■ Equation 7 is called a *reduction formula* because the exponent n has been *reduced* to $n-1$ and $n-2$.

EXAMPLE 6 Prove the reduction formula

$$\boxed{7} \qquad \int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

where $n \geqslant 2$ is an integer.

SOLUTION Let $\quad u = \sin^{n-1}x \qquad dv = \sin x\,dx$

Then $\quad du = (n-1)\sin^{n-2}x\cos x\,dx \qquad v = -\cos x$

so integration by parts gives

$$\int \sin^n x\,dx = -\cos x\sin^{n-1}x + (n-1)\int \sin^{n-2}x\cos^2 x\,dx$$

Since $\cos^2 x = 1 - \sin^2 x$, we have

$$\int \sin^n x\,dx = -\cos x\sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx$$

As in Example 4, we solve this equation for the desired integral by taking the last term on the right side to the left side. Thus we have

$$n\int \sin^n x\,dx = -\cos x\sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx$$

or

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x\sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,dx \qquad \square$$

The reduction formula (7) is useful because by using it repeatedly we could eventually express $\int \sin^n x\,dx$ in terms of $\int \sin x\,dx$ (if n is odd) or $\int (\sin x)^0\,dx = \int dx$ (if n is even).

8.1 EXERCISES

1–2 Evaluate the integral using integration by parts with the indicated choices of u and dv.

1. $\int x^2 \ln x\,dx;\quad u = \ln x,\ dv = x^2\,dx$

2. $\int \theta\cos\theta\,d\theta;\quad u = \theta,\ dv = \cos\theta\,d\theta$

3–32 Evaluate the integral.

3. $\int x\cos 5x\,dx$

4. $\int xe^{-x}\,dx$

5. $\int re^{r/2}\,dr$

6. $\int t\sin 2t\,dt$

7. $\int x^2\cos 3x\,dx$

8. $\int x^2\sin ax\,dx$

9. $\int \ln(2x+1)\,dx$

10. $\int \sin^{-1}x\,dx$

11. $\int \arctan 4t\,dt$

12. $\int p^5\ln p\,dp$

13. $\int t\sec^2 2t\,dt$

14. $\int s2^s\,ds$

15. $\int (\ln x)^2\,dx$

16. $\int t\sinh mt\,dt$

17. $\int e^{2\theta}\sin 3\theta\,d\theta$

18. $\int e^{-\theta}\cos 2\theta\,d\theta$

19. $\int_0^{\pi} t\sin 3t\,dt$

20. $\int_0^1 (x^2+1)e^{-x}\,dx$

21. $\int_0^1 t\cosh t\,dt$

22. $\int_4^9 \frac{\ln y}{\sqrt{y}}\,dy$

23. $\int_1^2 \frac{\ln x}{x^2}\,dx$

24. $\int_0^{\pi} x^3\cos x\,dx$

25. $\displaystyle\int_0^1 \frac{y}{e^{2y}}\,dy$

26. $\displaystyle\int_1^{\sqrt{3}} \arctan(1/x)\,dx$

27. $\displaystyle\int_0^{1/2} \cos^{-1}x\,dx$

28. $\displaystyle\int_1^2 \frac{(\ln x)^2}{x^3}\,dx$

29. $\displaystyle\int \cos x\,\ln(\sin x)\,dx$

30. $\displaystyle\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr$

31. $\displaystyle\int_1^2 x^4(\ln x)^2\,dx$

32. $\displaystyle\int_0^t e^s\sin(t-s)\,ds$

33–38 First make a substitution and then use integration by parts to evaluate the integral.

33. $\displaystyle\int \cos\sqrt{x}\,dx$

34. $\displaystyle\int t^3e^{-t^2}\,dt$

35. $\displaystyle\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3\cos(\theta^2)\,d\theta$

36. $\displaystyle\int_0^{\pi} e^{\cos t}\sin 2t\,dt$

37. $\displaystyle\int x\ln(1+x)\,dx$

38. $\displaystyle\int \sin(\ln x)\,dx$

39–42 Evaluate the indefinite integral. Illustrate, and check that your answer is reasonable, by graphing both the function and its antiderivative (take $C=0$).

39. $\displaystyle\int (2x+3)e^x\,dx$

40. $\displaystyle\int x^{3/2}\ln x\,dx$

41. $\displaystyle\int x^3\sqrt{1+x^2}\,dx$

42. $\displaystyle\int x^2\sin 2x\,dx$

43. (a) Use the reduction formula in Example 6 to show that

$$\int \sin^2x\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b) Use part (a) and the reduction formula to evaluate $\int \sin^4x\,dx$.

44. (a) Prove the reduction formula

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\,\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

(b) Use part (a) to evaluate $\int \cos^2x\,dx$.
(c) Use parts (a) and (b) to evaluate $\int \cos^4x\,dx$.

45. (a) Use the reduction formula in Example 6 to show that

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}x\,dx$$

where $n \geqslant 2$ is an integer.

(b) Use part (a) to evaluate $\int_0^{\pi/2}\sin^3x\,dx$ and $\int_0^{\pi/2}\sin^5x\,dx$.
(c) Use part (a) to show that, for odd powers of sine,

$$\int_0^{\pi/2} \sin^{2n+1}x\,dx = \frac{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)}$$

46. Prove that, for even powers of sine,

$$\int_0^{\pi/2} \sin^{2n}x\,dx = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}\,\frac{\pi}{2}$$

47–50 Use integration by parts to prove the reduction formula.

47. $\displaystyle\int (\ln x)^n\,dx = x(\ln x)^n - n\int(\ln x)^{n-1}\,dx$

48. $\displaystyle\int x^ne^x\,dx = x^ne^x - n\int x^{n-1}e^x\,dx$

49. $\displaystyle\tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx \quad (n\neq 1)$

50. $\displaystyle\int \sec^n x\,dx = \frac{\tan x\,\sec^{n-2}x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx \quad (n\neq 1)$

51. Use Exercise 47 to find $\int(\ln x)^3\,dx$.

52. Use Exercise 48 to find $\int x^4e^x\,dx$.

53–54 Find the area of the region bounded by the given curves.

53. $y = xe^{-0.4x},\quad y=0,\quad x=5$

54. $y = 5\ln x,\quad y = x\ln x$

55–56 Use a graph to find approximate x-coordinates of the points of intersection of the given curves. Then find (approximately) the area of the region bounded by the curves.

55. $y = x\sin x,\quad y=(x-2)^2$

56. $y = \arctan 3x,\quad y=\frac{1}{2}x$

57–60 Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the specified axis.

57. $y=\cos(\pi x/2),\ y=0,\ 0\leqslant x\leqslant 1;$ about the y-axis

58. $y=e^x,\ y=e^{-x},\ x=1;$ about the y-axis

59. $y=e^{-x},\ y=0,\ x=-1,\ x=0;$ about $x=1$

60. $y=e^x,\ x=0,\ y=\pi;$ about the x-axis

61. Find the average value of $f(x) = x^2 \ln x$ on the interval $[1, 3]$.

62. A rocket accelerates by burning its onboard fuel, so its mass decreases with time. Suppose the initial mass of the rocket at liftoff (including its fuel) is m, the fuel is consumed at rate r, and the exhaust gases are ejected with constant velocity v_e (relative to the rocket). A model for the velocity of the rocket at time t is given by the equation

$$v(t) = -gt - v_e \ln \frac{m - rt}{m}$$

where g is the acceleration due to gravity and t is not too large. If $g = 9.8 \text{ m/s}^2$, $m = 30{,}000$ kg, $r = 160$ kg/s, and $v_e = 3000$ m/s, find the height of the rocket one minute after liftoff.

63. A particle that moves along a straight line has velocity $v(t) = t^2 e^{-t}$ meters per second after t seconds. How far will it travel during the first t seconds?

64. If $f(0) = g(0) = 0$ and f'' and g'' are continuous, show that

$$\int_0^a f(x)g''(x)\,dx = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)\,dx$$

65. Suppose that $f(1) = 2$, $f(4) = 7$, $f'(1) = 5$, $f'(4) = 3$, and f'' is continuous. Find the value of $\int_1^4 x f''(x)\,dx$.

66. (a) Use integration by parts to show that

$$\int f(x)\,dx = xf(x) - \int xf'(x)\,dx$$

(b) If f and g are inverse functions and f' is continuous, prove that

$$\int_a^b f(x)\,dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)\,dy$$

[*Hint:* Use part (a) and make the substitution $y = f(x)$.]

(c) In the case where f and g are positive functions and $b > a > 0$, draw a diagram to give a geometric interpretation of part (b).

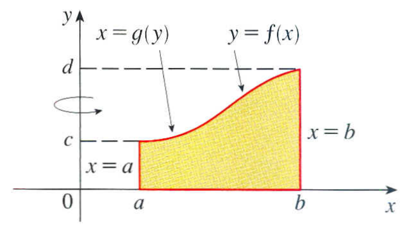
(d) Use part (b) to evaluate $\int_1^e \ln x\,dx$.

67. We arrived at Formula 6.3.2, $V = \int_a^b 2\pi x f(x)\,dx$, by using cylindrical shells, but now we can use integration by parts to prove it using the slicing method of Section 6.2, at least for the case where f is one-to-one and therefore has an inverse function g. Use the figure to show that

$$V = \pi b^2 d - \pi a^2 c - \int_c^d \pi [g(y)]^2\,dy$$

Make the substitution $y = f(x)$ and then use integration by parts on the resulting integral to prove that

$$V = \int_a^b 2\pi x f(x)\,dx$$



68. Let $I_n = \int_0^{\pi/2} \sin^n x\,dx$.

(a) Show that $I_{2n+2} \leqslant I_{2n+1} \leqslant I_{2n}$.

(b) Use Exercise 46 to show that

$$\frac{I_{2n+2}}{I_{2n}} = \frac{2n+1}{2n+2}$$

(c) Use parts (a) and (b) to show that

$$\frac{2n+1}{2n+2} \leqslant \frac{I_{2n+1}}{I_{2n}} \leqslant 1$$

and deduce that $\lim_{n\to\infty} I_{2n+1}/I_{2n} = 1$.

(d) Use part (c) and Exercises 45 and 46 to show that

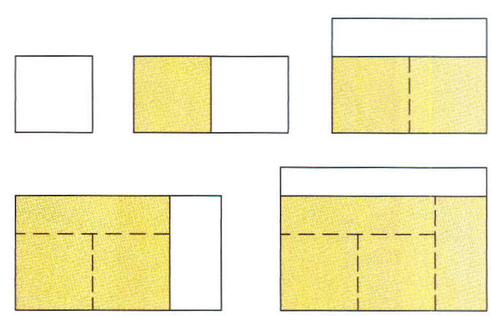
$$\lim_{n\to\infty} \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1} = \frac{\pi}{2}$$

This formula is usually written as an infinite product:

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots$$

and is called the *Wallis product*.

(e) We construct rectangles as follows. Start with a square of area 1 and attach rectangles of area 1 alternately beside or on top of the previous rectangle (see the figure). Find the limit of the ratios of width to height of these rectangles.



Then
$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x \tan^2 x\,dx$$
$$= \sec x \tan x - \int \sec x (\sec^2 x - 1)\,dx$$
$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

Using Formula 1 and solving for the required integral, we get

$$\int \sec^3 x\,dx = \tfrac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C \qquad \square$$

Integrals such as the one in the preceding example may seem very special but they occur frequently in applications of integration, as we will see in Chapter 9. Integrals of the form $\int \cot^m x \csc^n x\,dx$ can be found by similar methods because of the identity $1 + \cot^2 x = \csc^2 x$.

Finally, we can make use of another set of trigonometric identities:

■ These product identities are discussed in Appendix D.

2 To evaluate the integrals (a) $\int \sin mx \cos nx\,dx$, (b) $\int \sin mx \sin nx\,dx$, or (c) $\int \cos mx \cos nx\,dx$, use the corresponding identity:

(a) $\sin A \cos B = \tfrac{1}{2}[\sin(A - B) + \sin(A + B)]$

(b) $\sin A \sin B = \tfrac{1}{2}[\cos(A - B) - \cos(A + B)]$

(c) $\cos A \cos B = \tfrac{1}{2}[\cos(A - B) + \cos(A + B)]$

EXAMPLE 9 Evaluate $\int \sin 4x \cos 5x\,dx$.

SOLUTION This integral could be evaluated using integration by parts, but it's easier to use the identity in Equation 2(a) as follows:

$$\int \sin 4x \cos 5x\,dx = \int \tfrac{1}{2}[\sin(-x) + \sin 9x]\,dx$$
$$= \tfrac{1}{2}\int (-\sin x + \sin 9x)\,dx$$
$$= \tfrac{1}{2}(\cos x - \tfrac{1}{9}\cos 9x) + C \qquad \square$$

8.2 EXERCISES

1–49 Evaluate the integral.

1. $\int \sin^3 x \cos^2 x\,dx$

2. $\int \sin^6 x \cos^3 x\,dx$

3. $\int_{\pi/2}^{3\pi/4} \sin^5 x \cos^3 x\,dx$

4. $\int_0^{\pi/2} \cos^5 x\,dx$

5. $\int \sin^2(\pi x) \cos^5(\pi x)\,dx$

6. $\int \frac{\sin^3(\sqrt{x})}{\sqrt{x}}\,dx$

7. $\int_0^{\pi/2} \cos^2\theta\,d\theta$

8. $\int_0^{\pi/2} \sin^2(2\theta)\,d\theta$

9. $\int \cos^4 t\,dt$

10. $\int \sin^6 \pi x\,dx$

11. $\int (1 + \cos\theta)^2\,d\theta$

12. $\int x \cos^2 x\,dx$

13. $\int_0^{\pi/2} \sin^2 x \cos^2 x\,dx$

14. $\int_0^{\pi} \sin^2 t \cos^4 t\,dt$

15. $\int \frac{\cos^5\alpha}{\sqrt{\sin\alpha}}\,d\alpha$

16. $\int \cos\theta \cos^5(\sin\theta)\,d\theta$

17. $\int \cos^2 x \tan^3 x\, dx$

18. $\int \cot^5\theta \sin^4\theta\, d\theta$

19. $\int \dfrac{\cos x + \sin 2x}{\sin x}\, dx$

20. $\int \cos^2 x \sin 2x\, dx$

21. $\int \sec^2 x \tan x\, dx$

22. $\int_0^{\pi/2} \sec^4(t/2)\, dt$

23. $\int \tan^2 x\, dx$

24. $\int (\tan^2 x + \tan^4 x)\, dx$

25. $\int \sec^6 t\, dt$

26. $\int_0^{\pi/4} \sec^4\theta \tan^4\theta\, d\theta$

27. $\int_0^{\pi/3} \tan^5 x \sec^4 x\, dx$

28. $\int \tan^3(2x) \sec^5(2x)\, dx$

29. $\int \tan^3 x \sec x\, dx$

30. $\int_0^{\pi/3} \tan^5 x \sec^6 x\, dx$

31. $\int \tan^5 x\, dx$

32. $\int \tan^6(ay)\, dy$

33. $\int \dfrac{\tan^3\theta}{\cos^4\theta}\, d\theta$

34. $\int \tan^2 x \sec x\, dx$

35. $\int x \sec x \tan x\, dx$

36. $\int \dfrac{\sin\phi}{\cos^3\phi}\, d\phi$

37. $\int_{\pi/6}^{\pi/2} \cot^2 x\, dx$

38. $\int_{\pi/4}^{\pi/2} \cot^3 x\, dx$

39. $\int \cot^3\alpha \csc^3\alpha\, d\alpha$

40. $\int \csc^4 x \cot^6 x\, dx$

41. $\int \csc x\, dx$

42. $\int_{\pi/6}^{\pi/3} \csc^3 x\, dx$

43. $\int \sin 8x \cos 5x\, dx$

44. $\int \sin 3x \cos x\, dx$

45. $\int \cos 7\theta \cos 5\theta\, d\theta$

46. $\int \dfrac{\cos x + \sin x}{\sin 2x}\, dx$

47. $\int \dfrac{1 - \tan^2 x}{\sec^2 x}\, dx$

48. $\int \dfrac{dx}{\cos x - 1}$

49. $\int t \sec^2(t^2) \tan^4(t^2)\, dt$

50. If $\int_0^{\pi/4} \tan^6 x \sec x\, dx = I$, express the value of $\int_0^{\pi/4} \tan^8 x \sec x\, dx$ in terms of I.

51–54 Evaluate the indefinite integral. Illustrate, and check that your answer is reasonable, by graphing both the integrand and its antiderivative (taking $C = 0$).

51. $\int x \sin^2(x^2)\, dx$

52. $\int \sin^3 x \cos^4 x\, dx$

53. $\int \sin 3x \sin 6x\, dx$

54. $\int \sec^4 \dfrac{x}{2}\, dx$

55. Find the average value of the function $f(x) = \sin^2 x \cos^3 x$ on the interval $[-\pi, \pi]$.

56. Evaluate $\int \sin x \cos x\, dx$ by four methods:
(a) the substitution $u = \cos x$
(b) the substitution $u = \sin x$
(c) the identity $\sin 2x = 2 \sin x \cos x$
(d) integration by parts

Explain the different appearances of the answers.

57–58 Find the area of the region bounded by the given curves.

57. $y = \sin^2 x,\quad y = \cos^2 x,\quad -\pi/4 \leqslant x \leqslant \pi/4$

58. $y = \sin^3 x,\quad y = \cos^3 x,\quad \pi/4 \leqslant x \leqslant 5\pi/4$

59–60 Use a graph of the integrand to guess the value of the integral. Then use the methods of this section to prove that your guess is correct.

59. $\int_0^{2\pi} \cos^3 x\, dx$

60. $\int_0^2 \sin 2\pi x \cos 5\pi x\, dx$

61–64 Find the volume obtained by rotating the region bounded by the given curves about the specified axis.

61. $y = \sin x,\ y = 0,\ \pi/2 \leqslant x \leqslant \pi;$ about the x-axis

62. $y = \sin^2 x,\ y = 0,\ 0 \leqslant x \leqslant \pi;$ about the x-axis

63. $y = \sin x,\ y = \cos x,\ 0 \leqslant x \leqslant \pi/4;$ about $y = 1$

64. $y = \sec x,\ y = \cos x,\ 0 \leqslant x \leqslant \pi/3;$ about $y = -1$

65. A particle moves on a straight line with velocity function $v(t) = \sin \omega t \cos^2 \omega t$. Find its position function $s = f(t)$ if $f(0) = 0$.

66. Household electricity is supplied in the form of alternating current that varies from 155 V to -155 V with a frequency of 60 cycles per second (Hz). The voltage is thus given by the equation

$$E(t) = 155 \sin(120\pi t)$$

where t is the time in seconds. Voltmeters read the RMS (root-mean-square) voltage, which is the square root of the average value of $[E(t)]^2$ over one cycle.
(a) Calculate the RMS voltage of household current.
(b) Many electric stoves require an RMS voltage of 220 V. Find the corresponding amplitude A needed for the voltage $E(t) = A \sin(120\pi t)$.

67–69 Prove the formula, where m and n are positive integers.

67. $\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$

68. $\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$

69. $\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$

70. A *finite Fourier series* is given by the sum

$$\begin{aligned} f(x) &= \sum_{n=1}^{N} a_n \sin nx \\ &= a_1 \sin x + a_2 \sin 2x + \cdots + a_N \sin Nx \end{aligned}$$

Show that the mth coefficient a_m is given by the formula

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx$$

8.3 TRIGONOMETRIC SUBSTITUTION

In finding the area of a circle or an ellipse, an integral of the form $\int \sqrt{a^2 - x^2}\, dx$ arises, where $a > 0$. If it were $\int x\sqrt{a^2 - x^2}\, dx$, the substitution $u = a^2 - x^2$ would be effective but, as it stands, $\int \sqrt{a^2 - x^2}\, dx$ is more difficult. If we change the variable from x to θ by the substitution $x = a \sin \theta$, then the identity $1 - \sin^2\theta = \cos^2\theta$ allows us to get rid of the root sign because

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2 \cos^2\theta} = a|\cos \theta|$$

Notice the difference between the substitution $u = a^2 - x^2$ (in which the new variable is a function of the old one) and the substitution $x = a \sin \theta$ (the old variable is a function of the new one).

In general we can make a substitution of the form $x = g(t)$ by using the Substitution Rule in reverse. To make our calculations simpler, we assume that g has an inverse function; that is, g is one-to-one. In this case, if we replace u by x and x by t in the Substitution Rule (Equation 5.5.4), we obtain

$$\int f(x)\, dx = \int f(g(t))g'(t)\, dt$$

This kind of substitution is called *inverse substitution.*

We can make the inverse substitution $x = a \sin \theta$ provided that it defines a one-to-one function. This can be accomplished by restricting θ to lie in the interval $[-\pi/2, \pi/2]$.

In the following table we list trigonometric substitutions that are effective for the given radical expressions because of the specified trigonometric identities. In each case the restriction on θ is imposed to ensure that the function that defines the substitution is one-to-one. (These are the same intervals used in Section 7.6 in defining the inverse functions.)

TABLE OF TRIGONOMETRIC SUBSTITUTIONS

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

■ Figure 5 shows the graphs of the integrand in Example 7 and its indefinite integral (with $C = 0$). Which is which?

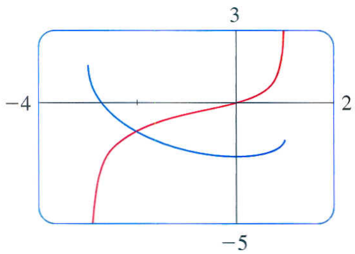


FIGURE 5

We now substitute $u = 2\sin\theta$, giving $du = 2\cos\theta\, d\theta$ and $\sqrt{4 - u^2} = 2\cos\theta$, so

$$\begin{aligned}
\int \frac{x}{\sqrt{3 - 2x - x^2}}\,dx &= \int \frac{2\sin\theta - 1}{2\cos\theta}\, 2\cos\theta\, d\theta \\
&= \int (2\sin\theta - 1)\, d\theta \\
&= -2\cos\theta - \theta + C \\
&= -\sqrt{4 - u^2} - \sin^{-1}\left(\frac{u}{2}\right) + C \\
&= -\sqrt{3 - 2x - x^2} - \sin^{-1}\left(\frac{x + 1}{2}\right) + C
\end{aligned}$$

□

8.3 EXERCISES

1–3 Evaluate the integral using the indicated trigonometric substitution. Sketch and label the associated right triangle.

1. $\displaystyle\int \frac{1}{x^2\sqrt{x^2 - 9}}\,dx; \quad x = 3\sec\theta$

2. $\displaystyle\int x^3\sqrt{9 - x^2}\,dx; \quad x = 3\sin\theta$

3. $\displaystyle\int \frac{x^3}{\sqrt{x^2 + 9}}\,dx; \quad x = 3\tan\theta$

4–30 Evaluate the integral.

4. $\displaystyle\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16 - x^2}}\,dx$

5. $\displaystyle\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2 - 1}}\,dt$

6. $\displaystyle\int_0^2 x^3\sqrt{x^2 + 4}\,dx$

7. $\displaystyle\int \frac{1}{x^2\sqrt{25 - x^2}}\,dx$

8. $\displaystyle\int \frac{\sqrt{x^2 - a^2}}{x^4}\,dx$

9. $\displaystyle\int \frac{dx}{\sqrt{x^2 + 16}}$

10. $\displaystyle\int \frac{t^5}{\sqrt{t^2 + 2}}\,dt$

11. $\displaystyle\int \sqrt{1 - 4x^2}\,dx$

12. $\displaystyle\int_0^1 x\sqrt{x^2 + 4}\,dx$

13. $\displaystyle\int \frac{\sqrt{x^2 - 9}}{x^3}\,dx$

14. $\displaystyle\int \frac{du}{u\sqrt{5 - u^2}}$

15. $\displaystyle\int_0^a x^2\sqrt{a^2 - x^2}\,dx$

16. $\displaystyle\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5\sqrt{9x^2 - 1}}$

17. $\displaystyle\int \frac{x}{\sqrt{x^2 - 7}}\,dx$

18. $\displaystyle\int \frac{dx}{[(ax)^2 - b^2]^{3/2}}$

19. $\displaystyle\int \frac{\sqrt{1 + x^2}}{x}\,dx$

20. $\displaystyle\int \frac{t}{\sqrt{25 - t^2}}\,dt$

21. $\displaystyle\int_0^{0.6} \frac{x^2}{\sqrt{9 - 25x^2}}\,dx$

22. $\displaystyle\int_0^1 \sqrt{x^2 + 1}\,dx$

23. $\displaystyle\int \sqrt{5 + 4x - x^2}\,dx$

24. $\displaystyle\int \frac{dt}{\sqrt{t^2 - 6t + 13}}$

25. $\displaystyle\int \frac{x}{\sqrt{x^2 + x + 1}}\,dx$

26. $\displaystyle\int \frac{x^2}{(3 + 4x - 4x^2)^{3/2}}\,dx$

27. $\displaystyle\int \sqrt{x^2 + 2x}\,dx$

28. $\displaystyle\int \frac{x^2 + 1}{(x^2 - 2x + 2)^2}\,dx$

29. $\displaystyle\int x\sqrt{1 - x^4}\,dx$

30. $\displaystyle\int_0^{\pi/2} \frac{\cos t}{\sqrt{1 + \sin^2 t}}\,dt$

31. (a) Use trigonometric substitution to show that

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

(b) Use the hyperbolic substitution $x = a\sinh t$ to show that

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

These formulas are connected by Formula 7.7.3.

32. Evaluate

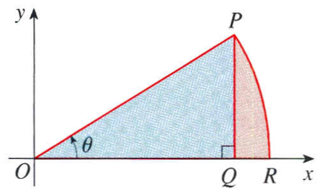
$$\int \frac{x^2}{(x^2 + a^2)^{3/2}}\,dx$$

(a) by trigonometric substitution.
(b) by the hyperbolic substitution $x = a\sinh t$.

33. Find the average value of $f(x) = \sqrt{x^2 - 1}/x$, $1 \leqslant x \leqslant 7$.

34. Find the area of the region bounded by the hyperbola $9x^2 - 4y^2 = 36$ and the line $x = 3$.

35. Prove the formula $A = \frac{1}{2}r^2\theta$ for the area of a sector of a circle with radius r and central angle θ. [*Hint:* Assume $0 < \theta < \pi/2$ and place the center of the circle at the origin so it has the equation $x^2 + y^2 = r^2$. Then A is the sum of the area of the triangle POQ and the area of the region PQR in the figure.]



36. Evaluate the integral

$$\int \frac{dx}{x^4\sqrt{x^2 - 2}}$$

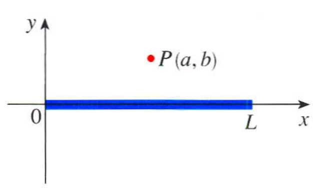
Graph the integrand and its indefinite integral on the same screen and check that your answer is reasonable.

37. Use a graph to approximate the roots of the equation $x^2\sqrt{4 - x^2} = 2 - x$. Then approximate the area bounded by the curve $y = x^2\sqrt{4 - x^2}$ and the line $y = 2 - x$.

38. A charged rod of length L produces an electric field at point $P(a, b)$ given by

$$E(P) = \int_{-a}^{L-a} \frac{\lambda b}{4\pi\varepsilon_0(x^2 + b^2)^{3/2}}\,dx$$

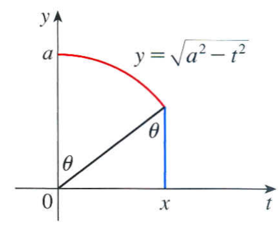
where λ is the charge density per unit length on the rod and ε_0 is the free space permittivity (see the figure). Evaluate the integral to determine an expression for the electric field $E(P)$.



39. (a) Use trigonometric substitution to verify that

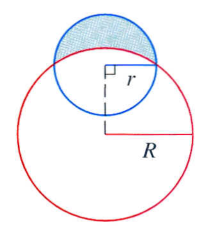
$$\int_0^x \sqrt{a^2 - t^2}\,dt = \tfrac{1}{2}a^2\sin^{-1}(x/a) + \tfrac{1}{2}x\sqrt{a^2 - x^2}$$

(b) Use the figure to give trigonometric interpretations of both terms on the right side of the equation in part (a).



40. The parabola $y = \frac{1}{2}x^2$ divides the disk $x^2 + y^2 \leqslant 8$ into two parts. Find the areas of both parts.

41. Find the area of the crescent-shaped region (called a *lune*) bounded by arcs of circles with radii r and R. (See the figure.)



42. A water storage tank has the shape of a cylinder with diameter 10 m. It is mounted so that the circular cross-sections are vertical. If the depth of the water is 7 m, what percentage of the total capacity is being used?

43. A torus is generated by rotating the circle $x^2 + (y - R)^2 = r^2$ about the x-axis. Find the volume enclosed by the torus.

8.4 INTEGRATION OF RATIONAL FUNCTIONS BY PARTIAL FRACTIONS

In this section we show how to integrate any rational function (a ratio of polynomials) by expressing it as a sum of simpler fractions, called *partial fractions,* that we already know how to integrate. To illustrate the method, observe that by taking the fractions $2/(x - 1)$ and $1/(x + 2)$ to a common denominator we obtain

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2) - (x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2}$$

If we now reverse the procedure, we see how to integrate the function on the right side of could be evaluated by the method of Case III, it's much easier to observe that if $u = x(x^2 + 3) = x^3 + 3x$, then $du = (3x^2 + 3)\,dx$ and so

$$\int \frac{x^2 + 1}{x(x^2 + 3)}\,dx = \tfrac{1}{3}\ln|x^3 + 3x| + C$$

RATIONALIZING SUBSTITUTIONS

Some nonrational functions can be changed into rational functions by means of appropriate substitutions. In particular, when an integrand contains an expression of the form $\sqrt[n]{g(x)}$, then the substitution $u = \sqrt[n]{g(x)}$ may be effective. Other instances appear in the exercises.

EXAMPLE 9 Evaluate $\displaystyle\int \frac{\sqrt{x + 4}}{x}\,dx$.

SOLUTION Let $u = \sqrt{x + 4}$. Then $u^2 = x + 4$, so $x = u^2 - 4$ and $dx = 2u\,du$. Therefore

$$\int \frac{\sqrt{x+4}}{x}\,dx = \int \frac{u}{u^2 - 4}\,2u\,du = 2\int \frac{u^2}{u^2 - 4}\,du$$
$$= 2\int \left(1 + \frac{4}{u^2 - 4}\right) du$$

We can evaluate this integral either by factoring $u^2 - 4$ as $(u - 2)(u + 2)$ and using partial fractions or by using Formula 6 with $a = 2$:

$$\int \frac{\sqrt{x+4}}{x}\,dx = 2\int du + 8\int \frac{du}{u^2 - 4}$$
$$= 2u + 8 \cdot \frac{1}{2 \cdot 2} \ln\left|\frac{u - 2}{u + 2}\right| + C$$
$$= 2\sqrt{x+4} + 2\ln\left|\frac{\sqrt{x+4} - 2}{\sqrt{x+4} + 2}\right| + C \qquad \square$$

8.4 EXERCISES

1–6 Write out the form of the partial fraction decomposition of the function (as in Example 7). Do not determine the numerical values of the coefficients.

1. (a) $\dfrac{2x}{(x+3)(3x+1)}$ (b) $\dfrac{1}{x^3 + 2x^2 + x}$

2. (a) $\dfrac{x - 1}{x^3 + x^2}$ (b) $\dfrac{x - 1}{x^3 + x}$

3. (a) $\dfrac{2}{x^2 + 3x - 4}$ (b) $\dfrac{x^2}{(x - 1)(x^2 + x + 1)}$

4. (a) $\dfrac{x^3}{x^2 + 4x + 3}$ (b) $\dfrac{2x + 1}{(x + 1)^3(x^2 + 4)^2}$

5. (a) $\dfrac{x^4}{x^4 - 1}$ (b) $\dfrac{t^4 + t^2 + 1}{(t^2 + 1)(t^2 + 4)^2}$

6. (a) $\dfrac{x^4}{(x^3 + x)(x^2 - x + 3)}$ (b) $\dfrac{1}{x^6 - x^3}$

7–38 Evaluate the integral.

7. $\displaystyle\int \frac{x^2}{x + 1}\,dx$

8. $\displaystyle\int \frac{y}{y + 2}\,dy$

9. $\displaystyle\int \frac{x - 9}{(x + 5)(x - 2)}\,dx$

10. $\displaystyle\int \frac{1}{(t + 4)(t - 1)}\,dt$

11. $\displaystyle\int_2^3 \frac{1}{x^2-1}\,dx$

12. $\displaystyle\int_0^1 \frac{x-1}{x^2+3x+2}\,dx$

13. $\displaystyle\int \frac{ax}{x^2-bx}\,dx$

14. $\displaystyle\int \frac{1}{(x+a)(x+b)}\,dx$

15. $\displaystyle\int_3^4 \frac{x^3-2x^2-4}{x^3-2x^2}\,dx$

16. $\displaystyle\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx$

17. $\displaystyle\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy$

18. $\displaystyle\int \frac{x^2+2x-1}{x^3-x}\,dx$

19. $\displaystyle\int \frac{1}{(x+5)^2(x-1)}\,dx$

20. $\displaystyle\int \frac{x^2-5x+16}{(2x+1)(x-2)^2}\,dx$

21. $\displaystyle\int \frac{x^3+4}{x^2+4}\,dx$

22. $\displaystyle\int \frac{ds}{s^2(s-1)^2}$

23. $\displaystyle\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx$

24. $\displaystyle\int \frac{x^2-x+6}{x^3+3x}\,dx$

25. $\displaystyle\int \frac{10}{(x-1)(x^2+9)}\,dx$

26. $\displaystyle\int \frac{x^2+x+1}{(x^2+1)^2}\,dx$

27. $\displaystyle\int \frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)}\,dx$

28. $\displaystyle\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}\,dx$

29. $\displaystyle\int \frac{x+4}{x^2+2x+5}\,dx$

30. $\displaystyle\int \frac{3x^2+x+4}{x^4+3x^2+2}\,dx$

31. $\displaystyle\int \frac{1}{x^3-1}\,dx$

32. $\displaystyle\int_0^1 \frac{x}{x^2+4x+13}\,dx$

33. $\displaystyle\int_2^5 \frac{x^2+2x}{x^3+3x^2+4}\,dx$

34. $\displaystyle\int \frac{x^3}{x^3+1}\,dx$

35. $\displaystyle\int \frac{dx}{x^4-x^2}$

36. $\displaystyle\int \frac{x^4+3x^2+1}{x^5+5x^3+5x}\,dx$

37. $\displaystyle\int \frac{x^2-3x+7}{(x^2-4x+6)^2}\,dx$

38. $\displaystyle\int \frac{x^3+2x^2+3x-2}{(x^2+2x+2)^2}\,dx$

39–50 Make a substitution to express the integrand as a rational function and then evaluate the integral.

39. $\displaystyle\int \frac{1}{x\sqrt{x+1}}\,dx$

40. $\displaystyle\int \frac{dx}{2\sqrt{x+3}+x}$

41. $\displaystyle\int_9^{16} \frac{\sqrt{x}}{x-4}\,dx$

42. $\displaystyle\int_0^1 \frac{1}{1+\sqrt[3]{x}}\,dx$

43. $\displaystyle\int \frac{x^3}{\sqrt[3]{x^2+1}}\,dx$

44. $\displaystyle\int_{1/3}^3 \frac{\sqrt{x}}{x^2+x}\,dx$

45. $\displaystyle\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx$ [*Hint:* Substitute $u = \sqrt[6]{x}$.]

46. $\displaystyle\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx$

47. $\displaystyle\int \frac{e^{2x}}{e^{2x}+3e^x+2}\,dx$

48. $\displaystyle\int \frac{\cos x}{\sin^2 x+\sin x}\,dx$

49. $\displaystyle\int \frac{\sec^2 t}{\tan^2 t+3\tan t+2}\,dt$

50. $\displaystyle\int \frac{e^x}{(e^x-2)(e^{2x}+1)}\,dx$

51–52 Use integration by parts, together with the techniques of this section, to evaluate the integral.

51. $\displaystyle\int \ln(x^2-x+2)\,dx$

52. $\displaystyle\int x\tan^{-1}x\,dx$

53. Use a graph of $f(x) = 1/(x^2-2x-3)$ to decide whether $\int_0^2 f(x)\,dx$ is positive or negative. Use the graph to give a rough estimate of the value of the integral and then use partial fractions to find the exact value.

54. Graph both $y = 1/(x^3-2x^2)$ and an antiderivative on the same screen.

55–56 Evaluate the integral by completing the square and using Formula 6.

55. $\displaystyle\int \frac{dx}{x^2-2x}$

56. $\displaystyle\int \frac{2x+1}{4x^2+12x-7}\,dx$

57. The German mathematician Karl Weierstrass (1815–1897) noticed that the substitution $t = \tan(x/2)$ will convert any rational function of $\sin x$ and $\cos x$ into an ordinary rational function of t.

(a) If $t = \tan(x/2)$, $-\pi < x < \pi$, sketch a right triangle or use trigonometric identities to show that

$$\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1+t^2}} \quad \text{and} \quad \sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1+t^2}}$$

(b) Show that

$$\cos x = \frac{1-t^2}{1+t^2} \quad \text{and} \quad \sin x = \frac{2t}{1+t^2}$$

(c) Show that

$$dx = \frac{2}{1+t^2}\,dt$$

58–61 Use the substitution in Exercise 57 to transform the integrand into a rational function of t and then evaluate the integral.

58. $\displaystyle\int \frac{dx}{3-5\sin x}$

59. $\displaystyle\int \frac{1}{3\sin x-4\cos x}\,dx$

60. $\displaystyle\int_{\pi/3}^{\pi/2} \frac{1}{1+\sin x-\cos x}\,dx$

61. $\displaystyle\int_0^{\pi/2} \frac{\sin 2x}{2 + \cos x}\, dx$

62–63 Find the area of the region under the given curve from 1 to 2.

62. $y = \dfrac{1}{x^3 + x}$

63. $y = \dfrac{x^2 + 1}{3x - x^2}$

64. Find the volume of the resulting solid if the region under the curve $y = 1/(x^2 + 3x + 2)$ from $x = 0$ to $x = 1$ is rotated about (a) the x-axis and (b) the y-axis.

65. One method of slowing the growth of an insect population without using pesticides is to introduce into the population a number of sterile males that mate with fertile females but produce no offspring. If P represents the number of female insects in a population, S the number of sterile males introduced each generation, and r the population's natural growth rate, then the female population is related to time t by

$$t = \int \frac{P + S}{P[(r - 1)P - S]}\, dP$$

Suppose an insect population with 10,000 females grows at a rate of $r = 0.10$ and 900 sterile males are added. Evaluate the integral to give an equation relating the female population to time. (Note that the resulting equation can't be solved explicitly for P.)

66. Factor $x^4 + 1$ as a difference of squares by first adding and subtracting the same quantity. Use this factorization to evaluate $\int 1/(x^4 + 1)\, dx$.

CAS **67.** (a) Use a computer algebra system to find the partial fraction decomposition of the function

$$f(x) = \frac{4x^3 - 27x^2 + 5x - 32}{30x^5 - 13x^4 + 50x^3 - 286x^2 - 299x - 70}$$

(b) Use part (a) to find $\int f(x)\, dx$ (by hand) and compare with the result of using the CAS to integrate f directly. Comment on any discrepancy.

CAS **68.** (a) Find the partial fraction decomposition of the function

$$f(x) = \frac{12x^5 - 7x^3 - 13x^2 + 8}{100x^6 - 80x^5 + 116x^4 - 80x^3 + 41x^2 - 20x + 4}$$

(b) Use part (a) to find $\int f(x)\, dx$ and graph f and its indefinite integral on the same screen.

(c) Use the graph of f to discover the main features of the graph of $\int f(x)\, dx$.

69. Suppose that F, G, and Q are polynomials and

$$\frac{F(x)}{Q(x)} = \frac{G(x)}{Q(x)}$$

for all x except when $Q(x) = 0$. Prove that $F(x) = G(x)$ for all x. [*Hint:* Use continuity.]

70. If f is a quadratic function such that $f(0) = 1$ and

$$\int \frac{f(x)}{x^2(x + 1)^3}\, dx$$

is a rational function, find the value of $f'(0)$.

8.5 STRATEGY FOR INTEGRATION

As we have seen, integration is more challenging than differentiation. In finding the derivative of a function it is obvious which differentiation formula we should apply. But it may not be obvious which technique we should use to integrate a given function.

Until now individual techniques have been applied in each section. For instance, we usually used substitution in Exercises 5.5, integration by parts in Exercises 8.1, and partial fractions in Exercises 8.4. But in this section we present a collection of miscellaneous integrals in random order and the main challenge is to recognize which technique or formula to use. No hard and fast rules can be given as to which method applies in a given situation, but we give some advice on strategy that you may find useful.

A prerequisite for strategy selection is a knowledge of the basic integration formulas. In the following table we have collected the integrals from our previous list together with several additional formulas that we have learned in this chapter. Most of them should be memorized. It is useful to know them all, but the ones marked with an asterisk need not be then we know from Part 1 of the Fundamental Theorem of Calculus that

$$F'(x) = e^{x^2}$$

Thus, $f(x) = e^{x^2}$ has an antiderivative F, but it has been proved that F is not an elementary function. This means that no matter how hard we try, we will never succeed in evaluating $\int e^{x^2}\,dx$ in terms of the functions we know. (In Chapter 12, however, we will see how to express $\int e^{x^2}\,dx$ as an infinite series.) The same can be said of the following integrals:

$$\int \frac{e^x}{x}\,dx \qquad \int \sin(x^2)\,dx \qquad \int \cos(e^x)\,dx$$

$$\int \sqrt{x^3+1}\,dx \qquad \int \frac{1}{\ln x}\,dx \qquad \int \frac{\sin x}{x}\,dx$$

In fact, the majority of elementary functions don't have elementary antiderivatives. You may be assured, though, that the integrals in the following exercises are all elementary functions.

8.5 EXERCISES

1–80 Evaluate the integral.

1. $\int \cos x\,(1+\sin^2 x)\,dx$

2. $\int \frac{\sin^3 x}{\cos x}\,dx$

3. $\int \frac{\cos x}{1+\sin^2 x}\,dx$

4. $\int \tan^3\theta\,d\theta$

5. $\int_0^2 \frac{2t}{(t-3)^2}\,dt$

6. $\int \frac{x}{\sqrt{3-x^4}}\,dx$

7. $\int_{-1}^1 \frac{e^{\arctan y}}{1+y^2}\,dy$

8. $\int x\csc x\cot x\,dx$

9. $\int_1^3 r^4\ln r\,dr$

10. $\int_0^4 \frac{x-1}{x^2-4x-5}\,dx$

11. $\int \frac{x-1}{x^2-4x+5}\,dx$

12. $\int \frac{x}{x^4+x^2+1}\,dx$

13. $\int \sin^3\theta\cos^5\theta\,d\theta$

14. $\int \frac{x^3}{\sqrt{1+x^2}}\,dx$

15. $\int \frac{dx}{(1-x^2)^{3/2}}$

16. $\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1-x^2}}\,dx$

17. $\int x\sin^2 x\,dx$

18. $\int \frac{e^{2t}}{1+e^{4t}}\,dt$

19. $\int e^{x+e^x}\,dx$

20. $\int \frac{1+\cos x}{\sin x}\,dx$

21. $\int \arctan\sqrt{x}\,dx$

22. $\int \frac{\ln x}{x\sqrt{1+(\ln x)^2}}\,dx$

23. $\int_0^1 (1+\sqrt{x})^8\,dx$

24. $\int \ln(x^2-1)\,dx$

25. $\int \frac{3x^2-2}{x^2-2x-8}\,dx$

26. $\int \frac{3x^2-2}{x^3-2x-8}\,dx$

27. $\int \frac{dx}{1+e^x}$

28. $\int \sin\sqrt{at}\,dt$

29. $\int_0^5 \frac{3w-1}{w+2}\,dw$

30. $\int_{-2}^2 |x^2-4x|\,dx$

31. $\int \sqrt{\frac{1+x}{1-x}}\,dx$

32. $\int \frac{\sqrt{2x-1}}{2x+3}\,dx$

33. $\int \sqrt{3-2x-x^2}\,dx$

34. $\int_{\pi/4}^{\pi/2} \frac{1+4\cot x}{4-\cot x}\,dx$

35. $\int_{-1}^1 x^8\sin x\,dx$

36. $\int \sin 4x\cos 3x\,dx$

37. $\int_0^{\pi/4} \cos^2\theta\tan^2\theta\,d\theta$

38. $\int_0^{\pi/4} \tan^5\theta\sec^3\theta\,d\theta$

39. $\int \frac{\sec\theta\tan\theta}{\sec^2\theta-\sec\theta}\,d\theta$

40. $\int \frac{1}{\sqrt{4y^2-4y-3}}\,dy$

41. $\int \theta\tan^2\theta\,d\theta$

42. $\int \frac{\tan^{-1}x}{x^2}\,dx$

43. $\int e^x\sqrt{1+e^x}\,dx$

44. $\int \sqrt{1+e^x}\,dx$

45. $\int x^5 e^{-x^3}\,dx$

46. $\int \frac{1+\sin x}{1-\sin x}\,dx$

47. $\int x^3(x-1)^{-4}\,dx$

48. $\int \frac{x}{x^4-a^4}\,dx$

49. $\int \frac{1}{x\sqrt{4x+1}}\,dx$

50. $\int \frac{1}{x^2\sqrt{4x+1}}\,dx$

51. $\int \frac{1}{x\sqrt{4x^2+1}}\,dx$

52. $\int \frac{dx}{x(x^4+1)}$

53. $\int x^2 \sinh mx\,dx$

54. $\int (x+\sin x)^2\,dx$

55. $\int \frac{dx}{x+x\sqrt{x}}$

56. $\int \frac{dx}{\sqrt{x}+x\sqrt{x}}$

57. $\int x\sqrt[3]{x+c}\,dx$

58. $\int \frac{x\ln x}{\sqrt{x^2-1}}\,dx$

59. $\int \cos x \cos^3(\sin x)\,dx$

60. $\int \frac{dx}{x^2\sqrt{4x^2-1}}$

61. $\int \sqrt{x}\,e^{\sqrt{x}}\,dx$

62. $\int \frac{1}{x+\sqrt[3]{x}}\,dx$

63. $\int \frac{\sin 2x}{1+\cos^4 x}\,dx$

64. $\int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)}{\sin x \cos x}\,dx$

65. $\int \frac{1}{\sqrt{x+1}+\sqrt{x}}\,dx$

66. $\int_2^3 \frac{u^3+1}{u^3-u^2}\,du$

67. $\int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x^2}\,dx$

68. $\int \frac{1}{1+2e^x-e^{-x}}\,dx$

69. $\int \frac{e^{2x}}{1+e^x}\,dx$

70. $\int \frac{\ln(x+1)}{x^2}\,dx$

71. $\int \frac{x+\arcsin x}{\sqrt{1-x^2}}\,dx$

72. $\int \frac{4^x+10^x}{2^x}\,dx$

73. $\int \frac{1}{(x-2)(x^2+4)}\,dx$

74. $\int \frac{dx}{\sqrt{x}\,(2+\sqrt{x}\,)^4}$

75. $\int \frac{xe^x}{\sqrt{1+e^x}}\,dx$

76. $\int (x^2-bx)\sin 2x\,dx$

77. $\int \frac{\sqrt{x}}{1+x^3}\,dx$

78. $\int \frac{\sec x \cos 2x}{\sin x+\sec x}\,dx$

79. $\int x\sin^2 x\cos x\,dx$

80. $\int \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx$

81. The functions $y = e^{x^2}$ and $y = x^2e^{x^2}$ don't have elementary antiderivatives, but $y = (2x^2+1)e^{x^2}$ does. Evaluate $\int (2x^2+1)e^{x^2}\,dx$.

8.6 INTEGRATION USING TABLES AND COMPUTER ALGEBRA SYSTEMS

In this section we describe how to use tables and computer algebra systems to integrate functions that have elementary antiderivatives. You should bear in mind, though, that even the most powerful computer algebra systems can't find explicit formulas for the antiderivatives of functions like e^{x^2} or the other functions described at the end of Section 8.5.

TABLES OF INTEGRALS

Tables of indefinite integrals are very useful when we are confronted by an integral that is difficult to evaluate by hand and we don't have access to a computer algebra system. A relatively brief table of 120 integrals, categorized by form, is provided on the Reference Pages at the back of the book. More extensive tables are available in *CRC Standard Mathematical Tables and Formulae,* 31st ed. by Daniel Zwillinger (Boca Raton, FL: CRC Press, 2002) (709 entries) or in Gradshteyn and Ryzhik's *Table of Integrals, Series, and Products,* 6e (San Diego: Academic Press, 2000), which contains hundreds of pages of integrals. It should be remembered, however, that integrals do not often occur in exactly the form listed in a table. Usually we need to use substitution or algebraic manipulation to transform a given integral into one of the forms in the table.

EXAMPLE 1 The region bounded by the curves $y = \arctan x$, $y = 0$, and $x = 1$ is rotated about the y-axis. Find the volume of the resulting solid.

SOLUTION Using the method of cylindrical shells, we see that the volume is

$$V = \int_0^1 2\pi x \arctan x\,dx$$

It's clear that both systems must have expanded $(x^2+5)^8$ by the Binomial Theorem and then integrated each term.

If we integrate by hand instead, using the substitution $u = x^2 + 5$, we get

■ Derive and the TI-89/92 also give this answer.

$$\int x(x^2+5)^8\,dx = \tfrac{1}{18}(x^2+5)^9 + C$$

For most purposes, this is a more convenient form of the answer. □

EXAMPLE 7 Use a CAS to find $\int \sin^5 x \cos^2 x\,dx$.

SOLUTION In Example 2 in Section 8.2 we found that

$$\int \sin^5 x \cos^2 x\,dx = -\tfrac{1}{3}\cos^3 x + \tfrac{2}{5}\cos^5 x - \tfrac{1}{7}\cos^7 x + C \tag{1}$$

Derive and Maple report the answer

$$-\tfrac{1}{7}\sin^4 x\cos^3 x - \tfrac{4}{35}\sin^2 x\cos^3 x - \tfrac{8}{105}\cos^3 x$$

whereas Mathematica produces

$$-\tfrac{5}{64}\cos x - \tfrac{1}{192}\cos 3x + \tfrac{3}{320}\cos 5x - \tfrac{1}{448}\cos 7x$$

We suspect that there are trigonometric identities which show these three answers are equivalent. Indeed, if we ask Derive, Maple, and Mathematica to simplify their expressions using trigonometric identities, they ultimately produce the same form of the answer as in Equation 1. □

8.6 EXERCISES

1–4 Use the indicated entry in the Table of Integrals on the Reference Pages to evaluate the integral.

1. $\displaystyle\int \frac{\sqrt{7-2x^2}}{x^2}\,dx$; entry 33

2. $\displaystyle\int \frac{3x}{\sqrt{3-2x}}\,dx$; entry 55

3. $\displaystyle\int \sec^3(\pi x)\,dx$; entry 71

4. $\displaystyle\int e^{2\theta}\sin 3\theta\,d\theta$; entry 98

5–30 Use the Table of Integrals on Reference Pages 6–10 to evaluate the integral.

5. $\displaystyle\int_0^1 2x\cos^{-1}x\,dx$

6. $\displaystyle\int_2^3 \frac{1}{x^2\sqrt{4x^2-7}}\,dx$

7. $\displaystyle\int \tan^3(\pi x)\,dx$

8. $\displaystyle\int \frac{\ln(1+\sqrt{x})}{\sqrt{x}}\,dx$

9. $\displaystyle\int x\sin^{-1}(x^2)\,dx$

10. $\displaystyle\int \frac{\sqrt{2y^2-3}}{y^2}\,dy$

11. $\displaystyle\int_{-1}^0 t^2e^{-t}\,dt$

12. $\displaystyle\int x^2\operatorname{csch}(x^3+1)\,dx$

13. $\displaystyle\int \frac{\tan^3(1/z)}{z^2}\,dz$

14. $\displaystyle\int \sin^{-1}\sqrt{x}\,dx$

15. $\displaystyle\int e^{2x}\arctan(e^x)\,dx$

16. $\displaystyle\int x\sin(x^2)\cos(3x^2)\,dx$

17. $\displaystyle\int y\sqrt{6+4y-4y^2}\,dy$

18. $\displaystyle\int \frac{dx}{2x^3-3x^2}$

19. $\displaystyle\int \sin^2 x\cos x\ln(\sin x)\,dx$

20. $\displaystyle\int \frac{\sin 2\theta}{\sqrt{5-\sin\theta}}\,d\theta$

21. $\displaystyle\int \frac{e^x}{3-e^{2x}}\,dx$

22. $\displaystyle\int_0^2 x^3\sqrt{4x^2-x^4}\,dx$

23. $\displaystyle\int \sec^5 x\,dx$

24. $\displaystyle\int \sin^6 2x\,dx$

25. $\displaystyle\int \frac{\sqrt{4+(\ln x)^2}}{x}\,dx$

26. $\displaystyle\int_0^1 x^4e^{-x}\,dx$

27. $\int \sqrt{e^{2x} - 1}\, dx$

28. $\int e^t \sin(\alpha t - 3)\, dt$

29. $\int \frac{x^4\, dx}{\sqrt{x^{10} - 2}}$

30. $\int \frac{\sec^2\theta \tan^2\theta}{\sqrt{9 - \tan^2\theta}}\, d\theta$

31. Find the volume of the solid obtained when the region under the curve $y = x\sqrt{4 - x^2}$, $0 \leqslant x \leqslant 2$, is rotated about the y-axis.

32. The region under the curve $y = \tan^2 x$ from 0 to $\pi/4$ is rotated about the x-axis. Find the volume of the resulting solid.

33. Verify Formula 53 in the Table of Integrals (a) by differentiation and (b) by using the substitution $t = a + bu$.

34. Verify Formula 31 (a) by differentiation and (b) by substituting $u = a \sin \theta$.

CAS **35–42** Use a computer algebra system to evaluate the integral. Compare the answer with the result of using tables. If the answers are not the same, show that they are equivalent.

35. $\int \sec^4 x\, dx$

36. $\int x^2(1 + x^3)^4\, dx$

37. $\int \sin^3 x \cos^2 x\, dx$

38. $\int \frac{dx}{e^x(3e^x + 2)}$

39. $\int x\sqrt{1 + 2x}\, dx$

40. $\int \sin^4 x\, dx$

41. $\int \tan^5 x\, dx$

42. $\int \frac{1}{\sqrt{1 + \sqrt[3]{x}}}\, dx$

CAS 43. (a) Use the table of integrals to evaluate $F(x) = \int f(x)\, dx$, where

$$f(x) = \frac{1}{x\sqrt{1 - x^2}}$$

What is the domain of f and F?

(b) Use a CAS to evaluate $F(x)$. What is the domain of the function F that the CAS produces? Is there a discrepancy between this domain and the domain of the function F that you found in part (a)?

CAS 44. Computer algebra systems sometimes need a helping hand from human beings. Try to evaluate

$$\int (1 + \ln x)\sqrt{1 + (x \ln x)^2}\, dx$$

with a computer algebra system. If it doesn't return an answer, make a substitution that changes the integral into one that the CAS *can* evaluate.

CAS **45–48** Use a CAS to find an antiderivative F of f such that $F(0) = 0$. Graph f and F and locate approximately the x-coordinates of the extreme points and inflection points of F.

45. $f(x) = \frac{x^2 - 1}{x^4 + x^2 + 1}$

46. $f(x) = xe^{-x}\sin x, \quad -5 \leqslant x \leqslant 5$

47. $f(x) = \sin^4 x \cos^6 x, \quad 0 \leqslant x \leqslant \pi$

48. $f(x) = \frac{x^3 - x}{x^6 + 1}$

DISCOVERY PROJECT

CAS PATTERNS IN INTEGRALS

In this project a computer algebra system is used to investigate indefinite integrals of families of functions. By observing the patterns that occur in the integrals of several members of the family, you will first guess, and then prove, a general formula for the integral of any member of the family.

1. (a) Use a computer algebra system to evaluate the following integrals.

(i) $\int \frac{1}{(x + 2)(x + 3)}\, dx$ (ii) $\int \frac{1}{(x + 1)(x + 5)}\, dx$

(iii) $\int \frac{1}{(x + 2)(x - 5)}\, dx$ (iv) $\int \frac{1}{(x + 2)^2}\, dx$

(b) Based on the pattern of your responses in part (a), guess the value of the integral

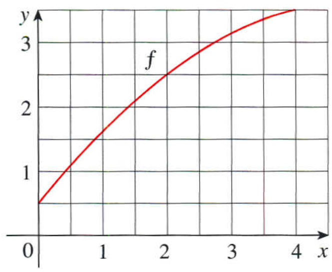
$$\int \frac{1}{(x + a)(x + b)}\, dx$$

if $a \neq b$. What if $a = b$?

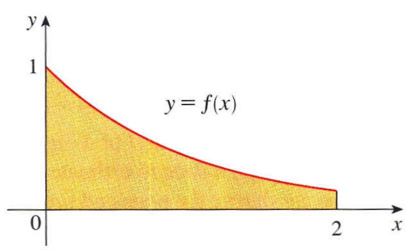
(c) Check your guess by asking your CAS to evaluate the integral in part (b). Then prove it using partial fractions.

8.7 EXERCISES

1. Let $I = \int_0^4 f(x)\,dx$, where f is the function whose graph is shown.
(a) Use the graph to find L_2, R_2, and M_2.
(b) Are these underestimates or overestimates of I?
(c) Use the graph to find T_2. How does it compare with I?
(d) For any value of n, list the numbers L_n, R_n, M_n, T_n, and I in increasing order.



2. The left, right, Trapezoidal, and Midpoint Rule approximations were used to estimate $\int_0^2 f(x)\,dx$, where f is the function whose graph is shown. The estimates were 0.7811, 0.8675, 0.8632, and 0.9540, and the same number of subintervals were used in each case.
(a) Which rule produced which estimate?
(b) Between which two approximations does the true value of $\int_0^2 f(x)\,dx$ lie?



3. Estimate $\int_0^1 \cos(x^2)\,dx$ using (a) the Trapezoidal Rule and (b) the Midpoint Rule, each with $n = 4$. From a graph of the integrand, decide whether your answers are underestimates or overestimates. What can you conclude about the true value of the integral?

4. Draw the graph of $f(x) = \sin(\frac{1}{2}x^2)$ in the viewing rectangle $[0, 1]$ by $[0, 0.5]$ and let $I = \int_0^1 f(x)\,dx$.
(a) Use the graph to decide whether L_2, R_2, M_2, and T_2 underestimate or overestimate I.
(b) For any value of n, list the numbers L_n, R_n, M_n, T_n, and I in increasing order.
(c) Compute L_5, R_5, M_5, and T_5. From the graph, which do you think gives the best estimate of I?

5–6 Use (a) the Midpoint Rule and (b) Simpson's Rule to approximate the given integral with the specified value of n. (Round your answers to six decimal places.) Compare your results to the actual value to determine the error in each approximation.

5. $\displaystyle\int_0^{\pi} x^2 \sin x\,dx, \quad n = 8$

6. $\displaystyle\int_0^1 e^{-\sqrt{x}}\,dx, \quad n = 6$

7–18 Use (a) the Trapezoidal Rule, (b) the Midpoint Rule, and (c) Simpson's Rule to approximate the given integral with the specified value of n. (Round your answers to six decimal places.)

7. $\displaystyle\int_0^1 e^{-x^2}\,dx, \quad n = 10$

8. $\displaystyle\int_0^2 \frac{1}{\sqrt{1+x^3}}\,dx, \quad n = 10$

9. $\displaystyle\int_1^2 \frac{\ln x}{1+x}\,dx, \quad n = 10$

10. $\displaystyle\int_0^3 \frac{dt}{1+t^2+t^4}, \quad n = 6$

11. $\displaystyle\int_0^{1/2} \sin(e^{t/2})\,dt, \quad n = 8$

12. $\displaystyle\int_0^4 \sqrt{1+\sqrt{x}}\,dx, \quad n = 8$

13. $\displaystyle\int_0^4 e^{\sqrt{t}} \sin t\,dt, \quad n = 8$

14. $\displaystyle\int_0^1 \sqrt{z}\,e^{-z}\,dz, \quad n = 10$

15. $\displaystyle\int_1^5 \frac{\cos x}{x}\,dx, \quad n = 8$

16. $\displaystyle\int_4^6 \ln(x^3+2)\,dx, \quad n = 10$

17. $\displaystyle\int_0^3 \frac{1}{1+y^5}\,dy, \quad n = 6$

18. $\displaystyle\int_0^4 \cos\sqrt{x}\,dx, \quad n = 10$

19. (a) Find the approximations T_8 and M_8 for the integral $\int_0^1 \cos(x^2)\,dx$.
(b) Estimate the errors in the approximations of part (a).
(c) How large do we have to choose n so that the approximations T_n and M_n to the integral in part (a) are accurate to within 0.0001?

20. (a) Find the approximations T_{10} and M_{10} for $\int_1^2 e^{1/x}\,dx$.
(b) Estimate the errors in the approximations of part (a).
(c) How large do we have to choose n so that the approximations T_n and M_n to the integral in part (a) are accurate to within 0.0001?

21. (a) Find the approximations T_{10}, M_{10}, and S_{10} for $\int_0^{\pi} \sin x\,dx$ and the corresponding errors E_T, E_M, and E_S.
(b) Compare the actual errors in part (a) with the error estimates given by (3) and (4).
(c) How large do we have to choose n so that the approximations T_n, M_n, and S_n to the integral in part (a) are accurate to within 0.00001?

22. How large should n be to guarantee that the Simpson's Rule approximation to $\int_0^1 e^{x^2}\,dx$ is accurate to within 0.00001?

CAS **23.** The trouble with the error estimates is that it is often very difficult to compute four derivatives and obtain a good upper bound K for $|f^{(4)}(x)|$ by hand. But computer algebra systems have no problem computing $f^{(4)}$ and graphing it, so we can easily find a value for K from a machine graph. This exercise deals with approximations to the integral $I = \int_0^{2\pi} f(x)\,dx$, where $f(x) = e^{\cos x}$.

(a) Use a graph to get a good upper bound for $|f''(x)|$.
(b) Use M_{10} to approximate I.
(c) Use part (a) to estimate the error in part (b).
(d) Use the built-in numerical integration capability of your CAS to approximate I.
(e) How does the actual error compare with the error estimate in part (c)?
(f) Use a graph to get a good upper bound for $|f^{(4)}(x)|$.
(g) Use S_{10} to approximate I.
(h) Use part (f) to estimate the error in part (g).
(i) How does the actual error compare with the error estimate in part (h)?
(j) How large should n be to guarantee that the size of the error in using S_n is less than 0.0001?

CAS **24.** Repeat Exercise 23 for the integral $\int_{-1}^{1} \sqrt{4 - x^3}\,dx$.

25–26 Find the approximations L_n, R_n, T_n, and M_n for $n = 5$, 10, and 20. Then compute the corresponding errors E_L, E_R, E_T, and E_M. (Round your answers to six decimal places. You may wish to use the sum command on a computer algebra system.) What observations can you make? In particular, what happens to the errors when n is doubled?

25. $\int_0^1 xe^x\,dx$

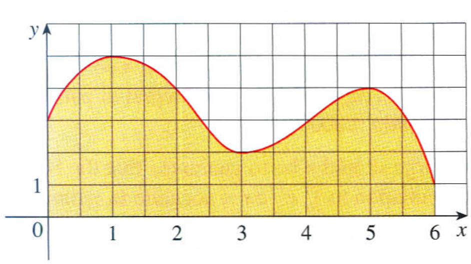
26. $\int_1^2 \frac{1}{x^2}\,dx$

27–28 Find the approximations T_n, M_n, and S_n for $n = 6$ and 12. Then compute the corresponding errors E_T, E_M, and E_S. (Round your answers to six decimal places. You may wish to use the sum command on a computer algebra system.) What observations can you make? In particular, what happens to the errors when n is doubled?

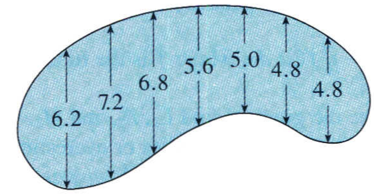
27. $\int_0^2 x^4\,dx$

28. $\int_1^4 \frac{1}{\sqrt{x}}\,dx$

29. Estimate the area under the graph in the figure by using (a) the Trapezoidal Rule, (b) the Midpoint Rule, and (c) Simpson's Rule, each with $n = 6$.



30. The widths (in meters) of a kidney-shaped swimming pool were measured at 2-meter intervals as indicated in the figure. Use Simpson's Rule to estimate the area of the pool.



31. (a) Use the Midpoint Rule and the given data to estimate the value of the integral $\int_0^{3.2} f(x)\,dx$.

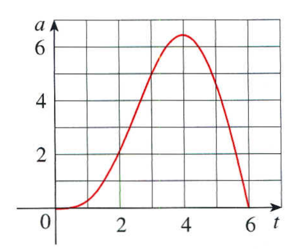
x	$f(x)$	x	$f(x)$
0.0	6.8	2.0	7.6
0.4	6.5	2.4	8.4
0.8	6.3	2.8	8.8
1.2	6.4	3.2	9.0
1.6	6.9		

(b) If it is known that $-4 \leqslant f''(x) \leqslant 1$ for all x, estimate the error involved in the approximation in part (a).

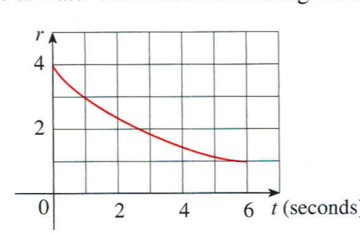
32. A radar gun was used to record the speed of a runner during the first 5 seconds of a race (see the table). Use Simpson's Rule to estimate the distance the runner covered during those 5 seconds.

t (s)	v (m/s)	t (s)	v (m/s)
0	0	3.0	10.51
0.5	4.67	3.5	10.67
1.0	7.34	4.0	10.76
1.5	8.86	4.5	10.81
2.0	9.73	5.0	10.81
2.5	10.22		

33. The graph of the acceleration $a(t)$ of a car measured in m/s^2 is shown. Use Simpson's Rule to estimate the increase in the velocity of the car during the 6-second time interval.



34. Water leaked from a tank at a rate of $r(t)$ liters per hour, where the graph of r is as shown. Use Simpson's Rule to estimate the total amount of water that leaked out during the first 6 hours.

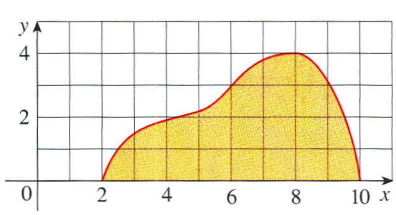


35. The table gives the power consumption in megawatts in Ontario from midnight to 6:00 A.M. on December 10, 2004. Use Simpson's Rule to estimate the energy used during that time period. (Use the fact that power is the derivative of energy.)

t	P	t	P
0:00	17,888	3:30	16,835
0:30	17,398	4:00	17,065
1:00	17,110	4:30	17,264
1:30	16,881	5:00	17,577
2:00	16,832	5:30	17,992
2:30	16,950	6:00	18,216
3:00	16,833		

36. Shown is the graph of traffic on an Internet service provider's T1 data line from midnight to 8:00 AM. D is the data throughput, measured in megabits per second. Use Simpson's Rule to estimate the total amount of data transmitted during that time period.

37. If the region shown in the figure is rotated about the y-axis to form a solid, use Simpson's Rule with $n = 8$ to estimate the volume of the solid.



38. The table shows values of a force function $f(x)$, where x is measured in meters and $f(x)$ in newtons. Use Simpson's Rule to estimate the work done by the force in moving an object a distance of 18 m.

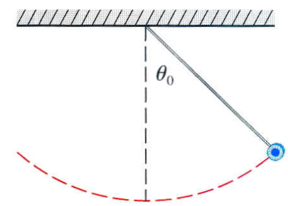
x	0	3	6	9	12	15	18
$f(x)$	9.8	9.1	8.5	8.0	7.7	7.5	7.4

39. The region bounded by the curves $y = e^{-1/x}$, $y = 0$, $x = 1$, and $x = 5$ is rotated about the x-axis. Use Simpson's Rule with $n = 8$ to estimate the volume of the resulting solid.

CAS **40.** The figure shows a pendulum with length L that makes a maximum angle θ_0 with the vertical. Using Newton's Second Law, it can be shown that the period T (the time for one complete swing) is given by

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

where $k = \sin(\frac{1}{2}\theta_0)$ and g is the acceleration due to gravity. If $L = 1$ m and $\theta_0 = 42°$, use Simpson's Rule with $n = 10$ to find the period.



41. The intensity of light with wavelength λ traveling through a diffraction grating with N slits at an angle θ is given by $I(\theta) = N^2 \sin^2 k/k^2$, where $k = (\pi N d \sin\theta)/\lambda$ and d is the distance between adjacent slits. A helium-neon laser with wavelength $\lambda = 632.8 \times 10^{-9}$ m is emitting a narrow band of light, given by $-10^{-6} < \theta < 10^{-6}$, through a grating with 10,000 slits spaced 10^{-4} m apart. Use the Midpoint Rule with $n = 10$ to estimate the total light intensity $\int_{-10^{-6}}^{10^{-6}} I(\theta)\, d\theta$ emerging from the grating.

42. Use the Trapezoidal Rule with $n = 10$ to approximate $\int_0^{20} \cos(\pi x)\, dx$. Compare your result to the actual value. Can you explain the discrepancy?

43. Sketch the graph of a continuous function on $[0, 2]$ for which the Trapezoidal Rule with $n = 2$ is more accurate than the Midpoint Rule.

44. Sketch the graph of a continuous function on $[0, 2]$ for which the right endpoint approximation with $n = 2$ is more accurate than Simpson's Rule.

45. If f is a positive function and $f''(x) < 0$ for $a \leqslant x \leqslant b$, show that

$$T_n < \int_a^b f(x)\, dx < M_n$$

46. Show that if f is a polynomial of degree 3 or lower, then Simpson's Rule gives the exact value of $\int_a^b f(x)\, dx$.

47. Show that $\frac{1}{2}(T_n + M_n) = T_{2n}$.

48. Show that $\frac{1}{3}T_n + \frac{2}{3}M_n = S_{2n}$.

Thus, taking $f(x) = e^{-x}$ and $g(x) = e^{-x^2}$ in the Comparison Theorem, we see that $\int_1^\infty e^{-x^2}\,dx$ is convergent. It follows that $\int_0^\infty e^{-x^2}\,dx$ is convergent. □

In Example 9 we showed that $\int_0^\infty e^{-x^2}\,dx$ is convergent without computing its value. In Exercise 70 we indicate how to show that its value is approximately 0.8862. In probability theory it is important to know the exact value of this improper integral, as we will see in Section 9.5; using the methods of multivariable calculus it can be shown that the exact value is $\sqrt{\pi}/2$. Table 1 illustrates the definition of an improper integral by showing how the (computer-generated) values of $\int_0^t e^{-x^2}\,dx$ approach $\sqrt{\pi}/2$ as t becomes large. In fact, these values converge quite quickly because $e^{-x^2} \to 0$ very rapidly as $x \to \infty$.

TABLE 1

t	$\int_0^t e^{-x^2}\,dx$
1	0.7468241328
2	0.8820813908
3	0.8862073483
4	0.8862269118
5	0.8862269255
6	0.8862269255

EXAMPLE 10 The integral $\displaystyle\int_1^\infty \frac{1 + e^{-x}}{x}\,dx$ is divergent by the Comparison Theorem because

$$\frac{1 + e^{-x}}{x} > \frac{1}{x}$$

and $\int_1^\infty (1/x)\,dx$ is divergent by Example 1 [or by (2) with $p = 1$]. □

TABLE 2

t	$\int_1^t [(1 + e^{-x})/x]\,dx$
2	0.8636306042
5	1.8276735512
10	2.5219648704
100	4.8245541204
1000	7.1271392134
10000	9.4297243064

Table 2 illustrates the divergence of the integral in Example 10. It appears that the values are not approaching any fixed number.

8.8 EXERCISES

1. Explain why each of the following integrals is improper.

(a) $\displaystyle\int_1^\infty x^4 e^{-x^4}\,dx$ (b) $\displaystyle\int_0^{\pi/2} \sec x\,dx$

(c) $\displaystyle\int_0^2 \frac{x}{x^2 - 5x + 6}\,dx$ (d) $\displaystyle\int_{-\infty}^0 \frac{1}{x^2 + 5}\,dx$

2. Which of the following integrals are improper? Why?

(a) $\displaystyle\int_1^2 \frac{1}{2x - 1}\,dx$ (b) $\displaystyle\int_0^1 \frac{1}{2x - 1}\,dx$

(c) $\displaystyle\int_{-\infty}^\infty \frac{\sin x}{1 + x^2}\,dx$ (d) $\displaystyle\int_1^2 \ln(x - 1)\,dx$

3. Find the area under the curve $y = 1/x^3$ from $x = 1$ to $x = t$ and evaluate it for $t = 10$, 100, and 1000. Then find the total area under this curve for $x \geq 1$.

4. (a) Graph the functions $f(x) = 1/x^{1.1}$ and $g(x) = 1/x^{0.9}$ in the viewing rectangles $[0, 10]$ by $[0, 1]$ and $[0, 100]$ by $[0, 1]$.
(b) Find the areas under the graphs of f and g from $x = 1$ to $x = t$ and evaluate for $t = 10$, 100, 10^4, 10^6, 10^{10}, and 10^{20}.
(c) Find the total area under each curve for $x \geq 1$, if it exists.

5–40 Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

5. $\displaystyle\int_1^\infty \frac{1}{(3x + 1)^2}\,dx$

6. $\displaystyle\int_{-\infty}^0 \frac{1}{2x - 5}\,dx$

7. $\displaystyle\int_{-\infty}^{-1} \frac{1}{\sqrt{2 - w}}\,dw$

8. $\displaystyle\int_0^\infty \frac{x}{(x^2 + 2)^2}\,dx$

9. $\displaystyle\int_4^\infty e^{-y/2}\,dy$

10. $\displaystyle\int_{-\infty}^{-1} e^{-2t}\,dt$

11. $\displaystyle\int_{-\infty}^\infty \frac{x}{1 + x^2}\,dx$

12. $\displaystyle\int_{-\infty}^\infty (2 - v^4)\,dv$

13. $\displaystyle\int_{-\infty}^\infty x e^{-x^2}\,dx$

14. $\displaystyle\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

15. $\displaystyle\int_{2\pi}^\infty \sin\theta\,d\theta$

16. $\displaystyle\int_{-\infty}^{\pi/2} \sin 2\theta\,d\theta$

17. $\displaystyle\int_1^\infty \frac{x + 1}{x^2 + 2x}\,dx$

18. $\displaystyle\int_0^\infty \frac{dz}{z^2 + 3z + 2}$

19. $\displaystyle\int_{-\infty}^1 x e^{2x}\,dx$

20. $\displaystyle\int_{-\infty}^6 r e^{r/3}\,dr$

21. $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx$

22. $\displaystyle\int_{-\infty}^\infty x^3 e^{-x^4}\,dx$

23. $\displaystyle\int_{-\infty}^\infty \frac{x^2}{9 + x^6}\,dx$

24. $\displaystyle\int_0^\infty \frac{e^x}{e^{2x} + 3}\,dx$

25. $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^3}\,dx$

26. $\displaystyle\int_0^\infty \frac{x \arctan x}{(1 + x^2)^2}\,dx$

27. $\displaystyle\int_0^1 \frac{3}{x^5}\,dx$

28. $\displaystyle\int_2^3 \frac{1}{\sqrt{3 - x}}\,dx$

29. $\displaystyle\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}}$

30. $\displaystyle\int_{6}^{8} \frac{4}{(x-6)^3}\,dx$

31. $\displaystyle\int_{-2}^{3} \frac{1}{x^4}\,dx$

32. $\displaystyle\int_{0}^{1} \frac{dx}{\sqrt{1-x^2}}$

33. $\displaystyle\int_{0}^{33} (x-1)^{-1/5}\,dx$

34. $\displaystyle\int_{0}^{1} \frac{1}{4y-1}\,dy$

35. $\displaystyle\int_{0}^{3} \frac{dx}{x^2-6x+5}$

36. $\displaystyle\int_{\pi/2}^{\pi} \csc x\,dx$

37. $\displaystyle\int_{-1}^{0} \frac{e^{1/x}}{x^3}\,dx$

38. $\displaystyle\int_{0}^{1} \frac{e^{1/x}}{x^3}\,dx$

39. $\displaystyle\int_{0}^{2} z^2 \ln z\,dz$

40. $\displaystyle\int_{0}^{1} \frac{\ln x}{\sqrt{x}}\,dx$

41–46 Sketch the region and find its area (if the area is finite).

41. $S = \{(x, y) \mid x \leqslant 1,\ 0 \leqslant y \leqslant e^x\}$

42. $S = \{(x, y) \mid x \geqslant -2,\ 0 \leqslant y \leqslant e^{-x/2}\}$

43. $S = \{(x, y) \mid 0 \leqslant y \leqslant 2/(x^2+9)\}$

44. $S = \{(x, y) \mid x \geqslant 0,\ 0 \leqslant y \leqslant x/(x^2+9)\}$

45. $S = \{(x, y) \mid 0 \leqslant x < \pi/2,\ 0 \leqslant y \leqslant \sec^2 x\}$

46. $S = \left\{(x, y) \mid -2 < x \leqslant 0,\ 0 \leqslant y \leqslant 1/\sqrt{x+2}\right\}$

47. (a) If $g(x) = (\sin^2 x)/x^2$, use your calculator or computer to make a table of approximate values of $\int_1^t g(x)\,dx$ for $t = 2, 5, 10, 100, 1000$, and $10{,}000$. Does it appear that $\int_1^\infty g(x)\,dx$ is convergent?
(b) Use the Comparison Theorem with $f(x) = 1/x^2$ to show that $\int_1^\infty g(x)\,dx$ is convergent.
(c) Illustrate part (b) by graphing f and g on the same screen for $1 \leqslant x \leqslant 10$. Use your graph to explain intuitively why $\int_1^\infty g(x)\,dx$ is convergent.

48. (a) If $g(x) = 1/(\sqrt{x} - 1)$, use your calculator or computer to make a table of approximate values of $\int_2^t g(x)\,dx$ for $t = 5$, $10, 100, 1000$, and $10{,}000$. Does it appear that $\int_2^\infty g(x)\,dx$ is convergent or divergent?
(b) Use the Comparison Theorem with $f(x) = 1/\sqrt{x}$ to show that $\int_2^\infty g(x)\,dx$ is divergent.
(c) Illustrate part (b) by graphing f and g on the same screen for $2 \leqslant x \leqslant 20$. Use your graph to explain intuitively why $\int_2^\infty g(x)\,dx$ is divergent.

49–54 Use the Comparison Theorem to determine whether the integral is convergent or divergent.

49. $\displaystyle\int_{0}^{\infty} \frac{x}{x^3+1}\,dx$

50. $\displaystyle\int_{1}^{\infty} \frac{2+e^{-x}}{x}\,dx$

51. $\displaystyle\int_{1}^{\infty} \frac{x+1}{\sqrt{x^4-x}}\,dx$

52. $\displaystyle\int_{0}^{\infty} \frac{\arctan x}{2+e^x}\,dx$

53. $\displaystyle\int_{0}^{1} \frac{\sec^2 x}{x\sqrt{x}}\,dx$

54. $\displaystyle\int_{0}^{\pi} \frac{\sin^2 x}{\sqrt{x}}\,dx$

55. The integral

$$\int_0^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx$$

is improper for two reasons: The interval $[0, \infty)$ is infinite and the integrand has an infinite discontinuity at 0. Evaluate it by expressing it as a sum of improper integrals of Type 2 and Type 1 as follows:

$$\int_0^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx = \int_0^1 \frac{1}{\sqrt{x}\,(1+x)}\,dx + \int_1^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx$$

56. Evaluate

$$\int_2^\infty \frac{1}{x\sqrt{x^2-4}}\,dx$$

by the same method as in Exercise 55.

57–59 Find the values of p for which the integral converges and evaluate the integral for those values of p.

57. $\displaystyle\int_{0}^{1} \frac{1}{x^p}\,dx$

58. $\displaystyle\int_{e}^{\infty} \frac{1}{x(\ln x)^p}\,dx$

59. $\displaystyle\int_{0}^{1} x^p \ln x\,dx$

60. (a) Evaluate the integral $\int_0^\infty x^n e^{-x}\,dx$ for $n = 0, 1, 2$, and 3.
(b) Guess the value of $\int_0^\infty x^n e^{-x}\,dx$ when n is an arbitrary positive integer.
(c) Prove your guess using mathematical induction.

61. (a) Show that $\int_{-\infty}^{\infty} x\,dx$ is divergent.
(b) Show that

$$\lim_{t\to\infty} \int_{-t}^{t} x\,dx = 0$$

This shows that we can't define

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_{-t}^{t} f(x)\,dx$$

62. The *average speed* of molecules in an ideal gas is

$$\bar{v} = \frac{4}{\sqrt{\pi}} \left(\frac{M}{2RT}\right)^{3/2} \int_0^\infty v^3 e^{-Mv^2/(2RT)}\,dv$$

where M is the molecular weight of the gas, R is the gas constant, T is the gas temperature, and v is the molecular speed. Show that

$$\bar{v} = \sqrt{\frac{8RT}{\pi M}}$$

63. We know from Example 1 that the region $\mathcal{R} = \{(x, y) \mid x \geqslant 1,\ 0 \leqslant y \leqslant 1/x\}$ has infinite area. Show that by rotating $\mathcal{R}$ about the x-axis we obtain a solid with finite volume.

64. Use the information and data in Exercises 29 and 30 of Section 6.4 to find the work required to propel a 1000-kg satellite out of the earth's gravitational field.

65. Find the *escape velocity* v_0 that is needed to propel a rocket of mass m out of the gravitational field of a planet with mass M and radius R. Use Newton's Law of Gravitation (see Exercise 29 in Section 6.4) and the fact that the initial kinetic energy of $\frac{1}{2}mv_0^2$ supplies the needed work.

66. Astronomers use a technique called *stellar stereography* to determine the density of stars in a star cluster from the observed (two-dimensional) density that can be analyzed from a photograph. Suppose that in a spherical cluster of radius R the density of stars depends only on the distance r from the center of the cluster. If the perceived star density is given by $y(s)$, where s is the observed planar distance from the center of the cluster, and $x(r)$ is the actual density, it can be shown that

$$y(s) = \int_s^R \frac{2r}{\sqrt{r^2 - s^2}}\, x(r)\, dr$$

If the actual density of stars in a cluster is $x(r) = \frac{1}{2}(R - r)^2$, find the perceived density $y(s)$.

67. A manufacturer of lightbulbs wants to produce bulbs that last about 700 hours but, of course, some bulbs burn out faster than others. Let $F(t)$ be the fraction of the company's bulbs that burn out before t hours, so $F(t)$ always lies between 0 and 1.

(a) Make a rough sketch of what you think the graph of F might look like.
(b) What is the meaning of the derivative $r(t) = F'(t)$?
(c) What is the value of $\int_0^\infty r(t)\, dt$? Why?

68. As we saw in Section 7.5, a radioactive substance decays exponentially: The mass at time t is $m(t) = m(0)e^{kt}$, where $m(0)$ is the initial mass and k is a negative constant. The *mean life* M of an atom in the substance is

$$M = -k \int_0^\infty t e^{kt}\, dt$$

For the radioactive carbon isotope, ^{14}C, used in radiocarbon dating, the value of k is -0.000121. Find the mean life of a ^{14}C atom.

69. Determine how large the number a has to be so that

$$\int_a^\infty \frac{1}{x^2 + 1}\, dx < 0.001$$

70. Estimate the numerical value of $\int_0^\infty e^{-x^2}\, dx$ by writing it as the sum of $\int_0^4 e^{-x^2}\, dx$ and $\int_4^\infty e^{-x^2}\, dx$. Approximate the first integral by using Simpson's Rule with $n = 8$ and show that the second integral is smaller than $\int_4^\infty e^{-4x}\, dx$, which is less than 0.0000001.

71. If $f(t)$ is continuous for $t \geqslant 0$, the *Laplace transform* of f is the function F defined by

$$F(s) = \int_0^\infty f(t)e^{-st}\, dt$$

and the domain of F is the set consisting of all numbers s for which the integral converges. Find the Laplace transforms of the following functions.

(a) $f(t) = 1$ (b) $f(t) = e^t$ (c) $f(t) = t$

72. Show that if $0 \leqslant f(t) \leqslant Me^{at}$ for $t \geqslant 0$, where M and a are constants, then the Laplace transform $F(s)$ exists for $s > a$.

73. Suppose that $0 \leqslant f(t) \leqslant Me^{at}$ and $0 \leqslant f'(t) \leqslant Ke^{at}$ for $t \geqslant 0$, where f' is continuous. If the Laplace transform of $f(t)$ is $F(s)$ and the Laplace transform of $f'(t)$ is $G(s)$, show that

$$G(s) = sF(s) - f(0) \qquad s > a$$

74. If $\int_{-\infty}^\infty f(x)\, dx$ is convergent and a and b are real numbers, show that

$$\int_{-\infty}^a f(x)\, dx + \int_a^\infty f(x)\, dx = \int_{-\infty}^b f(x)\, dx + \int_b^\infty f(x)\, dx$$

75. Show that $\int_0^\infty x^2 e^{-x^2}\, dx = \frac{1}{2}\int_0^\infty e^{-x^2}\, dx$.

76. Show that $\int_0^\infty e^{-x^2}\, dx = \int_0^1 \sqrt{-\ln y}\, dy$ by interpreting the integrals as areas.

77. Find the value of the constant C for which the integral

$$\int_0^\infty \left(\frac{1}{\sqrt{x^2 + 4}} - \frac{C}{x + 2} \right) dx$$

converges. Evaluate the integral for this value of C.

78. Find the value of the constant C for which the integral

$$\int_0^\infty \left(\frac{x}{x^2 + 1} - \frac{C}{3x + 1} \right) dx$$

converges. Evaluate the integral for this value of C.

79. Suppose f is continuous on $[0, \infty)$ and $\lim_{x \to \infty} f(x) = 1$. Is it possible that $\int_0^\infty f(x)\, dx$ is convergent?

80. Show that if $a > -1$ and $b > a + 1$, then the following integral is convergent.

$$\int_0^\infty \frac{x^a}{1 + x^b}\, dx$$

8 REVIEW

CONCEPT CHECK

1. State the rule for integration by parts. In practice, how do you use it?

2. How do you evaluate $\int \sin^m x \cos^n x \, dx$ if m is odd? What if n is odd? What if m and n are both even?

3. If the expression $\sqrt{a^2 - x^2}$ occurs in an integral, what substitution might you try? What if $\sqrt{a^2 + x^2}$ occurs? What if $\sqrt{x^2 - a^2}$ occurs?

4. What is the form of the partial fraction expansion of a rational function $P(x)/Q(x)$ if the degree of P is less than the degree of Q and $Q(x)$ has only distinct linear factors? What if a linear factor is repeated? What if $Q(x)$ has an irreducible quadratic factor (not repeated)? What if the quadratic factor is repeated?

5. State the rules for approximating the definite integral $\int_a^b f(x)\,dx$ with the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule. Which would you expect to give the best estimate? How do you approximate the error for each rule?

6. Define the following improper integrals.
   (a) $\int_a^\infty f(x)\,dx$ (b) $\int_{-\infty}^b f(x)\,dx$ (c) $\int_{-\infty}^\infty f(x)\,dx$

7. Define the improper integral $\int_a^b f(x)\,dx$ for each of the following cases.
   (a) f has an infinite discontinuity at a.
   (b) f has an infinite discontinuity at b.
   (c) f has an infinite discontinuity at c, where $a < c < b$.

8. State the Comparison Theorem for improper integrals.

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. $\dfrac{x(x^2+4)}{x^2-4}$ can be put in the form $\dfrac{A}{x+2} + \dfrac{B}{x-2}$.

2. $\dfrac{x^2+4}{x(x^2-4)}$ can be put in the form $\dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{x-2}$.

3. $\dfrac{x^2+4}{x^2(x-4)}$ can be put in the form $\dfrac{A}{x^2} + \dfrac{B}{x-4}$.

4. $\dfrac{x^2-4}{x(x^2+4)}$ can be put in the form $\dfrac{A}{x} + \dfrac{B}{x^2+4}$.

5. $\displaystyle\int_0^4 \frac{x}{x^2-1}\,dx = \tfrac{1}{2}\ln 15$

6. $\displaystyle\int_1^\infty \frac{1}{x^{\sqrt{2}}}\,dx$ is convergent.

7. If f is continuous, then $\int_{-\infty}^\infty f(x)\,dx = \lim_{t\to\infty}\int_{-t}^t f(x)\,dx$.

8. The Midpoint Rule is always more accurate than the Trapezoidal Rule.

9. (a) Every elementary function has an elementary derivative.
   (b) Every elementary function has an elementary antiderivative.

10. If f is continuous on $[0, \infty)$ and $\int_1^\infty f(x)\,dx$ is convergent, then $\int_0^\infty f(x)\,dx$ is convergent.

11. If f is a continuous, decreasing function on $[1, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, then $\int_1^\infty f(x)\,dx$ is convergent.

12. If $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ are both convergent, then $\int_a^\infty [f(x) + g(x)]\,dx$ is convergent.

13. If $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ are both divergent, then $\int_a^\infty [f(x) + g(x)]\,dx$ is divergent.

14. If $f(x) \leqslant g(x)$ and $\int_0^\infty g(x)\,dx$ diverges, then $\int_0^\infty f(x)\,dx$ also diverges.

EXERCISES

Note: Additional practice in techniques of integration is provided in Exercises 8.5.

1–40 Evaluate the integral.

1. $\displaystyle\int_0^5 \frac{x}{x+10}\,dx$

2. $\displaystyle\int_0^5 ye^{-0.6y}\,dy$

3. $\displaystyle\int_0^{\pi/2} \frac{\cos\theta}{1+\sin\theta}\,d\theta$

4. $\displaystyle\int_1^4 \frac{dt}{(2t+1)^3}$

5. $\displaystyle\int_0^{\pi/2} \sin^3\theta\cos^2\theta\,d\theta$

6. $\displaystyle\int \frac{1}{y^2-4y-12}\,dy$

7. $\displaystyle\int \frac{\sin(\ln t)}{t}\,dt$

8. $\displaystyle\int \frac{dx}{\sqrt{e^x-1}}$

9. $\displaystyle\int_1^4 x^{3/2}\ln x\,dx$

10. $\displaystyle\int_0^1 \frac{\sqrt{\arctan x}}{1+x^2}\,dx$

11. $\displaystyle\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx$

12. $\displaystyle\int_{-1}^1 \frac{\sin x}{1+x^2}\,dx$

13. $\displaystyle\int e^{\sqrt[3]{x}}\,dx$

14. $\displaystyle\int \frac{x^2+2}{x+2}\,dx$

15. $\displaystyle\int \frac{x-1}{x^2+2x}\,dx$

16. $\displaystyle\int \frac{\sec^6\theta}{\tan^2\theta}\,d\theta$

17. $\displaystyle\int x\sec x\tan x\,dx$

18. $\displaystyle\int \frac{x^2+8x-3}{x^3+3x^2}\,dx$

19. $\displaystyle\int \frac{x+1}{9x^2+6x+5}\,dx$

20. $\displaystyle\int \tan^5\theta\sec^3\theta\,d\theta$

21. $\displaystyle\int \frac{dx}{\sqrt{x^2-4x}}$

22. $\displaystyle\int te^{\sqrt{t}}\,dt$

23. $\displaystyle\int \frac{dx}{x\sqrt{x^2+1}}$

24. $\displaystyle\int e^x\cos x\,dx$

25. $\displaystyle\int \frac{3x^3-x^2+6x-4}{(x^2+1)(x^2+2)}\,dx$

26. $\displaystyle\int x\sin x\cos x\,dx$

27. $\displaystyle\int_0^{\pi/2} \cos^3x\sin 2x\,dx$

28. $\displaystyle\int \frac{\sqrt[3]{x}+1}{\sqrt[3]{x}-1}\,dx$

29. $\displaystyle\int_{-1}^1 x^5\sec x\,dx$

30. $\displaystyle\int \frac{dx}{e^x\sqrt{1-e^{-2x}}}$

31. $\displaystyle\int_0^{\ln 10} \frac{e^x\sqrt{e^x-1}}{e^x+8}\,dx$

32. $\displaystyle\int_0^{\pi/4} \frac{x\sin x}{\cos^3x}\,dx$

33. $\displaystyle\int \frac{x^2}{(4-x^2)^{3/2}}\,dx$

34. $\displaystyle\int (\arcsin x)^2\,dx$

35. $\displaystyle\int \frac{1}{\sqrt{x+x^{3/2}}}\,dx$

36. $\displaystyle\int \frac{1-\tan\theta}{1+\tan\theta}\,d\theta$

37. $\displaystyle\int (\cos x+\sin x)^2\cos 2x\,dx$

38. $\displaystyle\int \frac{x^2}{(x+2)^3}\,dx$

39. $\displaystyle\int_0^{1/2} \frac{xe^{2x}}{(1+2x)^2}\,dx$

40. $\displaystyle\int_{\pi/4}^{\pi/3} \frac{\sqrt{\tan\theta}}{\sin 2\theta}\,d\theta$

41–50 Evaluate the integral or show that it is divergent.

41. $\displaystyle\int_1^\infty \frac{1}{(2x+1)^3}\,dx$

42. $\displaystyle\int_1^\infty \frac{\ln x}{x^4}\,dx$

43. $\displaystyle\int_2^\infty \frac{dx}{x\ln x}$

44. $\displaystyle\int_2^6 \frac{y}{\sqrt{y-2}}\,dy$

45. $\displaystyle\int_0^4 \frac{\ln x}{\sqrt{x}}\,dx$

46. $\displaystyle\int_0^1 \frac{1}{2-3x}\,dx$

47. $\displaystyle\int_0^1 \frac{x-1}{\sqrt{x}}\,dx$

48. $\displaystyle\int_{-1}^1 \frac{dx}{x^2-2x}$

49. $\displaystyle\int_{-\infty}^\infty \frac{dx}{4x^2+4x+5}$

50. $\displaystyle\int_1^\infty \frac{\tan^{-1}x}{x^2}\,dx$

51–52 Evaluate the indefinite integral. Illustrate and check that your answer is reasonable by graphing both the function and its antiderivative (take $C=0$).

51. $\displaystyle\int \ln(x^2+2x+2)\,dx$

52. $\displaystyle\int \frac{x^3}{\sqrt{x^2+1}}\,dx$

53. Graph the function $f(x)=\cos^2x\sin^3x$ and use the graph to guess the value of the integral $\int_0^{2\pi} f(x)\,dx$. Then evaluate the integral to confirm your guess.

CAS **54.** (a) How would you evaluate $\int x^5e^{-2x}\,dx$ by hand? (Don't actually carry out the integration.)
(b) How would you evaluate $\int x^5e^{-2x}\,dx$ using tables? (Don't actually do it.)
(c) Use a CAS to evaluate $\int x^5e^{-2x}\,dx$.
(d) Graph the integrand and the indefinite integral on the same screen.

55–58 Use the Table of Integrals on the Reference Pages to evaluate the integral.

55. $\displaystyle\int \sqrt{4x^2-4x-3}\,dx$

56. $\displaystyle\int \csc^5t\,dt$

57. $\displaystyle\int \cos x\sqrt{4+\sin^2x}\,dx$

58. $\displaystyle\int \frac{\cot x}{\sqrt{1+2\sin x}}\,dx$

59. Verify Formula 33 in the Table of Integrals (a) by differentiation and (b) by using a trigonometric substitution.

60. Verify Formula 62 in the Table of Integrals.

61. Is it possible to find a number n such that $\int_0^\infty x^n\,dx$ is convergent?

62. For what values of a is $\int_0^\infty e^{ax}\cos x\,dx$ convergent? Evaluate the integral for those values of a.

63–64 Use (a) the Trapezoidal Rule, (b) the Midpoint Rule, and (c) Simpson's Rule with $n=10$ to approximate the given integral. Round your answers to six decimal places.

63. $\displaystyle\int_2^4 \frac{1}{\ln x}\,dx$

64. $\displaystyle\int_1^4 \sqrt{x}\cos x\,dx$

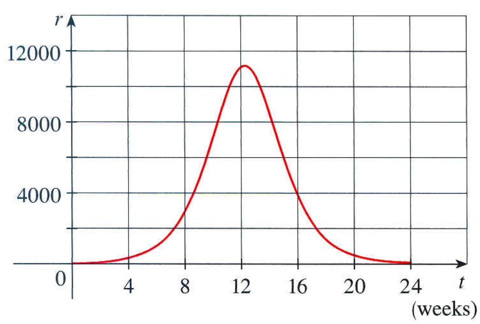
65. Estimate the errors involved in Exercise 63, parts (a) and (b). How large should n be in each case to guarantee an error of less than 0.00001?

66. Use Simpson's Rule with $n=6$ to estimate the area under the curve $y=e^x/x$ from $x=1$ to $x=4$.

67. The speedometer reading (v) on a car was observed at 1-minute intervals and recorded in the chart. Use Simpson's Rule to estimate the distance traveled by the car.

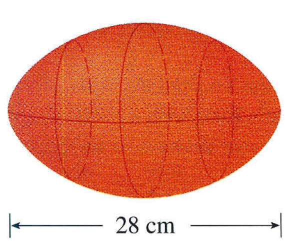
t (min)	v (km/h)	t (min)	v (km/h)
0	64	6	90
1	67	7	91
2	72	8	91
3	78	9	88
4	83	10	90
5	86		

68. A population of honeybees increased at a rate of $r(t)$ bees per week, where the graph of r is as shown. Use Simpson's Rule with six subintervals to estimate the increase in the bee population during the first 24 weeks.



CAS **69.** (a) If $f(x) = \sin(\sin x)$, use a graph to find an upper bound for $|f^{(4)}(x)|$.
(b) Use Simpson's Rule with $n = 10$ to approximate $\int_0^{\pi} f(x)\,dx$ and use part (a) to estimate the error.
(c) How large should n be to guarantee that the size of the error in using S_n is less than 0.00001?

70. Suppose you are asked to estimate the volume of a football. You measure and find that a football is 28 cm long. You use a piece of string and measure the circumference at its widest point to be 53 cm. The circumference 7 cm from each end is 45 cm. Use Simpson's Rule to make your estimate.



71. Use the Comparison Theorem to determine whether the integral

$$\int_1^{\infty} \frac{x^3}{x^5 + 2}\,dx$$

is convergent or divergent.

72. Find the area of the region bounded by the hyperbola $y^2 - x^2 = 1$ and the line $y = 3$.

73. Find the area bounded by the curves $y = \cos x$ and $y = \cos^2 x$ between $x = 0$ and $x = \pi$.

74. Find the area of the region bounded by the curves $y = 1/(2 + \sqrt{x})$, $y = 1/(2 - \sqrt{x})$, and $x = 1$.

75. The region under the curve $y = \cos^2 x$, $0 \leqslant x \leqslant \pi/2$, is rotated about the x-axis. Find the volume of the resulting solid.

76. The region in Exercise 75 is rotated about the y-axis. Find the volume of the resulting solid.

77. If f' is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, show that

$$\int_0^{\infty} f'(x)\,dx = -f(0)$$

78. We can extend our definition of average value of a continuous function to an infinite interval by defining the average value of f on the interval $[a, \infty)$ to be

$$\lim_{t\to\infty} \frac{1}{t - a} \int_a^t f(x)\,dx$$

(a) Find the average value of $y = \tan^{-1}x$ on the interval $[0, \infty)$.
(b) If $f(x) \geqslant 0$ and $\int_a^{\infty} f(x)\,dx$ is divergent, show that the average value of f on the interval $[a, \infty)$ is $\lim_{x\to\infty} f(x)$, if this limit exists.
(c) If $\int_a^{\infty} f(x)\,dx$ is convergent, what is the average value of f on the interval $[a, \infty)$?
(d) Find the average value of $y = \sin x$ on the interval $[0, \infty)$.

79. Use the substitution $u = 1/x$ to show that

$$\int_0^{\infty} \frac{\ln x}{1 + x^2}\,dx = 0$$

80. The magnitude of the repulsive force between two point charges with the same sign, one of size 1 and the other of size q, is

$$F = \frac{q}{4\pi\varepsilon_0 r^2}$$

where r is the distance between the charges and ε_0 is a constant. The *potential* V at a point P due to the charge q is defined to be the work expended in bringing a unit charge to P from infinity along the straight line that joins q and P. Find a formula for V.

PROBLEMS

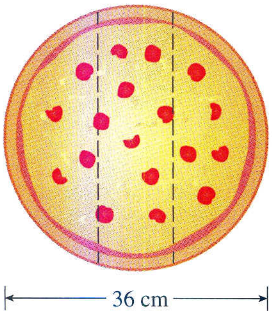


FIGURE FOR PROBLEM 1

1. Three mathematics students have ordered a 36-centimeter pizza. Instead of slicing it in the traditional way, they decide to slice it by parallel cuts, as shown in the figure. Being mathematics majors, they are able to determine where to slice so that each gets the same amount of pizza. Where are the cuts made?

2. Evaluate $\int \frac{1}{x^7 - x}\,dx$.

The straightforward approach would be to start with partial fractions, but that would be brutal. Try a substitution.

3. Evaluate $\int_0^1 \left(\sqrt[3]{1 - x^7} - \sqrt[7]{1 - x^3}\right) dx$.

4. The centers of two disks with radius 1 are one unit apart. Find the area of the union of the two disks.

5. An ellipse is cut out of a circle with radius a. The major axis of the ellipse coincides with a diameter of the circle and the minor axis has length $2b$. Prove that the area of the remaining part of the circle is the same as the area of an ellipse with semiaxes a and $a - b$.

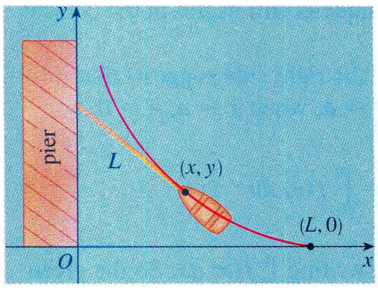


FIGURE FOR PROBLEM 6

6. A man initially standing at the point O walks along a pier pulling a rowboat by a rope of length L. The man keeps the rope straight and taut. The path followed by the boat is a curve called a *tractrix* and it has the property that the rope is always tangent to the curve (see the figure).

(a) Show that if the path followed by the boat is the graph of the function $y = f(x)$, then

$$f'(x) = \frac{dy}{dx} = \frac{-\sqrt{L^2 - x^2}}{x}$$

(b) Determine the function $y = f(x)$.

7. A function f is defined by

$$f(x) = \int_0^\pi \cos t \cos(x - t)\,dt \qquad 0 \leqslant x \leqslant 2\pi$$

Find the minimum value of f.

8. If n is a positive integer, prove that

$$\int_0^1 (\ln x)^n\,dx = (-1)^n n!$$

9. Show that

$$\int_0^1 (1 - x^2)^n\,dx = \frac{2^{2n}(n!)^2}{(2n + 1)!}$$

Hint: Start by showing that if I_n denotes the integral, then

$$I_{k+1} = \frac{2k + 2}{2k + 3} I_k$$

10. Suppose that f is a positive function such that f' is continuous.
(a) How is the graph of $y = f(x) \sin nx$ related to the graph of $y = f(x)$? What happens as $n \to \infty$?
(b) Make a guess as to the value of the limit

$$\lim_{n\to\infty} \int_0^1 f(x) \sin nx \, dx$$

based on graphs of the integrand.
(c) Using integration by parts, confirm the guess that you made in part (b). [Use the fact that, since f' is continuous, there is a constant M such that $|f'(x)| \leqslant M$ for $0 \leqslant x \leqslant 1$.]

11. If $0 < a < b$, find $\lim_{t\to 0} \left\{ \int_0^1 [bx + a(1-x)]^t \, dx \right\}^{1/t}$.

12. Graph $f(x) = \sin(e^x)$ and use the graph to estimate the value of t such that $\int_t^{t+1} f(x)\,dx$ is a maximum. Then find the exact value of t that maximizes this integral.

13. The circle with radius 1 shown in the figure touches the curve $y = |2x|$ twice. Find the area of the region that lies between the two curves.

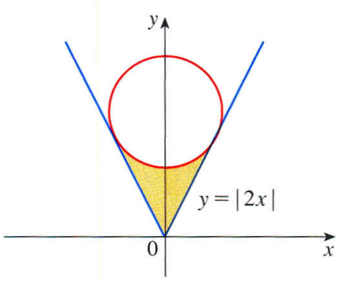


FIGURE FOR PROBLEM 13

14. A rocket is fired straight up, burning fuel at the constant rate of b kilograms per second. Let $v = v(t)$ be the velocity of the rocket at time t and suppose that the velocity u of the exhaust gas is constant. Let $M = M(t)$ be the mass of the rocket at time t and note that M decreases as the fuel burns. If we neglect air resistance, it follows from Newton's Second Law that

$$F = M\frac{dv}{dt} - ub$$

where the force $F = -Mg$. Thus

$$\boxed{1} \qquad M\frac{dv}{dt} - ub = -Mg$$

Let M_1 be the mass of the rocket without fuel, M_2 the initial mass of the fuel, and $M_0 = M_1 + M_2$. Then, until the fuel runs out at time $t = M_2 b$, the mass is $M = M_0 - bt$.
(a) Substitute $M = M_0 - bt$ into Equation 1 and solve the resulting equation for v. Use the initial condition $v(0) = 0$ to evaluate the constant.
(b) Determine the velocity of the rocket at time $t = M_2/b$. This is called the *burnout velocity*.
(c) Determine the height of the rocket $y = y(t)$ at the burnout time.
(d) Find the height of the rocket at any time t.

15. Use integration by parts to show that, for all $x > 0$,

$$0 < \int_0^\infty \frac{\sin t}{\ln(1 + x + t)}\,dt < \frac{2}{\ln(1+x)}$$

16. Suppose $f(1) = f'(1) = 0$, f'' is continuous on $[0, 1]$ and $|f''(x)| \leqslant 3$ for all x. Show that

$$\left| \int_0^1 f(x)\,dx \right| \leqslant \frac{1}{2}$$