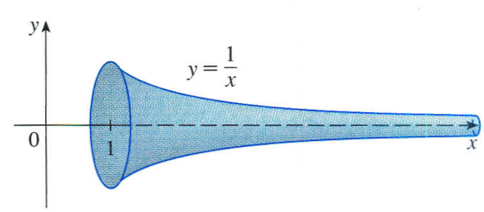
22. $y = \sqrt{x^2 + 1}, \quad 0 \leqslant x \leqslant 3$

CAS **23–24** Use a CAS to find the exact area of the surface obtained by rotating the curve about the y-axis. If your CAS has trouble evaluating the integral, express the surface area as an integral in the other variable.

23. $y = x^3, \quad 0 \leqslant y \leqslant 1$

24. $y = \ln(x + 1), \quad 0 \leqslant x \leqslant 1$

25. If the region $\mathcal{R} = \{(x, y) \mid x \geqslant 1,\ 0 \leqslant y \leqslant 1/x\}$ is rotated about the x-axis, the volume of the resulting solid is finite (see Exercise 63 in Section 8.8). Show that the surface area is infinite. (The surface is shown in the figure and is known as **Gabriel's horn**.)



26. If the infinite curve $y = e^{-x}$, $x \geqslant 0$, is rotated about the x-axis, find the area of the resulting surface.

27. (a) If $a > 0$, find the area of the surface generated by rotating the loop of the curve $3ay^2 = x(a - x)^2$ about the x-axis.
(b) Find the surface area if the loop is rotated about the y-axis.

28. A group of engineers is building a parabolic satellite dish whose shape will be formed by rotating the curve $y = ax^2$ about the y-axis. If the dish is to have a 6-m diameter and a maximum depth of 1 m, find the value of a and the surface area of the dish.

29. (a) The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a > b$$

is rotated about the x-axis to form a surface called an *ellipsoid, or prolate spheroid*. Find the surface area of this ellipsoid.
(b) If the ellipse in part (a) is rotated about its minor axis (the y-axis), the resulting ellipsoid is called an *oblate spheroid*. Find the surface area of this ellipsoid.

30. Find the surface area of the torus in Exercise 63 in Section 6.2.

31. If the curve $y = f(x)$, $a \leqslant x \leqslant b$, is rotated about the horizontal line $y = c$, where $f(x) \leqslant c$, find a formula for the area of the resulting surface.

CAS **32.** Use the result of Exercise 31 to set up an integral to find the area of the surface generated by rotating the curve $y = \sqrt{x}$, $0 \leqslant x \leqslant 4$, about the line $y = 4$. Then use a CAS to evaluate the integral.

33. Find the area of the surface obtained by rotating the circle $x^2 + y^2 = r^2$ about the line $y = r$.

34. Show that the surface area of a zone of a sphere that lies between two parallel planes is $S = \pi dh$, where d is the diameter of the sphere and h is the distance between the planes. (Notice that S depends only on the distance between the planes and not on their location, provided that both planes intersect the sphere.)

35. Formula 4 is valid only when $f(x) \geqslant 0$. Show that when $f(x)$ is not necessarily positive, the formula for surface area becomes

$$S = \int_a^b 2\pi |f(x)| \sqrt{1 + [f'(x)]^2}\, dx$$

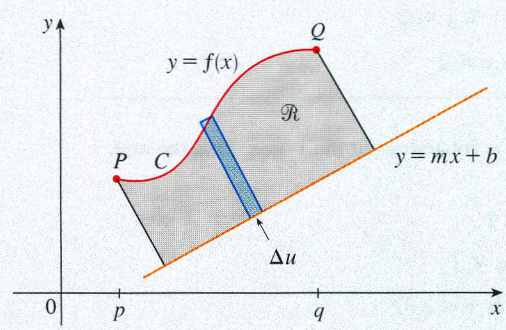
36. Let L be the length of the curve $y = f(x)$, $a \leqslant x \leqslant b$, where f is positive and has a continuous derivative. Let S_f be the surface area generated by rotating the curve about the x-axis. If c is a positive constant, define $g(x) = f(x) + c$ and let S_g be the corresponding surface area generated by the curve $y = g(x)$, $a \leqslant x \leqslant b$. Express S_g in terms of S_f and L.

DISCOVERY PROJECT

ROTATING ON A SLANT

We know how to find the volume of a solid of revolution obtained by rotating a region about a horizontal or vertical line (see Section 6.2). We also know how to find the surface area of a surface of revolution if we rotate a curve about a horizontal or vertical line (see Section 9.2). But what if we rotate about a slanted line, that is, a line that is neither horizontal nor vertical? In this project you are asked to discover formulas for the volume of a solid of revolution and for the area of a surface of revolution when the axis of rotation is a slanted line.

Let C be the arc of the curve $y = f(x)$ between the points $P(p, f(p))$ and $Q(q, f(q))$ and let $\mathcal{R}$ be the region bounded by C, by the line $y = mx + b$ (which lies entirely below C), and by the perpendiculars to the line from P and Q.



(see Section 6.3), we have

$$\begin{aligned} V &= \int_a^b 2\pi x[f(x) - g(x)]\,dx \\ &= 2\pi \int_a^b x[f(x) - g(x)]\,dx \\ &= 2\pi(\bar{x}A) \qquad \text{(by Formulas 9)} \\ &= (2\pi\bar{x})A = Ad \end{aligned}$$

where $d = 2\pi\bar{x}$ is the distance traveled by the centroid during one rotation about the y-axis. □

V EXAMPLE 7 A torus is formed by rotating a circle of radius r about a line in the plane of the circle that is a distance R ($> r$) from the center of the circle. Find the volume of the torus.

SOLUTION The circle has area $A = \pi r^2$. By the symmetry principle, its centroid is its center and so the distance traveled by the centroid during a rotation is $d = 2\pi R$. Therefore, by the Theorem of Pappus, the volume of the torus is

$$V = Ad = (2\pi R)(\pi r^2) = 2\pi^2 r^2 R$$ □

The method of Example 7 should be compared with the method of Exercise 63 in Section 6.2.

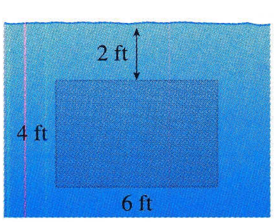
9.3 EXERCISES

1. An aquarium 5 ft long, 2 ft wide, and 3 ft deep is full of water. Find (a) the hydrostatic pressure on the bottom of the aquarium, (b) the hydrostatic force on the bottom, and (c) the hydrostatic force on one end of the aquarium.

2. A tank is 8 m long, 4 m wide, 2 m high, and contains kerosene with density 820 kg/m^3 to a depth of 1.5 m. Find (a) the hydrostatic pressure on the bottom of the tank, (b) the hydrostatic force on the bottom, and (c) the hydrostatic force on one end of the tank.

3–11 A vertical plate is submerged (or partially submerged) in water and has the indicated shape. Explain how to approximate the hydrostatic force against one side of the plate by a Riemann sum. Then express the force as an integral and evaluate it.

3.



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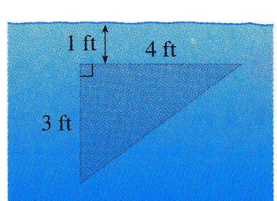

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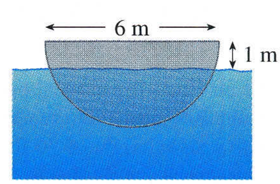

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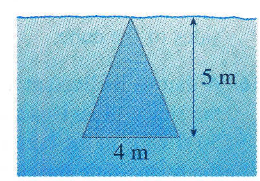

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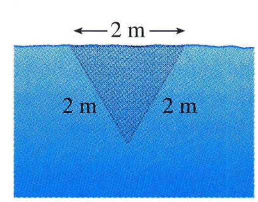

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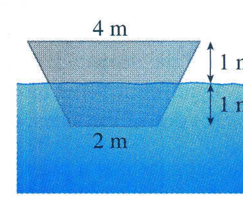

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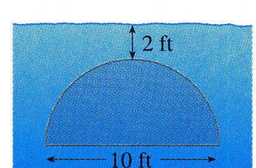

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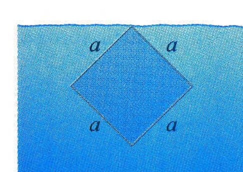

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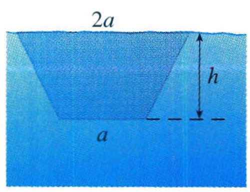
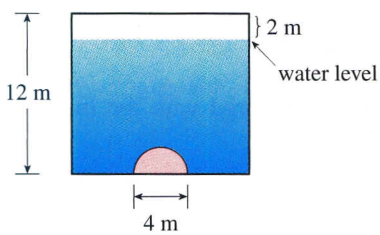


12. A large tank is designed with ends in the shape of the region between the curves $y = \frac{1}{2}x^2$ and $y = 12$, measured in feet. Find the hydrostatic force on one end of the tank if it is filled to a depth of 8 ft with gasoline. (Assume the gasoline's density is 42.0 lb/ft^3.)

13. A trough is filled with a liquid of density 840 kg/m^3. The ends of the trough are equilateral triangles with sides 8 m long and vertex at the bottom. Find the hydrostatic force on one end of the trough.

14. A vertical dam has a semicircular gate as shown in the figure. Find the hydrostatic force against the gate.



15. A cube with 20-cm-long sides is sitting on the bottom of an aquarium in which the water is one meter deep. Estimate the hydrostatic force on (a) the top of the cube and (b) one of the sides of the cube.

16. A dam is inclined at an angle of 30° from the vertical and has the shape of an isosceles trapezoid 100 ft wide at the top and 50 ft wide at the bottom and with a slant height of 70 ft. Find the hydrostatic force on the dam when it is full of water.

17. A swimming pool is 10 m wide and 20 m long and its bottom is an inclined plane, the shallow end having a depth of 1 m and the deep end, 3 m. If the pool is full of water, find the hydrostatic force on (a) the shallow end, (b) the deep end, (c) one of the sides, and (d) the bottom of the pool.

18. Suppose that a plate is immersed vertically in a fluid with density ρ and the width of the plate is $w(x)$ at a depth of x meters beneath the surface of the fluid. If the top of the plate is at depth a and the bottom is at depth b, show that the hydrostatic force on one side of the plate is

$$F = \int_a^b \rho g x w(x)\, dx$$

19. A vertical, irregularly shaped plate is submerged in water. The table shows measurements of its width, taken at the indicated depths. Use Simpson's Rule to estimate the force of the water against the plate.

Depth (m)	2.0	2.5	3.0	3.5	4.0	4.5	5.0
Plate width (m)	0	0.8	1.7	2.4	2.9	3.3	3.6

20. (a) Use the formula of Exercise 18 to show that

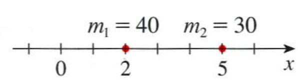
$$F = (\rho g \bar{x})A$$

where $\bar{x}$ is the x-coordinate of the centroid of the plate and A is its area. This equation shows that the hydrostatic force against a vertical plane region is the same as if the region were horizontal at the depth of the centroid of the region.

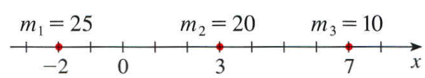
(b) Use the result of part (a) to give another solution to Exercise 10.

21–22 Point-masses m_i are located on the x-axis as shown. Find the moment M of the system about the origin and the center of mass $\bar{x}$.

21.



22.



23–24 The masses m_i are located at the points P_i. Find the moments M_x and M_y and the center of mass of the system.

23. $m_1 = 6,\ m_2 = 5,\ m_3 = 10;$
$P_1(1, 5),\ P_2(3, -2),\ P_3(-2, -1)$

24. $m_1 = 6,\ m_2 = 5,\ m_3 = 1,\ m_4 = 4;$
$P_1(1, -2),\ P_2(3, 4),\ P_3(-3, -7),\ P_4(6, -1)$

25–28 Sketch the region bounded by the curves, and visually estimate the location of the centroid. Then find the exact coordinates of the centroid.

25. $y = 4 - x^2, \quad y = 0$

26. $3x + 2y = 6, \quad y = 0, \quad x = 0$

27. $y = e^x, \quad y = 0, \quad x = 0, \quad x = 1$

28. $y = 1/x, \quad y = 0, \quad x = 1, \quad x = 2$

29–33 Find the centroid of the region bounded by the given curves.

29. $y = \sqrt{x}, \quad y = x$

30. $y = x + 2, \quad y = x^2$

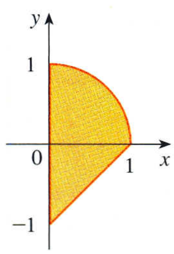
31. $y = \sin x, \quad y = \cos x, \quad x = 0, \quad x = \pi/4$

32. $y = x^3, \quad x + y = 2, \quad y = 0$

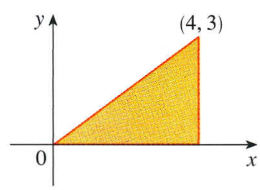
33. $x = 5 - y^2, \quad x = 0$

34–35 Calculate the moments M_x and M_y and the center of mass of a lamina with the given density and shape.

34. $\rho = 3$



35. $\rho = 10$



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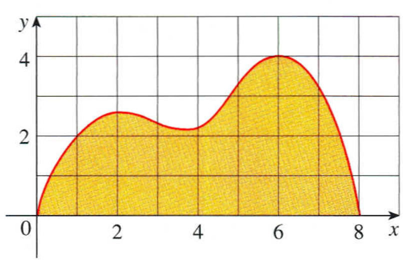
36. Use Simpson's Rule to estimate the centroid of the region shown.



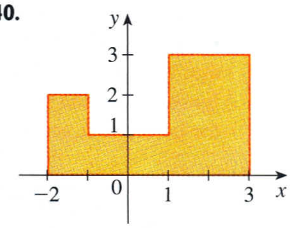
37. Find the centroid of the region bounded by the curves $y = 2^x$ and $y = x^2$, $0 \leqslant x \leqslant 2$, to three decimal places. Sketch the region and plot the centroid to see if your answer is reasonable.

38. Use a graph to find approximate x-coordinates of the points of intersection of the curves $y = x + \ln x$ and $y = x^3 - x$. Then find (approximately) the centroid of the region bounded by these curves.

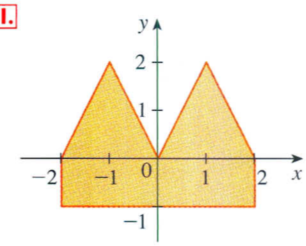
39. Prove that the centroid of any triangle is located at the point of intersection of the medians. [*Hints:* Place the axes so that the vertices are $(a, 0)$, $(0, b)$, and $(c, 0)$. Recall that a median is a line segment from a vertex to the midpoint of the opposite side. Recall also that the medians intersect at a point two-thirds of the way from each vertex (along the median) to the opposite side.]

40–41 Find the centroid of the region shown, not by integration, but by locating the centroids of the rectangles and triangles (from Exercise 39) and using additivity of moments.

40.



41.



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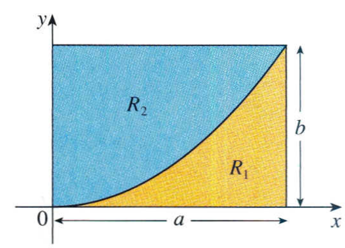
42. A rectangle R with sides a and b is divided into two parts R_1 and R_2 by an arc of a parabola that has its vertex at one corner of R and passes through the opposite corner. Find the centroids of both R_1 and R_2.



43. If $\bar{x}$ is the x-coordinate of the centroid of the region that lies under the graph of a continuous function f, where $a \leqslant x \leqslant b$, show that

$$\int_a^b (cx + d)f(x)\,dx = (c\bar{x} + d)\int_a^b f(x)\,dx$$

44–46 Use the Theorem of Pappus to find the volume of the given solid.

44. A sphere of radius r (Use Example 4.)

45. A cone with height h and base radius r

46. The solid obtained by rotating the triangle with vertices $(2, 3)$, $(2, 5)$, and $(5, 4)$ about the x-axis

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47. Prove Formulas 9.

48. Let $\mathcal{R}$ be the region that lies between the curves $y = x^m$ and $y = x^n$, $0 \leqslant x \leqslant 1$, where m and n are integers with $0 \leqslant n < m$.

(a) Sketch the region $\mathcal{R}$.

(b) Find the coordinates of the centroid of $\mathcal{R}$.

(c) Try to find values of m and n such that the centroid lies *outside* $\mathcal{R}$.

where F is the rate of flow that we are trying to determine. Thus the total amount of dye is approximately

$$\sum_{i=1}^{n} c(t_i)F\,\Delta t = F\sum_{i=1}^{n} c(t_i)\,\Delta t$$

and, letting $n \to \infty$, we find that the amount of dye is

$$A = F\int_0^T c(t)\,dt$$

Thus the cardiac output is given by

$$\boxed{3} \qquad F = \frac{A}{\int_0^T c(t)\,dt}$$

where the amount of dye A is known and the integral can be approximated from the concentration readings.

t	$c(t)$	t	$c(t)$
0	0	6	6.1
1	0.4	7	4.0
2	2.8	8	2.3
3	6.5	9	1.1
4	9.8	10	0
5	8.9		

EXAMPLE 2 A 5-mg bolus of dye is injected into a right atrium. The concentration of the dye (in milligrams per liter) is measured in the aorta at one-second intervals as shown in the chart. Estimate the cardiac output.

SOLUTION Here $A = 5$, $\Delta t = 1$, and $T = 10$. We use Simpson's Rule to approximate the integral of the concentration:

$$\int_0^{10} c(t)\,dt \approx \tfrac{1}{3}[0 + 4(0.4) + 2(2.8) + 4(6.5) + 2(9.8) + 4(8.9)$$
$$+ 2(6.1) + 4(4.0) + 2(2.3) + 4(1.1) + 0]$$
$$\approx 41.87$$

Thus Formula 3 gives the cardiac output to be

$$F = \frac{A}{\int_0^{10} c(t)\,dt} \approx \frac{5}{41.87} \approx 0.12 \text{ L/s} = 7.2 \text{ L/min}$$ □

9.4 EXERCISES

1. The marginal cost function $C'(x)$ was defined to be the derivative of the cost function. (See Sections 3.7 and 4.7.) If the marginal cost of maufacturing x meters of a fabric is $C'(x) = 5 - 0.008x + 0.000009x^2$ (measured in dollars per meter) and the fixed start-up cost is $C(0) = \$20,000$, use the Net Change Theorem to find the cost of producing the first 2000 units.

2. The marginal revenue from the sale of x units of a product is $12 - 0.0004x$. If the revenue from the sale of the first 1000 units is \$12,400, find the revenue from the sale of the first 5000 units.

3. The marginal cost of producing x units of a certain product is $74 + 1.1x - 0.002x^2 + 0.00004x^3$ (in dollars per unit). Find the increase in cost if the production level is raised from 1200 units to 1600 units.

4. The demand function for a certain commodity is $p = 20 - 0.05x$. Find the consumer surplus when the sales level is 300. Illustrate by drawing the demand curve and identifying the consumer surplus as an area.

5. A demand curve is given by $p = 450/(x + 8)$. Find the consumer surplus when the selling price is \$10.

6. The **supply function** $p_S(x)$ for a commodity gives the relation between the selling price and the number of units that manufacturers will produce at that price. For a higher price, manufacturers will produce more units, so p_S is an increasing function of x. Let X be the amount of the commodity currently produced and let $P = p_S(X)$ be the current price. Some producers would be willing to make and sell the commodity for a lower selling price and are therefore receiving more than their minimal price. The excess is called the **producer surplus**. An

argument similar to that for consumer surplus shows that the surplus is given by the integral

$$\int_0^X [P - p_S(x)]\, dx$$

Calculate the producer surplus for the supply function $p_S(x) = 3 + 0.01x^2$ at the sales level $X = 10$. Illustrate by drawing the supply curve and identifying the producer surplus as an area.

7. If a supply curve is modeled by the equation $p = 200 + 0.2x^{3/2}$, find the producer surplus when the selling price is \$400.

8. For a given commodity and pure competition, the number of units produced and the price per unit are determined as the coordinates of the point of intersection of the supply and demand curves. Given the demand curve $p = 50 - \frac{1}{20}x$ and the supply curve $p = 20 + \frac{1}{10}x$, find the consumer surplus and the producer surplus. Illustrate by sketching the supply and demand curves and identifying the surpluses as areas.

9. A company modeled the demand curve for its product (in dollars) by the equation

$$p = \frac{800{,}000e^{-x/5000}}{x + 20{,}000}$$

Use a graph to estimate the sales level when the selling price is \$16. Then find (approximately) the consumer surplus for this sales level.

10. A movie theater has been charging \$7.50 per person and selling about 400 tickets on a typical weeknight. After surveying their customers, the theater estimates that for every 50 cents that they lower the price, the number of moviegoers will increase by 35 per night. Find the demand function and calculate the consumer surplus when the tickets are priced at \$6.00.

11. If the amount of capital that a company has at time t is $f(t)$, then the derivative, $f'(t)$, is called the *net investment flow*. Suppose that the net investment flow is $\sqrt{t}$ million dollars per year (where t is measured in years). Find the increase in capital (the *capital formation*) from the fourth year to the eighth year.

12. If revenue flows into a company at a rate of $f(t) = 9000\sqrt{1 + 2t}$, where t is measured in years and $f(t)$ is measured in dollars per year, find the total revenue obtained in the first four years.

13. *Pareto's Law of Income* states that the number of people with incomes between $x = a$ and $x = b$ is $N = \int_a^b Ax^{-k}\, dx$, where A and k are constants with $A > 0$ and $k > 1$. The average income of these people is

$$\bar{x} = \frac{1}{N}\int_a^b Ax^{1-k}\, dx$$

Calculate $\bar{x}$.

14. A hot, wet summer is causing a mosquito population explosion in a lake resort area. The number of mosquitos is increasing at an estimated rate of $2200 + 10e^{0.8t}$ per week (where t is measured in weeks). By how much does the mosquito population increase between the fifth and ninth weeks of summer?

15. Use Poiseuille's Law to calculate the rate of flow in a small human artery where we can take $\eta = 0.027$, $R = 0.008$ cm, $l = 2$ cm, and $P = 4000$ dynes/cm^2.

16. High blood pressure results from constriction of the arteries. To maintain a normal flow rate (flux), the heart has to pump harder, thus increasing the blood pressure. Use Poiseuille's Law to show that if R_0 and P_0 are normal values of the radius and pressure in an artery and the constricted values are R and P, then for the flux to remain constant, P and R are related by the equation

$$\frac{P}{P_0} = \left(\frac{R_0}{R}\right)^4$$

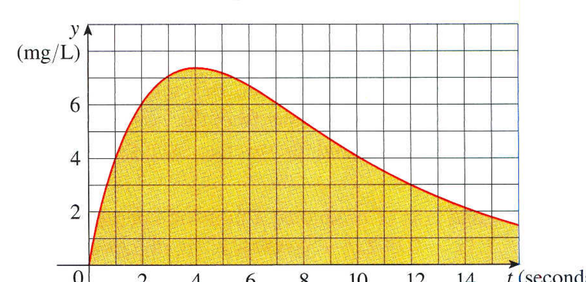
Deduce that if the radius of an artery is reduced to three-fourths of its former value, then the pressure is more than tripled.

17. The dye dilution method is used to measure cardiac output with 6 mg of dye. The dye concentrations, in mg/L, are modeled by $c(t) = 20te^{-0.6t}$, $0 \leq t \leq 10$, where t is measured in seconds. Find the cardiac output.

18. After an 8-mg injection of dye, the readings of dye concentration, in mg/L, at two-second intervals are as shown in the table. Use Simpson's Rule to estimate the cardiac output.

t	$c(t)$	t	$c(t)$
0	0	12	3.9
2	2.4	14	2.3
4	5.1	16	1.6
6	7.8	18	0.7
8	7.6	20	0
10	5.4		

19. The graph of the concentration function $c(t)$ is shown after a 7-mg injection of dye into a heart. Use Simpson's Rule to estimate the cardiac output.



SOLUTION

(a) Since IQ scores are normally distributed, we use the probability density function given by Equation 3 with $\mu = 100$ and $\sigma = 15$:

$$P(85 \leqslant X \leqslant 115) = \int_{85}^{115} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/(2\cdot 15^2)}\,dx$$

Recall from Section 8.5 that the function $y = e^{-x^2}$ doesn't have an elementary antiderivative, so we can't evaluate the integral exactly. But we can use the numerical integration capability of a calculator or computer (or the Midpoint Rule or Simpson's Rule) to estimate the integral. Doing so, we find that

$$P(85 \leqslant X \leqslant 115) \approx 0.68$$

So about 68% of the population has an IQ between 85 and 115, that is, within one standard deviation of the mean.

(b) The probability that the IQ score of a person chosen at random is more than 140 is

$$P(X > 140) = \int_{140}^{\infty} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/450}\,dx$$

To avoid the improper integral we could approximate it by the integral from 140 to 200. (It's quite safe to say that people with an IQ over 200 are extremely rare.) Then

$$P(X > 140) \approx \int_{140}^{200} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/450}\,dx \approx 0.0038$$

Therefore about 0.4% of the population has an IQ over 140. □

9.5 EXERCISES

1. Let $f(x)$ be the probability density function for the lifetime of a manufacturer's highest quality car tire, where x is measured in kilometers. Explain the meaning of each integral.

(a) $\int_{30{,}000}^{40{,}000} f(x)\,dx$ (b) $\int_{25{,}000}^{\infty} f(x)\,dx$

2. Let $f(t)$ be the probability density function for the time it takes you to drive to school in the morning, where t is measured in minutes. Express the following probabilities as integrals.

(a) The probability that you drive to school in less than 15 minutes

(b) The probability that it takes you more than half an hour to get to school

3. Let $f(x) = \frac{3}{64}x\sqrt{16 - x^2}$ for $0 \leqslant x \leqslant 4$ and $f(x) = 0$ for all other values of x.

(a) Verify that f is a probability density function.

(b) Find $P(X < 2)$.

4. Let $f(x) = xe^{-x}$ if $x \geqslant 0$ and $f(x) = 0$ if $x < 0$.

(a) Verify that f is a probability density function.

(b) Find $P(1 \leqslant X \leqslant 2)$.

5. Let $f(x) = c/(1 + x^2)$.

(a) For what value of c is f a probability density function?

(b) For that value of c, find $P(-1 < X < 1)$.

6. Let $f(x) = kx^2(1 - x)$ if $0 \leqslant x \leqslant 1$ and $f(x) = 0$ if $x < 0$ or $x > 1$.

(a) For what value of k is f a probability density function?

(b) For that value of k, find $P\left(X \geqslant \frac{1}{2}\right)$.

(c) Find the mean.

7. A spinner from a board game randomly indicates a real number between 0 and 10. The spinner is fair in the sense that it indicates a number in a given interval with the same probability as it indicates a number in any other interval of the same length.

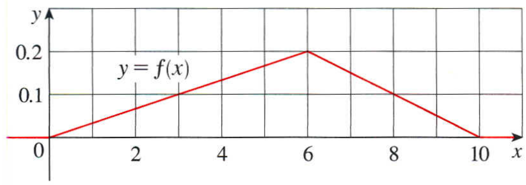
(a) Explain why the function

$$f(x) = \begin{cases} 0.1 & \text{if } 0 \leqslant x \leqslant 10 \\ 0 & \text{if } x < 0 \text{ or } x > 10 \end{cases}$$

is a probability density function for the spinner's values.

(b) What does your intuition tell you about the value of the mean? Check your guess by evaluating an integral.

8. (a) Explain why the function whose graph is shown is a probability density function.
(b) Use the graph to find the following probabilities:
(i) $P(X < 3)$ (ii) $P(3 \leqslant X \leqslant 8)$
(c) Calculate the mean.



9. Show that the median waiting time for a phone call to the company described in Example 4 is about 3.5 minutes.

10. (a) A type of lightbulb is labeled as having an average lifetime of 1000 hours. It's reasonable to model the probability of failure of these bulbs by an exponential density function with mean $\mu = 1000$. Use this model to find the probability that a bulb
(i) fails within the first 200 hours,
(ii) burns for more than 800 hours.
(b) What is the median lifetime of these lightbulbs?

11. The manager of a fast-food restaurant determines that the average time that her customers wait for service is 2.5 minutes.
(a) Find the probability that a customer has to wait more than 4 minutes.
(b) Find the probability that a customer is served within the first 2 minutes.
(c) The manager wants to advertise that anybody who isn't served within a certain number of minutes gets a free hamburger. But she doesn't want to give away free hamburgers to more than 2% of her customers. What should the advertisement say?

12. According to the National Health Survey, the heights of adult males in the United States are normally distributed with mean 69.0 inches and standard deviation 2.8 inches.
(a) What is the probability that an adult male chosen at random is between 65 inches and 73 inches tall?
(b) What percentage of the adult male population is more than 6 feet tall?

13. The "Garbage Project" at the University of Arizona reports that the amount of paper discarded by households per week is normally distributed with mean 4.3 kg and standard deviation 1.9 kg. What percentage of households throw out at least 5 kg of paper a week?

14. Boxes are labeled as containing 500 g of cereal. The machine filling the boxes produces weights that are normally distributed with standard deviation 12 g.
(a) If the target weight is 500 g, what is the probability that the machine produces a box with less than 480 g of cereal?
(b) Suppose a law states that no more than 5% of a manufacturer's cereal boxes can contain less than the stated weight of 500 g. At what target weight should the manufacturer set its filling machine?

15. The speeds of vehicles on a highway with speed limit 100 km/h are normally distributed with mean 112 km/h and standard deviation 8 km/h.
(a) What is the probability that a randomly chosen vehicle is traveling at a legal speed?
(b) If police are instructed to ticket motorists driving 125 km/h or more, what percentage of motorists are targeted?

16. Show that the probability density function for a normally distributed random variable has inflection points at $x = \mu \pm \sigma$.

17. For any normal distribution, find the probability that the random variable lies within two standard deviations of the mean.

18. The standard deviation for a random variable with probability density function f and mean μ is defined by

$$\sigma = \left[\int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, dx \right]^{1/2}$$

Find the standard deviation for an exponential density function with mean μ.

19. The hydrogen atom is composed of one proton in the nucleus and one electron, which moves about the nucleus. In the quantum theory of atomic structure, it is assumed that the electron does not move in a well-defined orbit. Instead, it occupies a state known as an *orbital,* which may be thought of as a "cloud" of negative charge surrounding the nucleus. At the state of lowest energy, called the *ground state,* or *1s-orbital,* the shape of this cloud is assumed to be a sphere centered at the nucleus. This sphere is described in terms of the probability density function

$$p(r) = \frac{4}{a_0^3} r^2 e^{-2r/a_0} \qquad r \geqslant 0$$

where a_0 is the *Bohr radius* ($a_0 \approx 5.59 \times 10^{-11}$ m). The integral

$$P(r) = \int_0^r \frac{4}{a_0^3} s^2 e^{-2s/a_0}\, ds$$

gives the probability that the electron will be found within the sphere of radius r meters centered at the nucleus.
(a) Verify that $p(r)$ is a probability density function.
(b) Find $\lim_{r \to \infty} p(r)$. For what value of r does $p(r)$ have its maximum value?
(c) Graph the density function.
(d) Find the probability that the electron will be within the sphere of radius $4a_0$ centered at the nucleus.
(e) Calculate the mean distance of the electron from the nucleus in the ground state of the hydrogen atom.

9 REVIEW

CONCEPT CHECK

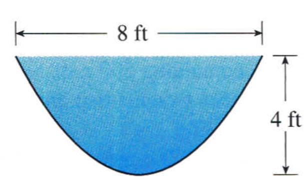
1. (a) How is the length of a curve defined?
   (b) Write an expression for the length of a smooth curve given by $y = f(x)$, $a \leqslant x \leqslant b$.
   (c) What if x is given as a function of y?
2. (a) Write an expression for the surface area of the surface obtained by rotating the curve $y = f(x)$, $a \leqslant x \leqslant b$, about the x-axis.
   (b) What if x is given as a function of y?
   (c) What if the curve is rotated about the y-axis?
3. Describe how we can find the hydrostatic force against a vertical wall submersed in a fluid.
4. (a) What is the physical significance of the center of mass of a thin plate?
   (b) If the plate lies between $y = f(x)$ and $y = 0$, where $a \leqslant x \leqslant b$, write expressions for the coordinates of the center of mass.
5. What does the Theorem of Pappus say?
6. Given a demand function $p(x)$, explain what is meant by the consumer surplus when the amount of a commodity currently available is X and the current selling price is P. Illustrate with a sketch.
7. (a) What is the cardiac output of the heart?
   (b) Explain how the cardiac output can be measured by the dye dilution method.
8. What is a probability density function? What properties does such a function have?
9. Suppose $f(x)$ is the probability density function for the weight of a female college student, where x is measured in kilograms.
   (a) What is the meaning of the integral $\int_0^{60} f(x)\,dx$?
   (b) Write an expression for the mean of this density function.
   (c) How can we find the median of this density function?
10. What is a normal distribution? What is the significance of the standard deviation?

EXERCISES

1–2 Find the length of the curve.

1. $y = \frac{1}{6}(x^2 + 4)^{3/2}$, $\quad 0 \leqslant x \leqslant 3$
2. $y = 2\ln\left(\sin \frac{1}{2}x\right)$, $\quad \pi/3 \leqslant x \leqslant \pi$

3. (a) Find the length of the curve
$$y = \frac{x^4}{16} + \frac{1}{2x^2} \qquad 1 \leqslant x \leqslant 2$$
   (b) Find the area of the surface obtained by rotating the curve in part (a) about the y-axis.
4. (a) The curve $y = x^2$, $0 \leqslant x \leqslant 1$, is rotated about the y-axis. Find the area of the resulting surface.
   (b) Find the area of the surface obtained by rotating the curve in part (a) about the x-axis.
5. Use Simpson's Rule with $n = 6$ to estimate the length of the curve $y = e^{-x^2}$, $0 \leqslant x \leqslant 3$.
6. Use Simpson's Rule with $n = 6$ to estimate the area of the surface obtained by rotating the curve in Exercise 5 about the x-axis.
7. Find the length of the curve
$$y = \int_1^x \sqrt{\sqrt{t} - 1}\,dt \qquad 1 \leqslant x \leqslant 16$$
8. Find the area of the surface obtained by rotating the curve in Exercise 7 about the y-axis.
9. A gate in an irrigation canal is constructed in the form of a trapezoid 1 m wide at the bottom, 2 m wide at the top, and 1 m high. It is placed vertically in the canal, with the water extending to its top. Find the hydrostatic force on one side of the gate.
10. A trough is filled with water and its vertical ends have the shape of the parabolic region in the figure. Find the hydrostatic force on one end of the trough.



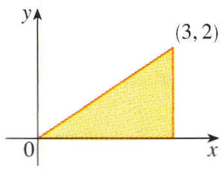
11–12 Find the centroid of the region bounded by the given curves.

11. $y = \frac{1}{2}x, \quad y = \sqrt{x}$

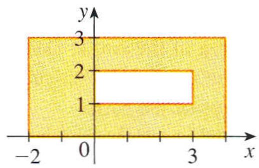
12. $y = \sin x, \quad y = 0, \quad x = \pi/4, \quad x = 3\pi/4$

13–14 Find the centroid of the region shown

13.



14.



15. Find the volume obtained when the circle of radius 1 with center (1, 0) is rotated about the y-axis.

16. Use the Theorem of Pappus and the fact that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ to find the centroid of the semicircular region bounded by the curve $y = \sqrt{r^2 - x^2}$ and the x-axis.

17. The demand function for a commodity is given by

$$p = 2000 - 0.1x - 0.01x^2$$

Find the consumer surplus when the sales level is 100.

18. After a 6-mg injection of dye into a heart, the readings of dye concentration at two-second intervals are as shown in the table. Use Simpson's Rule to estimate the cardiac output.

t	$c(t)$	t	$c(t)$
0	0	14	4.7
2	1.9	16	3.3
4	3.3	18	2.1
6	5.1	20	1.1
8	7.6	22	0.5
10	7.1	24	0
12	5.8		

19. (a) Explain why the function

$$f(x) = \begin{cases} \dfrac{\pi}{20}\sin\left(\dfrac{\pi x}{10}\right) & \text{if } 0 \leqslant x \leqslant 10 \\ 0 & \text{if } x < 0 \text{ or } x > 10 \end{cases}$$

is a probability density function.

(b) Find $P(X < 4)$.

(c) Calculate the mean. Is the value what you would expect?

20. Lengths of human pregnancies are normally distributed with mean 268 days and standard deviation 15 days. What percentage of pregnancies last between 250 days and 280 days?

21. The length of time spent waiting in line at a certain bank is modeled by an exponential density function with mean 8 minutes.

(a) What is the probability that a customer is served in the first 3 minutes?

(b) What is the probability that a customer has to wait more than 10 minutes?

(c) What is the median waiting time?

PROBLEMS PLUS

1. Find the area of the region $S = \{(x, y) \mid x \geqslant 0,\ y \leqslant 1,\ x^2 + y^2 \leqslant 4y\}$.

2. Find the centroid of the region enclosed by the loop of the curve $y^2 = x^3 - x^4$.

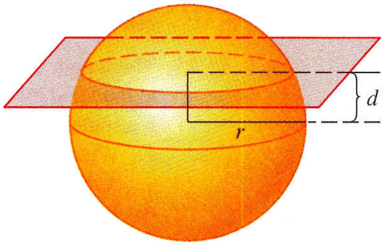
3. If a sphere of radius r is sliced by a plane whose distance from the center of the sphere is d, then the sphere is divided into two pieces called segments of one base. The corresponding surfaces are called *spherical zones of one base.*

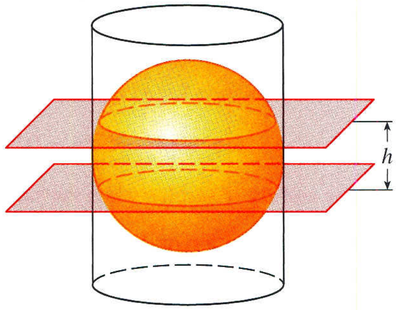
(a) Determine the surface areas of the two spherical zones indicated in the figure.

(b) Determine the approximate area of the Arctic Ocean by assuming that it is approximately circular in shape, with center at the North Pole and "circumference" at 75° north latitude. Use $r = 6370$ km for the radius of the earth.

(c) A sphere of radius r is inscribed in a right circular cylinder of radius r. Two planes perpendicular to the central axis of the cylinder and a distance h apart cut off a *spherical zone of two bases* on the sphere. Show that the surface area of the spherical zone equals the surface area of the region that the two planes cut off on the cylinder.

(d) The *Torrid Zone* is the region on the surface of the earth that is between the Tropic of Cancer (23.45° north latitude) and the Tropic of Capricorn (23.45° south latitude). What is the area of the Torrid Zone?





4. (a) Show that an observer at height H above the north pole of a sphere of radius r can see a part of the sphere that has area

$$\frac{2\pi r^2 H}{r + H}$$

(b) Two spheres with radii r and R are placed so that the distance between their centers is d, where $d > r + R$. Where should a light be placed on the line joining the centers of the spheres in order to illuminate the largest total surface?

5. Suppose that the density of seawater, $\rho = \rho(z)$, varies with the depth z below the surface.

(a) Show that the hydrostatic pressure is governed by the differential equation

$$\frac{dP}{dz} = \rho(z)g$$

where g is the acceleration due to gravity. Let P_0 and ρ_0 be the pressure and density at $z = 0$. Express the pressure at depth z as an integral.

(b) Suppose the density of seawater at depth z is given by $\rho = \rho_0 e^{z/H}$, where H is a positive constant. Find the total force, expressed as an integral, exerted on a vertical circular porthole of radius r whose center is located at a distance $L > r$ below the surface.

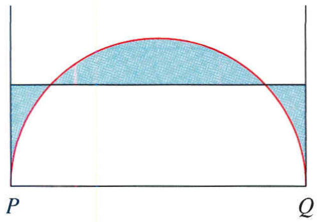


FIGURE FOR PROBLEM 6

6. The figure shows a semicircle with radius 1, horizontal diameter PQ, and tangent lines at P and Q. At what height above the diameter should the horizontal line be placed so as to minimize the shaded area?

7. Let P be a pyramid with a square base of side $2b$ and suppose that S is a sphere with its center on the base of P and S is tangent to all eight edges of P. Find the height of P. Then find the volume of the intersection of S and P.

8. Consider a flat metal plate to be placed vertically under water with its top 2 m below the surface of the water. Determine a shape for the plate so that if the plate is divided into any number of horizontal strips of equal height, the hydrostatic force on each strip is the same.

9. A uniform disk with radius 1 m is to be cut by a line so that the center of mass of the smaller piece lies halfway along a radius. How close to the center of the disk should the cut be made? (Express your answer correct to two decimal places.)

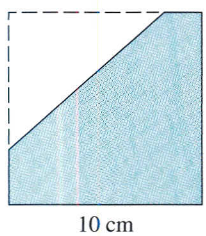


FIGURE FOR PROBLEM 10

10. A triangle with area 30 cm^2 is cut from a corner of a square with side 10 cm, as shown in the figure. If the centroid of the remaining region is 4 cm from the right side of the square, how far is it from the bottom of the square?

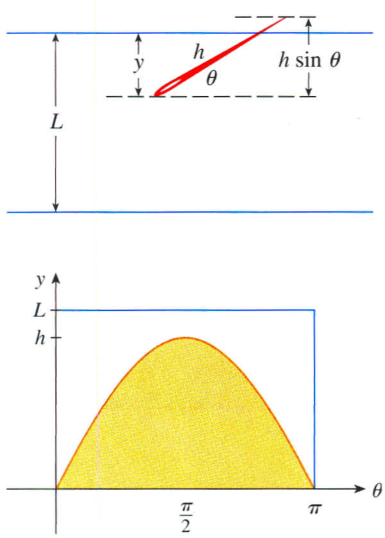


FIGURE FOR PROBLEM 11

11. In a famous 18th-century problem, known as *Buffon's needle problem*, a needle of length h is dropped onto a flat surface (for example, a table) on which parallel lines L units apart, $L \geqslant h$, have been drawn. The problem is to determine the probability that the needle will come to rest intersecting one of the lines. Assume that the lines run east-west, parallel to the x-axis in a rectangular coordinate system (as in the figure). Let y be the distance from the "southern" end of the needle to the nearest line to the north. (If the needle's southern end lies on a line, let $y = 0$. If the needle happens to lie east-west, let the "western" end be the "southern" end.) Let θ be the angle that the needle makes with a ray extending eastward from the "southern" end. Then $0 \leqslant y \leqslant L$ and $0 \leqslant \theta \leqslant \pi$. Note that the needle intersects one of the lines only when $y < h \sin\theta$. The total set of possibilities for the needle can be identified with the rectangular region $0 \leqslant y \leqslant L$, $0 \leqslant \theta \leqslant \pi$, and the proportion of times that the needle intersects a line is the ratio

$$\frac{\text{area under } y = h\sin\theta}{\text{area of rectangle}}$$

This ratio is the probability that the needle intersects a line. Find the probability that the needle will intersect a line if $h = L$. What if $h = \frac{1}{2}L$?

12. If the needle in Problem 11 has length $h > L$, it's possible for the needle to intersect more than one line.

(a) If $L = 4$, find the probability that a needle of length 7 will intersect at least one line. [*Hint:* Proceed as in Problem 11. Define y as before; then the total set of possibilities for the needle can be identified with the same rectangular region $0 \leqslant y \leqslant L$, $0 \leqslant \theta \leqslant \pi$. What portion of the rectangle corresponds to the needle intersecting a line?]

(b) If $L = 4$, find the probability that a needle of length 7 will intersect *two* lines.

(c) If $2L < h \leqslant 3L$, find a general formula for the probability that the needle intersects three lines.