7.1 EXERCISES

1. (a) What is a one-to-one function?
(b) How can you tell from the graph of a function whether it is one-to-one?

2. (a) Suppose f is a one-to-one function with domain A and range B. How is the inverse function f^{-1} defined? What is the domain of f^{-1}? What is the range of f^{-1}?
(b) If you are given a formula for f, how do you find a formula for f^{-1}?
(c) If you are given the graph of f, how do you find the graph of f^{-1}?

3–16 A function is given by a table of values, a graph, a formula, or a verbal description. Determine whether it is one-to-one.

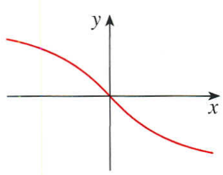
3.

x	1	2	3	4	5	6
$f(x)$	1.5	2.0	3.6	5.3	2.8	2.0

4.

x	1	2	3	4	5	6
$f(x)$	1	2	4	8	16	32

5.



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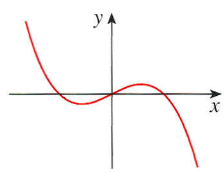

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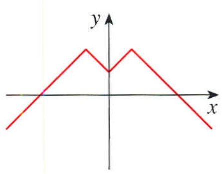

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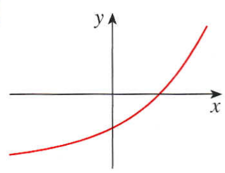


9. $f(x) = \frac{1}{2}(x + 5)$

10. $f(x) = 1 + 4x - x^2$

11. $g(x) = |x|$

12. $g(x) = \sqrt{x}$

13. $h(x) = 1 + \cos x$

14. $h(x) = 1 + \cos x, \quad 0 \leqslant x \leqslant \pi$

15. $f(t)$ is the height of a football t seconds after kickoff.

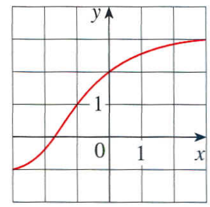
16. $f(t)$ is your height at age t.

17. If f is a one-to-one function such that $f(2) = 9$, what is $f^{-1}(9)$?

18. If $f(x) = x + \cos x$, find $f^{-1}(1)$.

19. If $h(x) = x + \sqrt{x}$, find $h^{-1}(6)$.

20. The graph of f is given.
(a) Why is f one-to-one?
(b) What are the domain and range of f^{-1}?
(c) What is the value of $f^{-1}(2)$?
(d) Estimate the value of $f^{-1}(0)$.



21. The formula $C = \frac{5}{9}(F - 32)$, where $F \geqslant -459.67$, expresses the Celsius temperature C as a function of the Fahrenheit temperature F. Find a formula for the inverse function and interpret it. What is the domain of the inverse function?

22. In the theory of relativity, the mass of a particle with speed v is

$$m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle and c is the speed of light in a vacuum. Find the inverse function of f and explain its meaning.

23–28 Find a formula for the inverse of the function.

23. $f(x) = 3 - 2x$

24. $f(x) = \dfrac{4x - 1}{2x + 3}$

25. $f(x) = \sqrt{10 - 3x}$

26. $y = 2x^3 + 3$

27. $y = \dfrac{1 - \sqrt{x}}{1 + \sqrt{x}}$

28. $f(x) = 2x^2 - 8x, \quad x \geqslant 2$

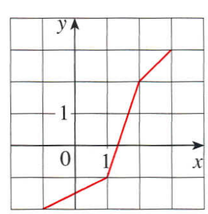
29–30 Find an explicit formula for f^{-1} and use it to graph f^{-1}, f, and the line $y = x$ on the same screen. To check your work, see whether the graphs of f and f^{-1} are reflections about this line.

29. $f(x) = x^4 + 1, \quad x \geqslant 0$

30. $f(x) = \sqrt{x^2 + 2x}, \quad x > 0$

31–32 Use the given graph of f to sketch the graph of f^{-1}.

31.



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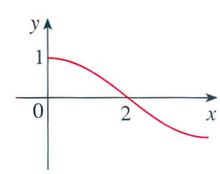


33–36
(a) Show that f is one-to-one.
(b) Use Theorem 7 to find $(f^{-1})'(a)$.
(c) Calculate $f^{-1}(x)$ and state the domain and range of f^{-1}.

(d) Calculate $(f^{-1})'(a)$ from the formula in part (c) and check that it agrees with the result of part (b).
(e) Sketch the graphs of f and f^{-1} on the same axes.

33. $f(x) = x^3, \quad a = 8$

34. $f(x) = \sqrt{x - 2}, \quad a = 2$

35. $f(x) = 9 - x^2, \quad 0 \leqslant x \leqslant 3, \quad a = 8$

36. $f(x) = 1/(x - 1), \quad x > 1, \quad a = 2$

37–40 Find $(f^{-1})'(a)$.

37. $f(x) = 2x^3 + 3x^2 + 7x + 4, \quad a = 4$

38. $f(x) = x^3 + 3 \sin x + 2 \cos x, \quad a = 2$

39. $f(x) = 3 + x^2 + \tan(\pi x/2), \quad -1 < x < 1, \quad a = 3$

40. $f(x) = \sqrt{x^3 + x^2 + x + 1}, \quad a = 2$

41. Suppose f^{-1} is the inverse function of a differentiable function f and $f(4) = 5$, $f'(4) = \frac{2}{3}$. Find $(f^{-1})'(5)$.

42. Suppose f^{-1} is the inverse function of a differentiable function f and let $G(x) = 1/f^{-1}(x)$. If $f(3) = 2$ and $f'(3) = \frac{1}{9}$, find $G'(2)$.

CAS **43.** Use a computer algebra system to find an explicit expression for the inverse function $f(x) = \sqrt{x^3 + x^2 + x + 1}$. (Your CAS will produce three possible expressions. Explain why two of them are irrelevant in this context.)

44. Show that $h(x) = \sin x$, $x \in \mathbb{R}$, is not one-to-one, but its restriction $f(x) = \sin x$, $-\pi/2 \leqslant x \leqslant \pi/2$, is one-to-one. Compute the derivative of $f^{-1} = \sin^{-1}$ by the method of Note 2.

45. (a) If we shift a curve to the left, what happens to its reflection about the line $y = x$? In view of this geometric principle, find an expression for the inverse of $g(x) = f(x + c)$, where f is a one-to-one function.
(b) Find an expression for the inverse of $h(x) = f(cx)$, where $c \neq 0$.

46. (a) If f is a one-to-one, twice differentiable function with inverse function g, show that

$$g''(x) = -\frac{f''(g(x))}{[f'(g(x))]^3}$$

(b) Deduce that if f is increasing and concave upward, then its inverse function is concave downward.

7.2 EXPONENTIAL FUNCTIONS AND THEIR DERIVATIVES

■ If your instructor has assigned Sections 7.2*, 7.3*, and 7.4*, you don't need to read Sections 7.2–7.4 (pp. 392–421).

The function $f(x) = 2^x$ is called an *exponential function* because the variable, x, is the exponent. It should not be confused with the power function $g(x) = x^2$, in which the variable is the base.

In general, an **exponential function** is a function of the form

$$f(x) = a^x$$

where a is a positive constant. Let's recall what this means.

If $x = n$, a positive integer, then

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$$

If $x = 0$, then $a^0 = 1$, and if $x = -n$, where n is a positive integer, then

$$a^{-n} = \frac{1}{a^n}$$

If x is a rational number, $x = p/q$, where p and q are integers and $q > 0$, then

$$a^x = a^{p/q} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$$

But what is the meaning of a^x if x is an irrational number? For instance, what is meant by $2^{\sqrt{3}}$ or 5^{π}?

To help us answer this question we first look at the graph of the function $y = 2^x$, where x is rational. A representation of this graph is shown in Figure 1. We want to enlarge the domain of $y = 2^x$ to include both rational and irrational numbers.

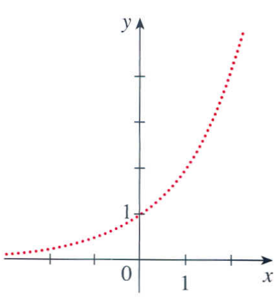


FIGURE 1
Representation of $y = 2^x$, x rational

7.2 EXERCISES

1. (a) Write an equation that defines the exponential function with base $a > 0$.
(b) What is the domain of this function?
(c) If $a \neq 1$, what is the range of this function?
(d) Sketch the general shape of the graph of the exponential function for each of the following cases.
(i) $a > 1$ (ii) $a = 1$ (iii) $0 < a < 1$

2. (a) How is the number e defined?
(b) What is an approximate value for e?
(c) What is the natural exponential function?

3–6 Graph the given functions on a common screen. How are these graphs related?

3. $y = 2^x$, $y = e^x$, $y = 5^x$, $y = 20^x$

4. $y = e^x$, $y = e^{-x}$, $y = 8^x$, $y = 8^{-x}$

5. $y = 3^x$, $y = 10^x$, $y = \left(\frac{1}{3}\right)^x$, $y = \left(\frac{1}{10}\right)^x$

6. $y = 0.9^x$, $y = 0.6^x$, $y = 0.3^x$, $y = 0.1^x$

7–12 Make a rough sketch of the graph of the function. Do not use a calculator. Just use the graphs given in Figures 3 and 12 and, if necessary, the transformations of Section 1.3.

7. $y = 4^x - 3$

8. $y = 4^{x-3}$

9. $y = -2^{-x}$

10. $y = 1 + 2e^x$

11. $y = 1 - \frac{1}{2}e^{-x}$

12. $y = 2 + 5(1 - e^{-x})$

13. Starting with the graph of $y = e^x$, write the equation of the graph that results from
(a) shifting 2 units downward
(b) shifting 2 units to the right
(c) reflecting about the x-axis
(d) reflecting about the y-axis
(e) reflecting about the x-axis and then about the y-axis

14. Starting with the graph of $y = e^x$, find the equation of the graph that results from
(a) reflecting about the line $y = 4$
(b) reflecting about the line $x = 2$

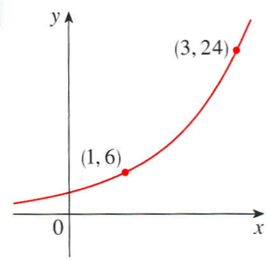
15–16 Find the domain of each function.

15. (a) $f(x) = \dfrac{1}{1 + e^x}$ (b) $f(x) = \dfrac{1}{1 - e^x}$

16. (a) $g(t) = \sin(e^{-t})$ (b) $g(t) = \sqrt{1 - 2^t}$

17–18 Find the exponential function $f(x) = Ca^x$ whose graph is given.

17.



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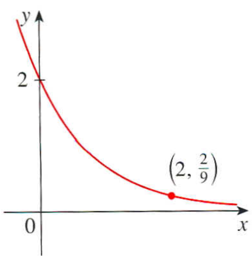

19. Suppose the graphs of $f(x) = x^2$ and $g(x) = 2^x$ are drawn on a coordinate grid where the unit of measurement is 1 cm. Show that, at a distance 1 m to the right of the origin, the height of the graph of f is 100 m but the height of the graph of g is more than 10^{25} km.

20. Compare the rates of growth of the functions $f(x) = x^5$ and $g(x) = 5^x$ by graphing both functions in several viewing rectangles. Find all points of intersection of the graphs correct to one decimal place.

21. Compare the functions $f(x) = x^{10}$ and $g(x) = e^x$ by graphing both f and g in several viewing rectangles. When does the graph of g finally surpass the graph of f?

22. Use a graph to estimate the values of x such that $e^x > 1{,}000{,}000{,}000$.

23–30 Find the limit.

23. $\lim_{x\to\infty} (1.001)^x$

24. $\lim_{x\to-\infty} (1.001)^x$

25. $\lim_{x\to\infty} \dfrac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$

26. $\lim_{x\to\infty} e^{-x^2}$

27. $\lim_{x\to 2^+} e^{3/(2-x)}$

28. $\lim_{x\to 2^-} e^{3/(2-x)}$

29. $\lim_{x\to\infty} (e^{-2x}\cos x)$

30. $\lim_{x\to(\pi/2)^+} e^{\tan x}$

31–46 Differentiate the function.

31. $f(x) = x^2 e^x$

32. $y = \dfrac{e^x}{1 + x}$

33. $y = e^{ax^3}$

34. $y = e^u(\cos u + cu)$

35. $f(u) = e^{1/u}$

36. $g(x) = \sqrt{x}\, e^x$

37. $F(t) = e^{t \sin 2t}$

38. $f(t) = \sin(e^t) + e^{\sin t}$

39. $y = \sqrt{1 + 2e^{3x}}$

40. $y = e^{k \tan \sqrt{x}}$

41. $y = e^{e^x}$

42. $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

43. $y = \dfrac{ae^x + b}{ce^x + d}$

44. $y = \sqrt{1 + xe^{-2x}}$

45. $y = \cos\left(\dfrac{1 - e^{2x}}{1 + e^{2x}}\right)$

46. $f(t) = \sin^2(e^{\sin^2 t})$

47–48 Find an equation of the tangent line to the curve at the given point.

47. $y = e^{2x} \cos \pi x, \quad (0, 1)$

48. $y = \dfrac{e^x}{x}, \quad (1, e)$

49. Find y' if $e^{x^2 y} = x + y$.

50. Find an equation of the tangent line to the curve $xe^y + ye^x = 1$ at the point $(0, 1)$.

51. Show that the function $y = e^x + e^{-x/2}$ satisfies the differential equation $2y'' - y' - y = 0$.

52. Show that the function $y = Ae^{-x} + Bxe^{-x}$ satisfies the differential equation $y'' + 2y' + y = 0$.

53. For what values of r does the function $y = e^{rx}$ satisfy the equation $y'' + 6y' + 8y = 0$?

54. Find the values of λ for which $y = e^{\lambda x}$ satisfies the equation $y + y' = y''$.

55. If $f(x) = e^{2x}$, find a formula for $f^{(n)}(x)$.

56. Find the thousandth derivative of $f(x) = xe^{-x}$.

57. (a) Use the Intermediate Value Theorem to show that there is a root of the equation $e^x + x = 0$.
(b) Use Newton's method to find the root of the equation in part (a) correct to six decimal places.

58. Use a graph to find an initial approximation (to one decimal place) to the root of the equation $4e^{-x^2} \sin x = x^2 - x + 1$. Then use Newton's method to find the root correct to eight decimal places.

59. Under certain circumstances a rumor spreads according to the equation

$$p(t) = \frac{1}{1 + ae^{-kt}}$$

where $p(t)$ is the proportion of the population that knows the rumor at time t and a and k are positive constants. [In Section 10.4 we will see that this is a reasonable model for $p(t)$.]
(a) Find $\lim_{t \to \infty} p(t)$.
(b) Find the rate of spread of the rumor.
(c) Graph p for the case $a = 10$, $k = 0.5$ with t measured in hours. Use the graph to estimate how long it will take for 80% of the population to hear the rumor.

60. An object is attached to the end of a vibrating spring and its displacement from its equilibrium position is $y = 8e^{-t/2} \sin 4t$, where t is measured in seconds and y is measured in centimeters.
(a) Graph the displacement function together with the functions $y = 8e^{-t/2}$ and $y = -8e^{-t/2}$. How are these graphs related? Can you explain why?
(b) Use the graph to estimate the maximum value of the displacement. Does it occur when the graph touches the graph of $y = 8e^{-t/2}$?
(c) What is the velocity of the object when it first returns to its equilibrium position?
(d) Use the graph to estimate the time after which the displacement is no more than 2 cm from equilibrium.

61. Find the absolute maximum value of the function $f(x) = x - e^x$.

62. Find the absolute minimum value of the function $g(x) = e^x/x$, $x > 0$.

63–64 Find the absolute maximum and absolute minimum values of f on the given interval.

63. $f(x) = xe^{-x^2/8}, \quad [-1, 4]$

64. $f(x) = x^2 e^{-x/2}, \quad [-1, 6]$

65–66 Find (a) the intervals of increase or decrease, (b) the intervals of concavity, and (c) the points of inflection.

65. $f(x) = (1 - x)e^{-x}$

66. $f(x) = \dfrac{e^x}{x^2}$

67–68 Discuss the curve using the guidelines of Section 4.5.

67. $y = e^{-1/(x+1)}$

68. $y = e^{-x} \sin x$

69. A *drug response curve* describes the level of medication in the bloodstream after a drug is administered. A surge function $S(t) = At^p e^{-kt}$ is often used to model the response curve, reflecting an initial surge in the drug level and then a more gradual decline. If, for a particular drug, $A = 0.01$, $p = 4$, $k = 0.07$, and t is measured in minutes, estimate the times corresponding to the inflection points and explain their significance. If you have a graphing device, use it to graph the drug response curve.

70–71 Draw a graph of f that shows all the important aspects of the curve. Estimate the local maximum and minimum values and then use calculus to find these values exactly. Use a graph of f'' to estimate the inflection points.

70. $f(x) = e^{\cos x}$

71. $f(x) = e^{x^3 - x}$

72. The family of bell-shaped curves

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

occurs in probability and statistics, where it is called the *normal density function.* The constant μ is called the *mean* and the positive constant σ is called the *standard deviation.* For simplicity, let's scale the function so as to remove the factor $1/(\sigma\sqrt{2\pi})$ and let's analyze the special case where $\mu = 0$. So we study the function

$$f(x) = e^{-x^2/(2\sigma^2)}$$

(a) Find the asymptote, maximum value, and inflection points of f.
(b) What role does σ play in the shape of the curve?
(c) Illustrate by graphing four members of this family on the same screen.

73–82 Evaluate the integral.

73. $\int_0^5 e^{-3x}\,dx$

74. $\int_0^1 xe^{-x^2}\,dx$

75. $\int e^x\sqrt{1 + e^x}\,dx$

76. $\int \frac{(1 + e^x)^2}{e^x}\,dx$

77. $\int \frac{e^x + 1}{e^x}\,dx$

78. $\int e^x(4 + e^x)^5\,dx$

79. $\int \sin x\, e^{\cos x}\,dx$

80. $\int \frac{e^{1/x}}{x^2}\,dx$

81. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

82. $\int e^x \sin(e^x)\,dx$

83. Find, correct to three decimal places, the area of the region bounded by the curves $y = e^x$, $y = e^{3x}$, and $x = 1$.

84. Find $f(x)$ if $f''(x) = 3e^x + 5\sin x$, $f(0) = 1$, and $f'(0) = 2$.

85. Find the volume of the solid obtained by rotating about the x-axis the region bounded by the curves $y = e^x$, $y = 0$, $x = 0$, and $x = 1$.

86. Find the volume of the solid obtained by rotating about the y-axis the region bounded by the curves $y = e^{-x^2}$, $y = 0$, $x = 0$, and $x = 1$.

87. The **error function**

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

is used in probability, statistics, and engineering.
(a) Show that $\int_a^b e^{-t^2}\,dt = \frac{1}{2}\sqrt{\pi}\,[\operatorname{erf}(b) - \operatorname{erf}(a)]$.
(b) Show that the function $y = e^{x^2}\operatorname{erf}(x)$ satisfies the differential equation $y' = 2xy + 2/\sqrt{\pi}$.

88. A bacteria population starts with 400 bacteria and grows at a rate of $r(t) = (450.268)e^{1.12567t}$ bacteria per hour. How many bacteria will there be after three hours?

89. If $f(x) = 3 + x + e^x$, find $(f^{-1})'(4)$.

90. Evaluate $\lim_{x \to \pi} \frac{e^{\sin x} - 1}{x - \pi}$.

91. If you graph the function

$$f(x) = \frac{1 - e^{1/x}}{1 + e^{1/x}}$$

you'll see that f appears to be an odd function. Prove it.

92. Graph several members of the family of functions

$$f(x) = \frac{1}{1 + ae^{bx}}$$

where $a > 0$. How does the graph change when b changes? How does it change when a changes?

93. (a) Show that $e^x \geqslant 1 + x$ if $x \geqslant 0$.
[*Hint:* Show that $f(x) = e^x - (1 + x)$ is increasing for $x > 0$.]
(b) Deduce that $\frac{4}{3} \leqslant \int_0^1 e^{x^2}\,dx \leqslant e$.

94. (a) Use the inequality of Exercise 93(a) to show that, for $x \geqslant 0$,

$$e^x \geqslant 1 + x + \tfrac{1}{2}x^2$$

(b) Use part (a) to improve the estimate of $\int_0^1 e^{x^2}\,dx$ given in Exercise 93(b).

95. (a) Use mathematical induction to prove that for $x \geqslant 0$ and any positive integer n,

$$e^x \geqslant 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

(b) Use part (a) to show that $e > 2.7$.
(c) Use part (a) to show that

$$\lim_{x \to \infty} \frac{e^x}{x^k} = \infty$$

for any positive integer k.

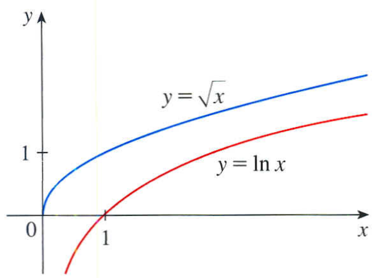

FIGURE 5

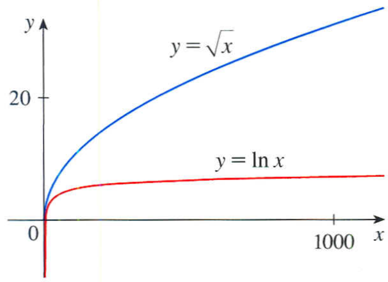

FIGURE 6

We have seen that $\ln x \to \infty$ as $x \to \infty$. But this happens *very* slowly. In fact, $\ln x$ grows more slowly than any positive power of x. To illustrate this fact, we compare approximate values of the functions $y = \ln x$ and $y = x^{1/2} = \sqrt{x}$ in the following table and we graph them in Figures 5 and 6.

x	1	2	5	10	50	100	500	1000	10,000	100,000
$\ln x$	0	0.69	1.61	2.30	3.91	4.6	6.2	6.9	9.2	11.5
$\sqrt{x}$	1	1.41	2.24	3.16	7.07	10.0	22.4	31.6	100	316
$\dfrac{\ln x}{\sqrt{x}}$	0	0.49	0.72	0.73	0.55	0.46	0.28	0.22	0.09	0.04

You can see that initially the graphs of $y = \sqrt{x}$ and $y = \ln x$ grow at comparable rates, but eventually the root function far surpasses the logarithm. In fact, we will be able to show in Section 7.8 that

$$\lim_{x\to\infty} \frac{\ln x}{x^p} = 0$$

for any positive power p. So for large x, the values of $\ln x$ are very small compared with x^p. (See Exercise 70.)

7.3 EXERCISES

1. (a) How is the logarithmic function $y = \log_a x$ defined?
(b) What is the domain of this function?
(c) What is the range of this function?
(d) Sketch the general shape of the graph of the function $y = \log_a x$ if $a > 1$.

2. (a) What is the natural logarithm?
(b) What is the common logarithm?
(c) Sketch the graphs of the natural logarithm function and the natural exponential function with a common set of axes.

3–8 Find the exact value of each expression.

3. (a) $\log_{10} 1000$ (b) $\log_{16} 4$

4. (a) $\ln e^{-100}$ (b) $\log_3 81$

5. (a) $\log_5 \frac{1}{25}$ (b) $e^{\ln 15}$

6. (a) $\log_{10} 0.1$ (b) $\log_8 320 - \log_8 5$

7. (a) $\log_2 6 - \log_2 15 + \log_2 20$
(b) $\log_3 100 - \log_3 18 - \log_3 50$

8. (a) $e^{-2\ln 5}$ (b) $\ln(\ln e^{e^{10}})$

9–12 Use the properties of logarithms to expand the quantity.

9. $\log_2\left(\dfrac{x^3 y}{z^2}\right)$

10. $\ln\sqrt{a(b^2+c^2)}$

11. $\ln(uv)^{10}$

12. $\ln\dfrac{3x^2}{(x+1)^5}$

13–18 Express the quantity as a single logarithm.

13. $\log_{10} a - \log_{10} b + \log_{10} c$

14. $\ln(x+y) + \ln(x-y) - 2\ln z$

15. $\ln 5 + 5\ln 3$

16. $\ln 3 + \frac{1}{3}\ln 8$

17. $\ln(1+x^2) + \frac{1}{2}\ln x - \ln \sin x$

18. $\ln(a+b) + \ln(a-b) - 2\ln c$

19. Use Formula 7 to evaluate each logarithm correct to six decimal places.
(a) $\log_{12} e$ (b) $\log_6 13.54$ (c) $\log_2 \pi$

20–22 Use Formula 7 to graph the given functions on a common screen. How are these graphs related?

20. $y = \log_2 x$, $y = \log_4 x$, $y = \log_6 x$, $y = \log_8 x$

21. $y = \log_{1.5} x$, $y = \ln x$, $y = \log_{10} x$, $y = \log_{50} x$

22. $y = \ln x$, $y = \log_{10} x$, $y = e^x$, $y = 10^x$

23–24 Make a rough sketch of the graph of each function. Do not use a calculator. Just use the graphs given in Figures 2 and 3 and, if necessary, the transformations of Section 1.3.

23. (a) $y = \log_{10}(x + 5)$ (b) $y = -\ln x$

24. (a) $y = \ln(-x)$ (b) $y = \ln|x|$

25–34 Solve each equation for x.

25. (a) $2 \ln x = 1$ (b) $e^{-x} = 5$

26. (a) $e^{2x+3} - 7 = 0$ (b) $\ln(5 - 2x) = -3$

27. (a) $2^{x-5} = 3$ (b) $\ln x + \ln(x - 1) = 1$

28. (a) $e^{3x+1} = k$ (b) $\log_2(mx) = c$

29. $3xe^x + x^2e^x = 0$

30. $10(1 + e^{-x})^{-1} = 3$

31. $\ln(\ln x) = 1$

32. $e^{e^x} = 10$

33. $e^{2x} - e^x - 6 = 0$

34. $\ln(2x + 1) = 2 - \ln x$

35–36 Find the solution of the equation correct to four decimal places.

35. (a) $e^{2+5x} = 100$ (b) $\ln(e^x - 2) = 3$

36. (a) $\ln(1 + \sqrt{x}) = 2$ (b) $3^{1/(x-4)} = 7$

37–38 Solve each inequality for x.

37. (a) $e^x < 10$ (b) $\ln x > -1$

38. (a) $2 < \ln x < 9$ (b) $e^{2-3x} > 4$

39. Suppose that the graph of $y = \log_2 x$ is drawn on a coordinate grid where the unit of measurement is a centimeter. How many kilometers to the right of the origin do we have to move before the height of the curve reaches 1 m?

40. The velocity of a particle that moves in a straight line under the influence of viscous forces is $v(t) = ce^{-kt}$, where c and k are positive constants.
(a) Show that the acceleration is proportional to the velocity.
(b) Explain the significance of the number c.
(c) At what time is the velocity equal to half the initial velocity?

41. The geologist C. F. Richter defined the magnitude of an earthquake to be $\log_{10}(I/S)$, where I is the intensity of the quake (measured by the amplitude of a seismograph 100 km from the epicenter) and S is the intensity of a "standard" earthquake (where the amplitude is only 1 micron $= 10^{-4}$ cm). The 1989 Loma Prieta earthquake that shook San Francisco had a magnitude of 7.1 on the Richter scale. The 1906 San Francisco earthquake was 16 times as intense. What was its magnitude on the Richter scale?

42. A sound so faint that it can just be heard has intensity $I_0 = 10^{-12}$ watt/m^2 at a frequency of 1000 hertz (Hz). The loudness, in decibels (dB), of a sound with intensity I is then defined to be $L = 10 \log_{10}(I/I_0)$. Amplified rock music is measured at 120 dB, whereas the noise from a motor-driven lawn mower is measured at 106 dB. Find the ratio of the intensity of the rock music to that of the mower.

43. If a bacteria population starts with 100 bacteria and doubles every three hours, then the number of bacteria after t hours is $n = f(t) = 100 \cdot 2^{t/3}$.
(a) Find the inverse of this function and explain its meaning.
(b) When will the population reach 50,000?

44. When a camera flash goes off, the batteries immediately begin to recharge the flash's capacitor, which stores electric charge given by

$$Q(t) = Q_0(1 - e^{-t/a})$$

(The maximum charge capacity is Q_0 and t is measured in seconds.)
(a) Find the inverse of this function and explain its meaning.
(b) How long does it take to recharge the capacitor to 90% of capacity if $a = 2$?

45–50 Find the limit.

45. $\lim_{x \to 3^+} \ln(x^2 - 9)$

46. $\lim_{x \to 2^-} \log_5(8x - x^4)$

47. $\lim_{x \to 0} \ln(\cos x)$

48. $\lim_{x \to 0^+} \ln(\sin x)$

49. $\lim_{x \to \infty} [\ln(1 + x^2) - \ln(1 + x)]$

50. $\lim_{x \to \infty} [\ln(2 + x) - \ln(1 + x)]$

51–52 Find the domain of the function.

51. $f(x) = \log_{10}(x^2 - 9)$

52. $f(x) = \ln x + \ln(2 - x)$

53–54 Find (a) the domain of f and (b) f^{-1} and its domain.

53. $f(x) = \sqrt{3 - e^{2x}}$

54. $f(x) = \ln(2 + \ln x)$

55–60 Find the inverse function.

55. $y = \ln(x + 3)$

56. $y = 2^{10^x}$

57. $f(x) = e^{x^3}$

58. $y = (\ln x)^2, \quad x \geqslant 1$

59. $y = \log_{10}\left(1 + \dfrac{1}{x}\right)$

60. $y = \dfrac{e^x}{1 + 2e^x}$

61. On what interval is the function $f(x) = e^{3x} - e^x$ increasing?

62. On what interval is the curve $y = 2e^x - e^{-3x}$ concave downward?

63. On what intervals is the curve $y = (x^2 - 2)e^{-x}$ concave upward?

64. For the period from 1980 to 2000, the percentage of households in the United States with at least one VCR has been modeled by the function

$$V(t) = \frac{85}{1 + 53e^{-0.5t}}$$

where the time t is measured in years since midyear 1980, so $0 \leq t \leq 20$. Use a graph to estimate the time at which the number of VCRs was increasing most rapidly. Then use derivatives to give a more accurate estimate.

65. (a) Show that the function $f(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$ is an odd function.
(b) Find the inverse function of f.

66. Find an equation of the tangent to the curve $y = e^{-x}$ that is perpendicular to the line $2x - y = 8$.

67. Show that the equation $x^{1/\ln x} = 2$ has no solution. What can you say about the function $f(x) = x^{1/\ln x}$?

68. Any function of the form $f(x) = [g(x)]^{h(x)}$, where $g(x) > 0$, can be analyzed as a power of e by writing $g(x) = e^{\ln g(x)}$ so that $f(x) = e^{h(x)\ln g(x)}$. Using this device, calculate each limit.
(a) $\lim_{x \to \infty} x^{\ln x}$ (b) $\lim_{x \to 0^+} x^{-\ln x}$
(c) $\lim_{x \to 0^+} x^{1/x}$ (d) $\lim_{x \to \infty} (\ln 2x)^{-\ln x}$

69. Let $a > 1$. Prove, using Definitions 4.4.6 and 4.4.7, that
(a) $\lim_{x \to -\infty} a^x = 0$ (b) $\lim_{x \to \infty} a^x = \infty$

70. (a) Compare the rates of growth of $f(x) = x^{0.1}$ and $g(x) = \ln x$ by graphing both f and g in several viewing rectangles. When does the graph of f finally surpass the graph of g?
(b) Graph the function $h(x) = (\ln x)/x^{0.1}$ in a viewing rectangle that displays the behavior of the function as $x \to \infty$.
(c) Find a number N such that

$$\text{if} \quad x > N \quad \text{then} \quad \frac{\ln x}{x^{0.1}} < 0.1$$

71. Solve the inequality $\ln(x^2 - 2x - 2) \leq 0$.

72. A **prime number** is a positive integer that has no factors other than 1 and itself. The first few primes are 2, 3, 5, 7, 11, 13, 17, ... We denote by $\pi(n)$ the number of primes that are less than or equal to n. For instance, $\pi(15) = 6$ because there are six primes smaller than 15.
(a) Calculate the numbers $\pi(25)$ and $\pi(100)$.
[*Hint:* To find $\pi(100)$, first compile a list of the primes up to 100 using the *sieve of Eratosthenes:* Write the numbers from 2 to 100 and cross out all multiples of 2. Then cross out all multiples of 3. The next remaining number is 5, so cross out all remaining multiples of it, and so on.]
(b) By inspecting tables of prime numbers and tables of logarithms, the great mathematician K. F. Gauss made the guess in 1792 (when he was 15) that the number of primes up to n is approximately $n/\ln n$ when n is large. More precisely, he conjectured that

$$\lim_{n \to \infty} \frac{\pi(n)}{n/\ln n} = 1$$

This was finally proved, a hundred years later, by Jacques Hadamard and Charles de la Vallée Poussin and is called the **Prime Number Theorem**. Provide evidence for the truth of this theorem by computing the ratio of $\pi(n)$ to $n/\ln n$ for $n = 100, 1000, 10^4, 10^5, 10^6$, and 10^7. Use the following data: $\pi(1000) = 168$, $\pi(10^4) = 1229$, $\pi(10^5) = 9592$, $\pi(10^6) = 78{,}498$, $\pi(10^7) = 664{,}579$.
(c) Use the Prime Number Theorem to estimate the number of primes up to a billion.

7.4 DERIVATIVES OF LOGARITHMIC FUNCTIONS

In this section we find the derivatives of the logarithmic functions $y = \log_a x$ and the exponential functions $y = a^x$. We start with the natural logarithmic function $y = \ln x$. We know that it is differentiable because it is the inverse of the differentiable function $y = e^x$.

1

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

PROOF Let $y = \ln x$. Then

$$e^y = x$$

Because $f'(1) = 1$, we have

$$\lim_{x \to 0} \ln(1 + x)^{1/x} = 1$$

Then, by Theorem 2.5.8 and the continuity of the exponential function, we have

$$e = e^1 = e^{\lim_{x \to 0} \ln(1+x)^{1/x}} = \lim_{x \to 0} e^{\ln(1+x)^{1/x}} = \lim_{x \to 0} (1 + x)^{1/x}$$

8 $$e = \lim_{x \to 0} (1 + x)^{1/x}$$

Formula 8 is illustrated by the graph of the function $y = (1 + x)^{1/x}$ in Figure 6 and a table of values for small values of x.

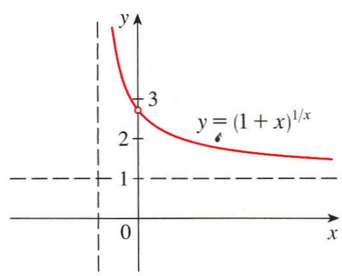

FIGURE 6

x	$(1 + x)^{1/x}$
0.1	2.59374246
0.01	2.70481383
0.001	2.71692393
0.0001	2.71814593
0.00001	2.71826824
0.000001	2.71828047
0.0000001	2.71828169
0.00000001	2.71828181

If we put $n = 1/x$ in Formula 8, then $n \to \infty$ as $x \to 0^+$ and so an alternative expression for e is

9 $$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

7.4 EXERCISES

1. Explain why the natural logarithmic function $y = \ln x$ is used much more frequently in calculus than the other logarithmic functions $y = \log_a x$.

2–26 Differentiate the function.

2. $f(x) = \ln(x^2 + 10)$

3. $f(\theta) = \ln(\cos \theta)$

4. $f(x) = \cos(\ln x)$

5. $f(x) = \log_2(1 - 3x)$

6. $f(x) = \log_5(xe^x)$

7. $f(x) = \sqrt[5]{\ln x}$

8. $f(x) = \ln \sqrt[5]{x}$

9. $f(x) = \sin x \ln(5x)$

10. $f(t) = \dfrac{1 + \ln t}{1 - \ln t}$

11. $F(t) = \ln \dfrac{(2t + 1)^3}{(3t - 1)^4}$

12. $h(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$

13. $g(x) = \ln\left(x\sqrt{x^2 - 1}\right)$

14. $F(y) = y \ln(1 + e^y)$

15. $f(u) = \dfrac{\ln u}{1 + \ln(2u)}$

16. $y = \ln(x^4 \sin^2 x)$

17. $h(t) = t^3 - 3^t$

18. $y = 10^{\tan \theta}$

19. $y = \ln|2 - x - 5x^2|$

20. $H(z) = \ln \sqrt{\dfrac{a^2 - z^2}{a^2 + z^2}}$

21. $y = \ln(e^{-x} + xe^{-x})$

22. $y = [\ln(1 + e^x)]^2$

23. $y = 2x \log_{10} \sqrt{x}$

24. $y = \log_2(e^{-x} \cos \pi x)$

25. $y = 5^{-1/x}$

26. $y = 2^{3^{x^2}}$

27–30 Find y' and y''.

27. $y = x^2 \ln(2x)$

28. $y = (\ln x)/x^2$

29. $y = \ln\left(x + \sqrt{1 + x^2}\right)$

30. $y = \ln(\sec x + \tan x)$

31–34 Differentiate f and find the domain of f.

31. $f(x) = \dfrac{x}{1 - \ln(x - 1)}$

32. $f(x) = \dfrac{1}{1 + \ln x}$

33. $f(x) = \ln(x^2 - 2x)$

34. $f(x) = \ln \ln \ln x$

35. If $f(x) = \dfrac{\ln x}{1 + x^2}$, find $f'(1)$.

36. If $f(x) = \ln(1 + e^{2x})$, find $f'(0)$.

37–38 Find an equation of the tangent line to the curve at the given point.

37. $y = \ln\left(xe^{x^2}\right)$, $(1, 1)$

38. $y = \ln(x^3 - 7)$, $(2, 0)$

39. If $f(x) = \sin x + \ln x$, find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

40. Find equations of the tangent lines to the curve $y = (\ln x)/x$ at the points $(1, 0)$ and $(e, 1/e)$. Illustrate by graphing the curve and its tangent lines.

41–52 Use logarithmic differentiation to find the derivative of the function.

41. $y = (2x + 1)^5(x^4 - 3)^6$

42. $y = \sqrt{x}\, e^{x^2}(x^2 + 1)^{10}$

43. $y = \dfrac{\sin^2 x \tan^4 x}{(x^2 + 1)^2}$

44. $y = \sqrt[4]{\dfrac{x^2 + 1}{x^2 - 1}}$

45. $y = x^x$

46. $y = x^{\cos x}$

47. $y = x^{\sin x}$

48. $y = \sqrt{x}^{\,x}$

49. $y = (\cos x)^x$

50. $y = (\sin x)^{\ln x}$

51. $y = (\tan x)^{1/x}$

52. $y = (\ln x)^{\cos x}$

53. Find y' if $y = \ln(x^2 + y^2)$.

54. Find y' if $x^y = y^x$.

55. Find a formula for $f^{(n)}(x)$ if $f(x) = \ln(x - 1)$.

56. Find $\dfrac{d^9}{dx^9}(x^8 \ln x)$.

57–58 Use a graph to estimate the roots of the equation correct to one decimal place. Then use these estimates as the initial approximations in Newton's method to find the roots correct to six decimal places.

57. $(x - 4)^2 = \ln x$

58. $\ln(4 - x^2) = x$

59. Find the intervals of concavity and the inflection points of the function $f(x) = (\ln x)/\sqrt{x}$.

60. Find the absolute minimum value of the function $f(x) = x \ln x$.

61–64 Discuss the curve under the guidelines of Section 4.5.

61. $y = \ln(\sin x)$

62. $y = \ln(\tan^2 x)$

63. $y = \ln(1 + x^2)$

64. $y = \ln(x^2 - 3x + 2)$

CAS **65.** If $f(x) = \ln(2x + x \sin x)$, use the graphs of f, f', and f'' to estimate the intervals of increase and the inflection points of f on the interval $(0, 15]$.

66. Investigate the family of curves $f(x) = \ln(x^2 + c)$. What happens to the inflection points and asymptotes as c changes? Graph several members of the family to illustrate what you discover.

67. The flash unit on a camera operates by storing charge on a capacitor and releasing it suddenly when the flash is set off. The following data describe the charge Q remaining on the capacitor (measured in microcoulombs, μC) at time t (measured in seconds).

t	0.00	0.02	0.04	0.06	0.08	0.10
Q	100.00	81.87	67.03	54.88	44.93	36.76

(a) Use a graphing calculator or computer to find an exponential model for the charge.

(b) The derivative $Q'(t)$ represents the electric current (measured in microamperes, μA) flowing from the capacitor to the flash bulb. Use part (a) to estimate the current when $t = 0.04$ s. Compare with the result of Example 2 in Section 2.1.

68. The table gives the population of Mexico (in millions) in the 20th century census years.

Year	Population	Year	Population
1900	13.6	1960	34.9
1910	15.2	1970	48.2
1920	14.3	1980	66.8
1930	16.6	1990	81.2
1940	19.7	2000	97.5
1950	25.8		

(a) Use a graphing calculator or computer to fit an exponential function to the data. Graph the data points and the exponential model. How good is the fit?

(b) Estimate the rates of population growth in 1950 and 1960 by averaging slopes of secant lines.

(c) Use your exponential model in part (a) to find a model for the rate of growth of the Mexican population in the 20th century.

(d) Use your model in part (c) to estimate the rates of growth in 1950 and 1960. Compare with your estimates in part (b).

69–80 Evaluate the integral.

69. $\int_2^4 \frac{3}{x}\,dx$

70. $\int_1^2 \frac{4+u^2}{u^3}\,du$

71. $\int_1^2 \frac{dt}{8-3t}$

72. $\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

73. $\int_1^e \frac{x^2+x+1}{x}\,dx$

74. $\int \frac{\sin(\ln x)}{x}\,dx$

75. $\int \frac{(\ln x)^2}{x}\,dx$

76. $\int \frac{\cos x}{2+\sin x}\,dx$

77. $\int \frac{\sin 2x}{1+\cos^2 x}\,dx$

78. $\int \frac{e^x}{e^x+1}\,dx$

79. $\int_1^2 10^t\,dt$

80. $\int x2^{x^2}\,dx$

81. Show that $\int \cot x\,dx = \ln|\sin x| + C$ by (a) differentiating the right side of the equation and (b) using the method of Example 11.

82. Find, correct to three decimal places, the area of the region above the hyperbola $y = 2/(x-2)$, below the x-axis, and between the lines $x = -4$ and $x = -1$.

83. Find the volume of the solid obtained by rotating the region under the curve

$$y = \frac{1}{\sqrt{x+1}}$$

from 0 to 1 about the x-axis.

84. Find the volume of the solid obtained by rotating the region under the curve

$$y = \frac{1}{x^2+1}$$

from 0 to 3 about the y-axis.

85. The work done by a gas when it expands from volume V_1 to volume V_2 is $W = \int_{V_1}^{V_2} P\,dV$, where $P = P(V)$ is the pressure as a function of the volume V. (See Exercise 27 in Section 6.4.) Boyle's Law states that when a quantity of gas expands at constant temperature, $PV = C$, where C is a constant. If the initial volume is 600 cm^3 and the initial pressure is 150 kPa, find the work done by the gas when it expands at constant temperature to 1000 cm^3.

86. Find f if $f''(x) = x^{-2}$, $x > 0$, $f(1) = 0$, and $f(2) = 0$.

87. If g is the inverse function of $f(x) = 2x + \ln x$, find $g'(2)$.

88. If $f(x) = e^x + \ln x$ and $h(x) = f^{-1}(x)$, find $h'(e)$.

89. For what values of m do the line $y = mx$ and the curve $y = x/(x^2+1)$ enclose a region? Find the area of the region.

90. (a) Find the linear approximation to $f(x) = \ln x$ near 1.
(b) Illustrate part (a) by graphing f and its linearization.
(c) For what values of x is the linear approximation accurate to within 0.1?

91. Use the definition of derivative to prove that

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$$

92. Show that $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for any $x > 0$.

7.2* THE NATURAL LOGARITHMIC FUNCTION

■ If your instructor has assigned Sections 7.2–7.4, you need not read Sections 7.2*, 7.3*, and 7.4* (pp. 421–446).

In this section we define the natural logarithm as an integral and then show that it obeys the usual laws of logarithms. The Fundamental Theorem makes it easy to differentiate this function.

1 DEFINITION The **natural logarithmic function** is the function defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt \qquad x > 0$$

The existence of this function depends on the fact that the integral of a continuous function always exists. If $x > 1$, then $\ln x$ can be interpreted geometrically as the area under

LOGARITHMIC DIFFERENTIATION

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called **logarithmic differentiation**.

EXAMPLE 14 Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

SOLUTION We take logarithms of both sides of the equation and use the Laws of Logarithms to simplify:

$$\ln y = \tfrac{3}{4}\ln x + \tfrac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - 5\cdot\frac{3}{3x+2}$$

Solving for dy/dx, we get

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

■ If we hadn't used logarithmic differentiation in Example 14, we would have had to use both the Quotient Rule and the Product Rule. The resulting calculation would have been horrendous.

Because we have an explicit expression for y, we can substitute and write

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$ □

STEPS IN LOGARITHMIC DIFFERENTIATION

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.

If $f(x) < 0$ for some values of x, then $\ln f(x)$ is not defined, but we can write $|y| = |f(x)|$ and use Equation 7.

7.2* EXERCISES

1–4 Use the Laws of Logarithms to expand the quantity.

1. $\ln \dfrac{r^2}{3\sqrt{s}}$
2. $\ln \sqrt{a(b^2+c^2)}$
3. $\ln (uv)^{10}$
4. $\ln \dfrac{3x^2}{(x+1)^5}$

5–8 Express the quantity as a single logarithm.

5. $\ln 5 + 5\ln 3$
6. $\ln 3 + \frac{1}{3}\ln 8$
7. $\frac{1}{2}\ln x - 5\ln(x^2+1)$
8. $\ln x + a\ln y - b\ln z$

9–12 Make a rough sketch of the graph of each function. Do not use a calculator. Just use the graph given in Figure 4 and, if necessary, the transformations of Section 1.3.

9. $y = -\ln x$
10. $y = \ln|x|$
11. $y = \ln(x+3)$
12. $y = 1 + \ln(x-2)$

13–14 Find the limit.

13. $\lim_{x\to 3^+} \ln(x^2 - 9)$

14. $\lim_{x\to\infty} [\ln(2 + x) - \ln(1 + x)]$

15–34 Differentiate the function.

15. $f(x) = \sqrt{x}\ \ln x$

16. $f(x) = \ln(x^2 + 10)$

17. $f(\theta) = \ln(\cos\theta)$

18. $f(x) = \cos(\ln x)$

19. $f(x) = \sqrt[5]{\ln x}$

20. $f(x) = \ln\sqrt[5]{x}$

21. $f(x) = \sin x \ln(5x)$

22. $h(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$

23. $g(x) = \ln\dfrac{a - x}{a + x}$

24. $f(t) = \dfrac{1 + \ln t}{1 - \ln t}$

25. $F(t) = \ln\dfrac{(2t + 1)^3}{(3t - 1)^4}$

26. $H(z) = \ln\sqrt{\dfrac{a^2 - z^2}{a^2 + z^2}}$

27. $g(x) = \ln\left(x\sqrt{x^2 - 1}\right)$

28. $y = \ln(x^4 \sin^2 x)$

29. $f(u) = \dfrac{\ln u}{1 + \ln(2u)}$

30. $y = (\ln\tan x)^2$

31. $y = \ln|2 - x - 5x^2|$

32. $y = \ln\tan^2 x$

33. $y = \tan[\ln(ax + b)]$

34. $y = \ln|\cos(\ln x)|$

35–36 Find y' and y''.

35. $y = x^2\ln(2x)$

36. $y = \ln(\sec x + \tan x)$

37–40 Differentiate f and find the domain of f.

37. $f(x) = \dfrac{x}{1 - \ln(x - 1)}$

38. $f(x) = \ln(x^2 - 2x)$

39. $f(x) = \sqrt{1 - \ln x}$

40. $f(x) = \ln\ln\ln x$

41. If $f(x) = \dfrac{\ln x}{1 + x^2}$, find $f'(1)$.

42. If $f(x) = \dfrac{\ln x}{x}$, find $f''(e)$.

43–44 Find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

43. $f(x) = \sin x + \ln x$

44. $f(x) = \ln(x^2 + x + 1)$

45–46 Find an equation of the tangent line to the curve at the given point.

45. $y = \sin(2\ln x)$, $(1, 0)$

46. $y = \ln(x^3 - 7)$, $(2, 0)$

47. Find y' if $y = \ln(x^2 + y^2)$.

48. Find y' if $\ln xy = y\sin x$.

49. Find a formula for $f^{(n)}(x)$ if $f(x) = \ln(x - 1)$.

50. Find $\dfrac{d^9}{dx^9}(x^8\ln x)$.

51–52 Use a graph to estimate the roots of the equation correct to one decimal place. Then use these estimates as the initial approximations in Newton's method to find the roots correct to six decimal places.

51. $(x - 4)^2 = \ln x$

52. $\ln(4 - x^2) = x$

53–56 Discuss the curve under the guidelines of Section 4.5.

53. $y = \ln(\sin x)$

54. $y = \ln(\tan^2 x)$

55. $y = \ln(1 + x^2)$

56. $y = \ln(x^2 - 3x + 2)$

CAS **57.** If $f(x) = \ln(2x + x\sin x)$, use the graphs of f, f', and f'' to estimate the intervals of increase and the inflection points of f on the interval $(0, 15]$.

58. Investigate the family of curves $f(x) = \ln(x^2 + c)$. What happens to the inflection points and asymptotes as c changes? Graph several members of the family to illustrate what you discover.

59–62 Use logarithmic differentiation to find the derivative of the function.

59. $y = (2x + 1)^5(x^4 - 3)^6$

60. $y = \dfrac{(x^3 + 1)^4\sin^2 x}{\sqrt[3]{x}}$

61. $y = \dfrac{\sin^2 x\tan^4 x}{(x^2 + 1)^2}$

62. $y = \sqrt[4]{\dfrac{x^2 + 1}{x^2 - 1}}$

63–72 Evaluate the integral.

63. $\displaystyle\int_2^4 \frac{3}{x}\,dx$

64. $\displaystyle\int_1^2 \frac{4 + u^2}{u^3}\,du$

65. $\displaystyle\int_1^2 \frac{dt}{8 - 3t}$

66. $\displaystyle\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

67. $\displaystyle\int_1^e \frac{x^2 + x + 1}{x}\,dx$

68. $\displaystyle\int_e^6 \frac{dx}{x\ln x}$

69. $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$

70. $\displaystyle\int \frac{\cos x}{2 + \sin x}\,dx$

71. $\displaystyle\int \frac{\sin 2x}{1 + \cos^2 x}\,dx$

72. $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx$

73. Show that $\int \cot x\,dx = \ln|\sin x| + C$ by (a) differentiating the right side of the equation and (b) using the method of Example 13.

74. Find, correct to three decimal places, the area of the region above the hyperbola $y = 2/(x - 2)$, below the x-axis, and between the lines $x = -4$ and $x = -1$.

75. Find the volume of the solid obtained by rotating the region under the curve $y = 1/\sqrt{x+1}$ from 0 to 1 about the x-axis.

76. Find the volume of the solid obtained by rotating the region under the curve

$$y = \frac{1}{x^2 + 1}$$

from 0 to 3 about the y-axis.

77. The work done by a gas when it expands from volume V_1 to volume V_2 is $W = \int_{V_1}^{V_2} P\,dV$, where $P = P(V)$ is the pressure as a function of the volume V. (See Exercise 27 in Section 6.4.) Boyle's Law states that when a quantity of gas expands at constant temperature, $PV = C$, where C is a constant. If the initial volume is 600 cm^3 and the initial pressure is 150 kPa, find the work done by the gas when it expands at constant temperature to 1000 cm^3.

78. Find f if $f''(x) = x^{-2}$, $x > 0$, $f(1) = 0$, and $f(2) = 0$.

79. If g is the inverse function of $f(x) = 2x + \ln x$, find $g'(2)$.

80. (a) Find the linear approximation to $f(x) = \ln x$ near 1.
(b) Illustrate part (a) by graphing f and its linearization.
(c) For what values of x is the linear approximation accurate to within 0.1?

81. (a) By comparing areas, show that

$$\tfrac{1}{3} < \ln 1.5 < \tfrac{5}{12}$$

(b) Use the Midpoint Rule with $n = 10$ to estimate ln 1.5.

82. Refer to Example 1.
(a) Find an equation of the tangent line to the curve $y = 1/t$ that is parallel to the secant line AD.
(b) Use part (a) to show that $\ln 2 > 0.66$.

83. By comparing areas, show that

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}$$

84. Prove the third law of logarithms. [*Hint:* Start by showing that both sides of the equation have the same derivative.]

85. For what values of m do the line $y = mx$ and the curve $y = x/(x^2 + 1)$ enclose a region? Find the area of the region.

86. (a) Compare the rates of growth of $f(x) = x^{0.1}$ and $g(x) = \ln x$ by graphing both f and g in several viewing rectangles. When does the graph of f finally surpass the graph of g?
(b) Graph the function $h(x) = (\ln x)/x^{0.1}$ in a viewing rectangle that displays the behavior of the function as $x \to \infty$.
(c) Find a number N such that

$$\text{if} \quad x > N \quad \text{then} \quad \frac{\ln x}{x^{0.1}} < 0.1$$

87. Use the definition of derivative to prove that

$$\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1$$

7.3* THE NATURAL EXPONENTIAL FUNCTION

Since ln is an increasing function, it is one-to-one and therefore has an inverse function, which we denote by exp. Thus, according to the definition of an inverse function,

$f^{-1}(x) = y \iff f(y) = x$

1 $$\exp(x) = y \iff \ln y = x$$

and the cancellation equations are

$f^{-1}(f(x)) = x$
$f(f^{-1}(x)) = x$

2 $$\exp(\ln x) = x \quad \text{and} \quad \ln(\exp x) = x$$

In particular, we have

$$\exp(0) = 1 \quad \text{since} \quad \ln 1 = 0$$
$$\exp(1) = e \quad \text{since} \quad \ln e = 1$$

We obtain the graph of $y = \exp x$ by reflecting the graph of $y = \ln x$ about the line

INTEGRATION

Because the exponential function $y = e^x$ has a simple derivative, its integral is also simple:

10
$$\int e^x\,dx = e^x + C$$

EXAMPLE 8 Evaluate $\int x^2 e^{x^3}\,dx$.

SOLUTION We substitute $u = x^3$. Then $du = 3x^2\,dx$, so $x^2\,dx = \frac{1}{3}\,du$ and

$$\int x^2 e^{x^3}\,dx = \tfrac{1}{3}\int e^u\,du = \tfrac{1}{3}e^u + C = \tfrac{1}{3}e^{x^3} + C$$ □

EXAMPLE 9 Find the area under the curve $y = e^{-3x}$ from 0 to 1.

SOLUTION The area is

$$A = \int_0^1 e^{-3x}\,dx = -\tfrac{1}{3}e^{-3x}\Big]_0^1 = \tfrac{1}{3}(1 - e^{-3})$$ □

7.3* EXERCISES

1. Sketch, by hand, the graph of the function $f(x) = e^x$ with particular attention to how the graph crosses the y-axis. What fact allows you to do this?

2–4 Simplify each expression.

2. (a) $e^{\ln 15}$ (b) $\ln(1/e)$

3. (a) $e^{-2\ln 5}$ (b) $\ln(\ln e^{e^{10}})$

4. (a) $\ln e^{\sin x}$ (b) $e^{x + \ln x}$

5–12 Solve each equation for x.

5. (a) $2\ln x = 1$ (b) $e^{-x} = 5$

6. (a) $e^{2x+3} - 7 = 0$ (b) $\ln(5 - 2x) = -3$

7. (a) $e^{3x+1} = k$ (b) $\ln x + \ln(x - 1) = 1$

8. (a) $\ln(\ln x) = 1$ (b) $e^{e^x} = 10$

9. $3xe^x + x^2e^x = 0$

10. $10(1 + e^{-x})^{-1} = 3$

11. $e^{2x} - e^x - 6 = 0$

12. $\ln(2x + 1) = 2 - \ln x$

13–14 Find the solution of the equation correct to four decimal places.

13. (a) $e^{2+5x} = 100$ (b) $\ln(e^x - 2) = 3$

14. (a) $\ln(1 + \sqrt{x}) = 2$ (b) $e^{1/(x-4)} = 7$

15–16 Solve each inequality for x.

15. (a) $e^x < 10$ (b) $\ln x > -1$

16. (a) $2 < \ln x < 9$ (b) $e^{2-3x} > 4$

17–20 Make a rough sketch of the graph of the function. Do not use a calculator. Just use the graph given in Figure 2 and, if necessary, the transformations of Section 1.3.

17. $y = e^{-x}$

18. $y = 1 + 2e^x$

19. $y = 1 - \frac{1}{2}e^{-x}$

20. $y = 2 + 5(1 - e^{-x})$

21–22 Find (a) the domain of f and (b) f^{-1} and its domain.

21. $f(x) = \sqrt{3 - e^{2x}}$

22. $f(x) = \ln(2 + \ln x)$

23–26 Find the inverse function.

23. $y = \ln(x + 3)$

24. $y = (\ln x)^2,\quad x \geqslant 1$

25. $f(x) = e^{x^3}$

26. $y = \dfrac{e^x}{1 + 2e^x}$

27–32 Find the limit.

27. $\displaystyle\lim_{x\to\infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$

28. $\displaystyle\lim_{x\to\infty} e^{-x^2}$

29. $\displaystyle\lim_{x\to 2^+} e^{3/(2-x)}$

30. $\displaystyle\lim_{x\to 2^-} e^{3/(2-x)}$

31. $\displaystyle\lim_{x\to\infty} (e^{-2x}\cos x)$

32. $\displaystyle\lim_{x\to(\pi/2)^+} e^{\tan x}$

33–48 Differentiate the function.

33. $f(x) = x^2 e^x$

34. $y = \dfrac{e^x}{1 + x}$

35. $y = e^{ax^3}$

36. $y = e^u(\cos u + cu)$

37. $f(u) = e^{1/u}$

38. $g(x) = \sqrt{x}\, e^x$

39. $F(t) = e^{t \sin 2t}$

40. $f(t) = \sin(e^t) + e^{\sin t}$

41. $y = \sqrt{1 + 2e^{3x}}$

42. $y = e^{k \tan \sqrt{x}}$

43. $y = e^{e^x}$

44. $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

45. $y = \dfrac{ae^x + b}{ce^x + d}$

46. $y = \sqrt{1 + xe^{-2x}}$

47. $y = \cos\left(\dfrac{1 - e^{2x}}{1 + e^{2x}}\right)$

48. $f(t) = \sin^2(e^{\sin^2 t})$

49–50 Find an equation of the tangent line to the curve at the given point.

49. $y = e^{2x} \cos \pi x, \quad (0, 1)$

50. $y = e^x/x, \quad (1, e)$

51. Find y' if $e^{x^2 y} = x + y$.

52. Find an equation of the tangent line to the curve $xe^y + ye^x = 1$ at the point $(0, 1)$.

53. Show that the function $y = e^x + e^{-x/2}$ satisfies the differential equation $2y'' - y' - y = 0$.

54. Show that the function $y = Ae^{-x} + Bxe^{-x}$ satisfies the differential equation $y'' + 2y' + y = 0$.

55. For what values of r does the function $y = e^{rx}$ satisfy the equation $y'' + 6y' + 8y = 0$?

56. Find the values of λ for which $y = e^{\lambda x}$ satisfies the equation $y + y' = y''$.

57. If $f(x) = e^{2x}$, find a formula for $f^{(n)}(x)$.

58. Find the thousandth derivative of $f(x) = xe^{-x}$.

59. (a) Use the Intermediate Value Theorem to show that there is a root of the equation $e^x + x = 0$.
(b) Use Newton's method to find the root of the equation in part (a) correct to six decimal places.

60. Use a graph to find an initial approximation (to one decimal place) to the root of the equation $4e^{-x^2} \sin x = x^2 - x + 1$. Then use Newton's method to find the root correct to eight decimal places.

61. Under certain circumstances a rumor spreads according to the equation

$$p(t) = \frac{1}{1 + ae^{-kt}}$$

where $p(t)$ is the proportion of the population that knows the rumor at time t and a and k are positive constants. [In Section 10.4 we will see that this is a reasonable equation for $p(t)$.]
(a) Find $\lim_{t\to\infty} p(t)$.
(b) Find the rate of spread of the rumor.
(c) Graph p for the case $a = 10$, $k = 0.5$ with t measured in hours. Use the graph to estimate how long it will take for 80% of the population to hear the rumor.

62. For the period from 1980 to 2000, the percentage of households in the United States with at least one VCR has been modeled by the function

$$V(t) = \frac{85}{1 + 53e^{-0.5t}}$$

where the time t is measured in years since midyear 1980, so $0 \leqslant t \leqslant 20$. Use a graph to estimate the time at which the number of VCRs was increasing most rapidly. Then use derivatives to give a more accurate estimate.

63. Find the absolute maximum value of the function $f(x) = x - e^x$.

64. Find the absolute minimum value of the function $g(x) = e^x/x$, $x > 0$.

65–66 Find the absolute maximum and absolute minimum values of f on the given interval.

65. $f(x) = xe^{-x^2/8}, \quad [-1, 4]$

66. $f(x) = x^2 e^{-x/2}, \quad [-1, 6]$

67–68 Find (a) the intervals of increase or decrease, (b) the intervals of concavity, and (c) the points of inflection.

67. $f(x) = (1 - x)e^{-x}$

68. $f(x) = \dfrac{e^x}{x^2}$

69–70 Discuss the curve using the guidelines of Section 4.5.

69. $y = e^{-1/(x+1)}$

70. $y = e^{2x} - e^x$

71. A *drug response curve* describes the level of medication in the bloodstream after a drug is administered. A surge function $S(t) = At^p e^{-kt}$ is often used to model the response curve, reflecting an initial surge in the drug level and then a more gradual decline. If, for a particular drug, $A = 0.01$, $p = 4$, $k = 0.07$, and t is measured in minutes, estimate the times corresponding to the inflection points and explain their significance. If you have a graphing device, use it to graph the drug response curve.

72–73 Draw a graph of f that shows all the important aspects of the curve. Estimate the local maximum and minimum values and then use calculus to find these values exactly. Use a graph of f'' to estimate the inflection points.

72. $f(x) = e^{\cos x}$

73. $f(x) = e^{x^3 - x}$

74. The family of bell-shaped curves

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

occurs in probability and statistics, where it is called the *normal density function*. The constant μ is called the *mean* and the positive constant σ is called the *standard deviation*. For simplicity, let's scale the function so as to remove the factor $1/(\sigma\sqrt{2\pi})$ and let's analyze the special case where $\mu = 0$. So we study the function

$$f(x) = e^{-x^2/(2\sigma^2)}$$

(a) Find the asymptote, maximum value, and inflection points of f.
(b) What role does σ play in the shape of the curve?
(c) Illustrate by graphing four members of this family on the same screen.

75–84 Evaluate the integral.

75. $\int_0^5 e^{-3x}\,dx$

76. $\int_0^1 xe^{-x^2}\,dx$

77. $\int e^x\sqrt{1+e^x}\,dx$

78. $\int \frac{(1+e^x)^2}{e^x}\,dx$

79. $\int (e^x + e^{-x})^2\,dx$

80. $\int e^x(4+e^x)^5\,dx$

81. $\int \sin x\, e^{\cos x}\,dx$

82. $\int \frac{e^{1/x}}{x^2}\,dx$

83. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

84. $\int e^x \sin(e^x)\,dx$

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85. Find, correct to three decimal places, the area of the region bounded by the curves $y = e^x$, $y = e^{3x}$, and $x = 1$.

86. Find $f(x)$ if $f''(x) = 3e^x + 5\sin x$, $f(0) = 1$, and $f'(0) = 2$.

87. Find the volume of the solid obtained by rotating about the x-axis the region bounded by the curves $y = e^x$, $y = 0$, $x = 0$, and $x = 1$.

88. Find the volume of the solid obtained by rotating about the y-axis the region bounded by the curves $y = e^{-x^2}$, $y = 0$, $x = 0$, and $x = 1$.

89. The **error function**

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

is used in probability, statistics, and engineering.
(a) Show that $\int_a^b e^{-t^2}\,dt = \frac{1}{2}\sqrt{\pi}\,[\operatorname{erf}(b) - \operatorname{erf}(a)]$.
(b) Show that the function $y = e^{x^2}\operatorname{erf}(x)$ satisfies the differential equation $y' = 2xy + 2/\sqrt{\pi}$.

90. A bacteria population starts with 400 bacteria and grows at a rate of $r(t) = (450.268)e^{1.12567t}$ bacteria per hour. How many bacteria will there be after three hours?

91. If $f(x) = 3 + x + e^x$, find $(f^{-1})'(4)$.

92. Evaluate $\lim_{x\to\pi} \frac{e^{\sin x} - 1}{x - \pi}$.

93. If you graph the function

$$f(x) = \frac{1 - e^{1/x}}{1 + e^{1/x}}$$

you'll see that f appears to be an odd function. Prove it.

94. Graph several members of the family of functions

$$f(x) = \frac{1}{1 + ae^{bx}}$$

where $a > 0$. How does the graph change when b changes? How does it change when a changes?

95. Prove the second law of exponents [see (7)].

96. Prove the third law of exponents [see (7)].

97. (a) Show that $e^x \geqslant 1 + x$ if $x \geqslant 0$.
[*Hint:* Show that $f(x) = e^x - (1 + x)$ is increasing for $x > 0$.]
(b) Deduce that $\frac{4}{3} \leqslant \int_0^1 e^{x^2}\,dx \leqslant e$.

98. (a) Use the inequality of Exercise 97(a) to show that, for $x \geqslant 0$,

$$e^x \geqslant 1 + x + \tfrac{1}{2}x^2$$

(b) Use part (a) to improve the estimate of $\int_0^1 e^{x^2}\,dx$ given in Exercise 97(b).

99. (a) Use mathematical induction to prove that for $x \geqslant 0$ and any positive integer n,

$$e^x \geqslant 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

(b) Use part (a) to show that $e > 2.7$.
(c) Use part (a) to show that

$$\lim_{x\to\infty} \frac{e^x}{x^k} = \infty$$

for any positive integer k.

100. This exercise illustrates Exercise 99(c) for the case $k = 10$.
(a) Compare the rates of growth of $f(x) = x^{10}$ and $g(x) = e^x$ by graphing both f and g in several viewing rectangles. When does the graph of g finally surpass the graph of f?
(b) Find a viewing rectangle that shows how the function $h(x) = e^x/x^{10}$ behaves for large x.
(c) Find a number N such that

$$\text{if} \quad x > N \quad \text{then} \quad \frac{e^x}{x^{10}} > 10^{10}$$

7.4* EXERCISES

1. (a) Write an equation that defines a^x when a is a positive number and x is a real number.
(b) What is the domain of the function $f(x) = a^x$?
(c) If $a \neq 1$, what is the range of this function?
(d) Sketch the general shape of the graph of the exponential function for each of the following cases.
(i) $a > 1$ (ii) $a = 1$ (iii) $0 < a < 1$

2. (a) If a is a positive number and $a \neq 1$, how is $\log_a x$ defined?
(b) What is the domain of the function $f(x) = \log_a x$?
(c) What is the range of this function?
(d) If $a > 1$, sketch the general shapes of the graphs of $y = \log_a x$ and $y = a^x$ with a common set of axes.

3–6 Write the expression as a power of e.

3. $5^{\sqrt{7}}$

4. 10^{x^2}

5. $(\cos x)^x$

6. $x^{\cos x}$

7–10 Evaluate the expression.

7. (a) $\log_{10} 1000$ (b) $\log_2 \frac{1}{16}$

8. $\log_{10} \sqrt{10}$ (b) $\log_8 320 - \log_8 5$

9. (a) $\log_2 6 - \log_2 15 + \log_2 20$
(b) $\log_3 100 - \log_3 18 - \log_3 50$

10. (a) $\log_a \dfrac{1}{a}$ (b) $10^{(\log_{10} 4 + \log_{10} 7)}$

11–12 Graph the given functions on a common screen. How are these graphs related?

11. $y = 2^x$, $y = e^x$, $y = 5^x$, $y = 20^x$

12. $y = 3^x$, $y = 10^x$, $y = \left(\frac{1}{3}\right)^x$, $y = \left(\frac{1}{10}\right)^x$

13. Use Formula 6 to evaluate each logarithm correct to six decimal places.
(a) $\log_{12} e$ (b) $\log_6 13.54$ (c) $\log_2 \pi$

14–16 Use Formula 6 to graph the given functions on a common screen. How are these graphs related?

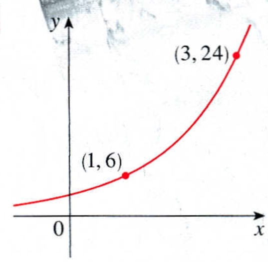
14. $y = \log_2 x$, $y = \log_4 x$, $y = \log_6 x$, $y = \log_8 x$

15. $y = \log_{1.5} x$, $y = \ln x$, $y = \log_{10} x$, $y = \log_{50} x$

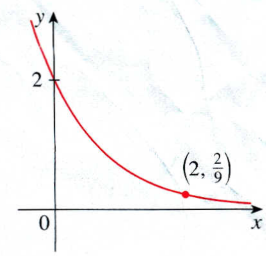
16. $y = \ln x$, $y = \log_{10} x$, $y = e^x$, $y = 10^x$

17–18 Find the exponential function $f(x) = Ca^x$ whose graph is given.

17.



18.



19. (a) Suppose the graphs of $f(x) = x^2$ and $g(x) = 2^x$ are drawn on a coordinate grid where the unit of measurement is 1 cm. Show that, at a distance 1 m to the right of the origin, the height of the graph of f is 100 m but the height of the graph of g is more than 10^{25} km.
(b) Suppose that the graph of $y = \log_2 x$ is drawn on a coordinate grid where the unit of measurement is a centimeter. How many kilometers to the right of the origin do we have to move before the height of the curve reaches 1 m?

20. Compare the rates of growth of the functions $f(x) = x^5$ and $g(x) = 5^x$ by graphing both functions in several viewing rectangles. Find all points of intersection of the graphs correct to one decimal place.

21–24 Find the limit.

21. $\lim_{x\to\infty} (1.001)^x$

22. $\lim_{x\to-\infty} (1.001)^x$

23. $\lim_{t\to\infty} 2^{-t^2}$

24. $\lim_{x\to3^+} \log_{10}(x^2 - 5x + 6)$

25–42 Differentiate the function.

25. $h(t) = t^3 - 3^t$

26. $g(x) = x^4 4^x$

27. $y = 5^{-1/x}$

28. $y = 10^{\tan\theta}$

29. $f(u) = (2^u + 2^{-u})^{10}$

30. $y = 2^{3^{x^2}}$

31. $f(x) = \log_2(1 - 3x)$

32. $f(x) = \log_5(xe^x)$

33. $y = 2x \log_{10} \sqrt{x}$

34. $y = \log_2(e^{-x} \cos \pi x)$

35. $y = x^x$

36. $y = x^{\cos x}$

37. $y = x^{\sin x}$

38. $y = \sqrt{x}^x$

39. $y = (\cos x)^x$

40. $y = (\sin x)^{\ln x}$

41. $y = (\tan x)^{1/x}$

42. $y = (\ln x)^{\cos x}$

43. Find an equation of the tangent line to the curve $y = 10^x$ at the point (1, 10).

44. If $f(x) = x^{\cos x}$, find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

45–50 Evaluate the integral.

45. $\int_1^2 10^t\,dt$

46. $\int (x^5 + 5^x)\,dx$

47. $\int \frac{\log_{10} x}{x}\,dx$

48. $\int x2^{x^2}\,dx$

49. $\int 3^{\sin\theta}\cos\theta\,d\theta$

50. $\int \frac{2^x}{2^x + 1}\,dx$

51. Find the area of the region bounded by the curves $y = 2^x$, $y = 5^x$, $x = -1$, and $x = 1$.

52. The region under the curve $y = 10^{-x}$ from $x = 0$ to $x = 1$ is rotated about the x-axis. Find the volume of the resulting solid.

53. Use a graph to find the root of the equation $2^x = 1 + 3^{-x}$ correct to one decimal place. Then use this estimate as the initial approximation in Newton's method to find the root correct to six decimal places.

54. Find y' if $x^y = y^x$.

55. Find the inverse function of $f(x) = \log_{10}\left(1 + \frac{1}{x}\right)$.

56. Calculate $\lim_{x\to\infty} x^{-\ln x}$.

57. The geologist C. F. Richter defined the magnitude of an earthquake to be $\log_{10}(I/S)$, where I is the intensity of the quake (measured by the amplitude of a seismograph 100 km from the epicenter) and S is the intensity of a "standard" earthquake (where the amplitude is only 1 micron $= 10^{-4}$ cm). The 1989 Loma Prieta earthquake that shook San Francisco had a magnitude of 7.1 on the Richter scale. The 1906 San Francisco earthquake was 16 times as intense. What was its magnitude on the Richter scale?

58. A sound so faint that it can just be heard has intensity $I_0 = 10^{-12}$ watt/m^2 at a frequency of 1000 hertz (Hz). The loudness, in decibels (dB), of a sound with intensity I is then defined to be $L = 10\log_{10}(I/I_0)$. Amplified rock music is measured at 120 dB, whereas the noise from a motor-driven lawn mower is measured at 106 dB. Find the ratio of the intensity of the rock music to that of the mower.

59. Referring to Exercise 58, find the rate of change of the loudness with respect to the intensity when the sound is measured at 50 dB (the level of ordinary conversation).

60. According to the Beer-Lambert Law, the light intensity at a depth of x meters below the surface of the ocean is $I(x) = I_0 a^x$, where I_0 is the light intensity at the surface and a is a constant such that $0 < a < 1$.
(a) Express the rate of change of $I(x)$ with respect to x in terms of $I(x)$.
(b) If $I_0 = 8$ and $a = 0.38$, find the rate of change of intensity with respect to depth at a depth of 20 m.
(c) Using the values from part (b), find the average light intensity between the surface and a depth of 20 m.

61. The flash unit on a camera operates by storing charge on a capacitor and releasing it suddenly when the flash is set off. The following data describe the charge Q remaining on the capacitor (measured in microcoulombs, μC) at time t (measured in seconds).

t	0.00	0.02	0.04	0.06	0.08	0.10
Q	100.00	81.87	67.03	54.88	44.93	36.76

(a) Use a graphing calculator or computer to find an exponential model for the charge.
(b) The derivative $Q'(t)$ represents the electric current (measured in microamperes, μA) flowing from the capacitor to the flash bulb. Use part (a) to estimate the current when $t = 0.04$ s. Compare with the result of Example 2 in Section 2.1.

62. The table gives the population of Mexico (in millions) in the 20th century census years.

Year	Population	Year	Population
1900	13.6	1960	34.9
1910	15.2	1970	48.2
1920	14.3	1980	66.3
1930	16.6	1990	81.2
1940	19.7	2000	97.5
1950	25.8		

(a) Use a graphing calculator or computer to fit an exponential function to the data. Graph the data points and the exponential model. How good is the fit?
(b) Estimate the rates of population growth in 1950 and 1960 by averaging slopes of secant lines.
(c) Use your exponential model in part (a) to find a model for the rate of growth of the Mexican population in the 20th century.
(d) Use your model in part (c) to estimate the rates of growth in 1950 and 1960. Compare with your estimates in part (b).

63. Prove the second law of exponents [see (3)].

64. Prove the fourth law of exponents [see (3)].

65. Deduce the following laws of logarithms from (3):
(a) $\log_a(xy) = \log_a x + \log_a y$
(b) $\log_a(x/y) = \log_a x - \log_a y$
(c) $\log_a(x^y) = y\log_a x$

66. Show that $\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x$ for any $x > 0$.

7.5 EXERCISES

1. A population of protozoa develops with a constant relative growth rate of 0.7944 per member per day. On day zero the population consists of two members. Find the population size after six days.

2. A common inhabitant of human intestines is the bacterium *Escherichia coli*. A cell of this bacterium in a nutrient-broth medium divides into two cells every 20 minutes. The initial population of a culture is 60 cells.
(a) Find the relative growth rate.
(b) Find an expression for the number of cells after t hours.
(c) Find the number of cells after 8 hours.
(d) Find the rate of growth after 8 hours.
(e) When will the population reach 20,000 cells?

3. A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour the population has increased to 420.
(a) Find an expression for the number of bacteria after t hours.
(b) Find the number of bacteria after 3 hours.
(c) Find the rate of growth after 3 hours.
(d) When will the population reach 10,000?

4. A bacteria culture grows with constant relative growth rate. After 2 hours there are 600 bacteria and after 8 hours the count is 75,000.
(a) Find the initial population.
(b) Find an expression for the population after t hours.
(c) Find the number of cells after 5 hours.
(d) Find the rate of growth after 5 hours.
(e) When will the population reach 200,000?

5. The table gives estimates of the world population, in millions, from 1750 to 2000:

Year	Population	Year	Population
1750	790	1900	1650
1800	980	1950	2560
1850	1260	2000	6080

(a) Use the exponential model and the population figures for 1750 and 1800 to predict the world population in 1900 and 1950. Compare with the actual figures.
(b) Use the exponential model and the population figures for 1850 and 1900 to predict the world population in 1950. Compare with the actual population.
(c) Use the exponential model and the population figures for 1900 and 1950 to predict the world population in 2000. Compare with the actual population and try to explain the discrepancy.

6. The table gives the population of India, in millions, for the second half of the 20th century.

Year	Population
1951	361
1961	439
1971	548
1981	683
1991	846
2001	1029

(a) Use the exponential model and the census figures for 1951 and 1961 to predict the population in 2001. Compare with the actual figure.
(b) Use the exponential model and the census figures for 1961 and 1981 to predict the population in 2001. Compare with the actual population. Then use this model to predict the population in the years 2010 and 2020.
(c) Graph both of the exponential functions in parts (a) and (b) together with a plot of the actual population. Are these models reasonable ones?

7. Experiments show that if the chemical reaction

$$N_2O_5 \rightarrow 2NO_2 + \tfrac{1}{2}O_2$$

takes place at 45°C, the rate of reaction of dinitrogen pentoxide is proportional to its concentration as follows:

$$-\frac{d[N_2O_5]}{dt} = 0.0005[N_2O_5]$$

(a) Find an expression for the concentration $[N_2O_5]$ after t seconds if the initial concentration is C.
(b) How long will the reaction take to reduce the concentration of N_2O_5 to 90% of its original value?

8. Bismuth-210 has a half-life of 5.0 days.
(a) A sample originally has a mass of 800 mg. Find a formula for the mass remaining after t days.
(b) Find the mass remaining after 30 days.
(c) When is the mass reduced to 1 mg?
(d) Sketch the graph of the mass function.

9. The half-life of cesium-137 is 30 years. Suppose we have a 100-mg sample.
(a) Find the mass that remains after t years.
(b) How much of the sample remains after 100 years?
(c) After how long will only 1 mg remain?

10. A sample of tritium-3 decayed to 94.5% of its original amount after a year.
(a) What is the half-life of tritium-3?
(b) How long would it take the sample to decay to 20% of its original amount?

11. Scientists can determine the age of ancient objects by the method of *radiocarbon dating*. The bombardment of the upper atmosphere by cosmic rays converts nitrogen to a radioactive isotope of carbon, ^{14}C, with a half-life of about 5730 years. Vegetation absorbs carbon dioxide through the atmosphere and animal life assimilates ^{14}C through food chains. When a plant or animal dies, it stops replacing its carbon and the amount of ^{14}C begins to decrease through radioactive decay. Therefore the level of radioactivity must also decay exponentially.

A parchment fragment was discovered that had about 74% as much ^{14}C radioactivity as does plant material on the earth today. Estimate the age of the parchment.

12. A curve passes through the point (0, 5) and has the property that the slope of the curve at every point P is twice the y-coordinate of P. What is the equation of the curve?

13. A roast turkey is taken from an oven when its temperature has reached 85°C and is placed on a table in a room where the temperature is 22°C.
(a) If the temperature of the turkey is 65°C after half an hour, what is the temperature after 45 minutes?
(b) When will the turkey have cooled to 40°C?

14. A thermometer is taken from a room where the temperature is 20°C to the outdoors, where the temperature is 5°C. After one minute the thermometer reads 12°C.
(a) What will the reading on the thermometer be after one more minute?
(b) When will the thermometer read 6°C?

15. When a cold drink is taken from a refrigerator, its temperature is 5°C. After 25 minutes in a 20°C room its temperature has increased to 10°C.
(a) What is the temperature of the drink after 50 minutes?
(b) When will its temperature be 15°C?

16. (a) A cup of coffee has temperature 95°C and takes 30 minutes to cool to 61°C in a room with temperature 20°C. Show that the temperature of the coffee after t minutes is

$$T(t) = 20 + 75e^{-kt}$$

where $k \approx 0.02$.
(b) What is the average temperature of the coffee during the first half hour?

17. The rate of change of atmospheric pressure P with respect to altitude h is proportional to P, provided that the temperature is constant. At 15°C the pressure is 101.3 kPa at sea level and 87.14 kPa at $h = 1000$ m.
(a) What is the pressure at an altitude of 3000 m?
(b) What is the pressure at the top of Mount McKinley, at an altitude of 6187 m?

18. (a) If \$1000 is borrowed at 8% interest, find the amounts due at the end of 3 years if the interest is compounded (i) annually, (ii) quarterly, (iii) monthly, (iv) weekly, (v) daily, (vi) hourly, and (viii) continuously.
(b) Suppose \$1000 is borrowed and the interest is compounded continuously. If $A(t)$ is the amount due after t years, where $0 \leqslant t \leqslant 3$, graph $A(t)$ for each of the interest rates 6%, 8%, and 10% on a common screen.

19. (a) If \$3000 is invested at 5% interest, find the value of the investment at the end of 5 years if the interest is compounded (i) annually, (ii) semiannually, (iii) monthly, (iv) weekly, (v) daily, and (vi) continuously.
(b) If $A(t)$ is the amount of the investment at time t for the case of continuous compounding, write a differential equation and an initial condition satisfied by $A(t)$.

20. (a) How long will it take an investment to double in value if the interest rate is 6% compounded continuously?
(b) What is the equivalent annual interest rate?

7.6 INVERSE TRIGONOMETRIC FUNCTIONS

In this section we apply the ideas of Section 7.1 to find the derivatives of the so-called inverse trigonometric functions. We have a slight difficulty in this task: Because the trigonometric functions are not one-to-one, they do not have inverse functions. The difficulty is overcome by restricting the domains of these functions so that they become one-to-one.

You can see from Figure 1 that the sine function $y = \sin x$ is not one-to-one (use the Horizontal Line Test). But the function $f(x) = \sin x$, $-\pi/2 \leqslant x \leqslant \pi/2$, *is* one-to-one (see Figure 2). The inverse function of this restricted sine function f exists and is denoted by $\sin^{-1}$ or arcsin. It is called the **inverse sine function** or the **arcsine function**.

7.6 EXERCISES

1–10 Find the exact value of each expression.

1. (a) $\sin^{-1}(\sqrt{3}/2)$ (b) $\cos^{-1}(-1)$

2. (a) $\tan^{-1}(1/\sqrt{3})$ (b) $\sec^{-1} 2$

3. (a) $\arctan 1$ (b) $\sin^{-1}(1/\sqrt{2})$

4. (a) $\sec^{-1}\sqrt{2}$ (b) $\arcsin 1$

5. (a) $\tan(\arctan 10)$ (b) $\sin^{-1}(\sin(7\pi/3))$

6. (a) $\tan^{-1}(\tan 3\pi/4)$ (b) $\cos(\arcsin \frac{1}{2})$

7. $\tan(\sin^{-1}(\frac{2}{3}))$

8. $\csc(\arccos \frac{3}{5})$

9. $\sin(2 \tan^{-1}\sqrt{2})$

10. $\cos(\tan^{-1} 2 + \tan^{-1} 3)$

11. Prove that $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$.

12–14 Simplify the expression.

12. $\tan(\sin^{-1}x)$

13. $\sin(\tan^{-1}x)$

14. $\cos(2 \tan^{-1}x)$

15–16 Graph the given functions on the same screen. How are these graphs related?

15. $y = \sin x,\ -\pi/2 \leqslant x \leqslant \pi/2;\quad y = \sin^{-1}x;\quad y = x$

16. $y = \tan x,\ -\pi/2 < x < \pi/2;\quad y = \tan^{-1}x;\quad y = x$

17. Prove Formula 6 for the derivative of $\cos^{-1}$ by the same method as for Formula 3.

18. (a) Prove that $\sin^{-1}x + \cos^{-1}x = \pi/2$.
(b) Use part (a) to prove Formula 6.

19. Prove that $\dfrac{d}{dx}(\cot^{-1}x) = -\dfrac{1}{1 + x^2}$.

20. Prove that $\dfrac{d}{dx}(\sec^{-1}x) = \dfrac{1}{x\sqrt{x^2 - 1}}$.

21. Prove that $\dfrac{d}{dx}(\csc^{-1}x) = -\dfrac{1}{x\sqrt{x^2 - 1}}$.

22–35 Find the derivative of the function. Simplify where possible.

22. $y = \sqrt{\tan^{-1}x}$

23. $y = \tan^{-1}\sqrt{x}$

24. $f(x) = x \ln(\arctan x)$

25. $y = \sin^{-1}(2x + 1)$

26. $g(x) = \sqrt{x^2 - 1} \sec^{-1}x$

27. $H(x) = (1 + x^2) \arctan x$

28. $F(\theta) = \arcsin \sqrt{\sin \theta}$

29. $y = \cos^{-1}(e^{2x})$

30. $y = \arctan \sqrt{\dfrac{1 - x}{1 + x}}$

31. $y = \arctan(\cos \theta)$

32. $y = \tan^{-1}(x - \sqrt{1 + x^2})$

33. $h(t) = \cot^{-1}(t) + \cot^{-1}(1/t)$

34. $y = \tan^{-1}\left(\dfrac{x}{a}\right) + \ln\sqrt{\dfrac{x - a}{x + a}}$

35. $y = \arccos\left(\dfrac{b + a \cos x}{a + b \cos x}\right),\quad 0 \leqslant x \leqslant \pi,\ a > b > 0$

36–37 Find the derivative of the function. Find the domains of the function and its derivative.

36. $f(x) = \arcsin(e^x)$

37. $g(x) = \cos^{-1}(3 - 2x)$

38. Find y' if $\tan^{-1}(xy) = 1 + x^2y$.

39. If $g(x) = x \sin^{-1}(x/4) + \sqrt{16 - x^2}$, find $g'(2)$.

40. Find an equation of the tangent line to the curve $y = 3 \arccos(x/2)$ at the point $(1, \pi)$.

41–42 Find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

41. $f(x) = \sqrt{1 - x^2} \arcsin x$

42. $f(x) = \arctan(x^2 - x)$

43–46 Find the limit.

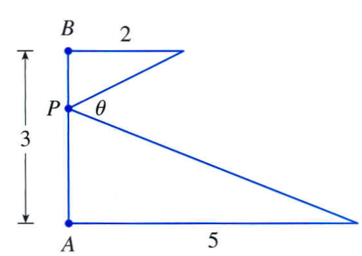
43. $\lim_{x \to -1^+} \sin^{-1}x$

44. $\lim_{x \to \infty} \arccos\left(\dfrac{1 + x^2}{1 + 2x^2}\right)$

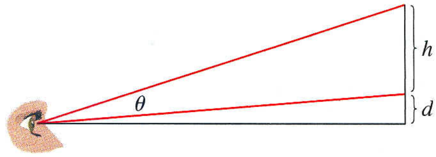
45. $\lim_{x \to \infty} \arctan(e^x)$

46. $\lim_{x \to 0^+} \tan^{-1}(\ln x)$

47. Where should the point P be chosen on the line segment AB so as to maximize the angle θ?



48. A painting in an art gallery has height h and is hung so that its lower edge is a distance d above the eye of an observer (as in the figure on page 462). How far from the wall should the observer stand to get the best view? (In other words, where should the observer stand so as to maximize the angle θ subtended at her eye by the painting?)



49. A ladder 5 m long leans against a vertical wall. If the bottom of the ladder slides away from the base of the wall at a speed of 1 m/s, how fast is the angle between the ladder and the wall changing when the bottom of the ladder is 3 m from the base of the wall?

50. A lighthouse is located on a small island, 3 km away from the nearest point P on a straight shoreline, and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P?

51–54 Sketch the curve using the guidelines of Section 4.5.

51. $y = \sin^{-1}\left(\dfrac{x}{x+1}\right)$

52. $y = \tan^{-1}\left(\dfrac{x-1}{x+1}\right)$

53. $y = x - \tan^{-1}x$

54. $y = \tan^{-1}(\ln x)$

CAS **55.** If $f(x) = \arctan(\cos(3 \arcsin x))$, use the graphs of f, f', and f'' to estimate the x-coordinates of the maximum and minimum points and inflection points of f.

56. Investigate the family of curves given by $f(x) = x - c\sin^{-1}x$. What happens to the number of maxima and minima as c changes? Graph several members of the family to illustrate what you discover.

57. Find the most general antiderivative of the function

$$f(x) = \frac{2 + x^2}{1 + x^2}$$

58. Find $f(x)$ if $f'(x) = 4/\sqrt{1 - x^2}$ and $f(\frac{1}{2}) = 1$.

59–70 Evaluate the integral.

59. $\displaystyle\int_{1/2}^{\sqrt{3}/2} \frac{6}{\sqrt{1 - t^2}}\, dt$

60. $\displaystyle\int \frac{\tan^{-1}x}{1 + x^2}\, dx$

61. $\displaystyle\int_0^{\sqrt{3}/4} \frac{dx}{1 + 16x^2}$

62. $\displaystyle\int \frac{dt}{\sqrt{1 - 4t^2}}$

63. $\displaystyle\int \frac{1 + x}{1 + x^2}\, dx$

64. $\displaystyle\int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x}\, dx$

65. $\displaystyle\int \frac{dx}{\sqrt{1 - x^2}\,\sin^{-1}x}$

66. $\displaystyle\int \frac{1}{x\sqrt{x^2 - 4}}\, dx$

67. $\displaystyle\int \frac{t^2}{\sqrt{1 - t^6}}\, dt$

68. $\displaystyle\int \frac{e^{2x}}{\sqrt{1 - e^{4x}}}\, dx$

69. $\displaystyle\int \frac{dx}{\sqrt{x}\,(1 + x)}$

70. $\displaystyle\int \frac{x}{1 + x^4}\, dx$

71. Use the method of Example 8 to show that, if $a > 0$,

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

72. The region under the curve $y = 1/\sqrt{x^2 + 4}$ from $x = 0$ to $x = 2$ is rotated about the x-axis. Find the volume of the resulting solid.

73. Evaluate $\int_0^1 \sin^{-1}x\, dx$ by interpreting it as an area and integrating with respect to y instead of x.

74. Prove that, for $xy \neq 1$,

$$\arctan x + \arctan y = \arctan \frac{x + y}{1 - xy}$$

if the left side lies between $-\pi/2$ and $\pi/2$.

75. Use the result of Exercise 74 to prove the following:
(a) $\arctan \frac{1}{2} + \arctan \frac{1}{3} = \pi/4$
(b) $2 \arctan \frac{1}{3} + \arctan \frac{1}{7} = \pi/4$

76. (a) Sketch the graph of the function $f(x) = \sin(\sin^{-1}x)$.
(b) Sketch the graph of the function $g(x) = \sin^{-1}(\sin x)$, $x \in \mathbb{R}$.
(c) Show that $g'(x) = \dfrac{\cos x}{|\cos x|}$.
(d) Sketch the graph of $h(x) = \cos^{-1}(\sin x)$, $x \in \mathbb{R}$, and find its derivative.

77. Use the method of Example 6 to prove the identity

$$2\sin^{-1}x = \cos^{-1}(1 - 2x^2) \qquad x \geqslant 0$$

78. Prove the identity

$$\arcsin \frac{x - 1}{x + 1} = 2 \arctan \sqrt{x} - \frac{\pi}{2}$$

79. Some authors define $y = \sec^{-1}x \iff \sec y = x$ and $y \in [0, \pi/2) \cup (\pi/2, \pi]$. Show that with this definition we have (instead of the formula given in Exercise 20)

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2 - 1}} \qquad |x| > 1$$

80. Let $f(x) = x \arctan(1/x)$ if $x \neq 0$ and $f(0) = 0$.
(a) Is f continuous at 0?
(b) Is f differentiable at 0?

SOLUTION 2 From Equation 3 (proved in Example 3), we have

$$\frac{d}{dx}(\sinh^{-1}x) = \frac{d}{dx}\ln\left(x + \sqrt{x^2+1}\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}}\frac{d}{dx}\left(x+\sqrt{x^2+1}\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left(1 + \frac{x}{\sqrt{x^2+1}}\right)$$

$$= \frac{\sqrt{x^2+1}+x}{\left(x+\sqrt{x^2+1}\right)\sqrt{x^2+1}}$$

$$= \frac{1}{\sqrt{x^2+1}} \qquad \square$$

EXAMPLE 5 Find $\frac{d}{dx}[\tanh^{-1}(\sin x)]$.

SOLUTION Using Table 6 and the Chain Rule, we have

$$\frac{d}{dx}[\tanh^{-1}(\sin x)] = \frac{1}{1-(\sin x)^2}\frac{d}{dx}(\sin x)$$

$$= \frac{1}{1-\sin^2 x}\cos x = \frac{\cos x}{\cos^2 x} = \sec x \qquad \square$$

EXAMPLE 6 Evaluate $\int_0^1 \frac{dx}{\sqrt{1+x^2}}$.

SOLUTION Using Table 6 (or Example 4) we know that an antiderivative of $1/\sqrt{1+x^2}$ is $\sinh^{-1}x$. Therefore

$$\int_0^1 \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1}x\Big]_0^1$$

$$= \sinh^{-1} 1$$

$$= \ln\left(1+\sqrt{2}\right) \qquad \text{(from Equation 3)} \qquad \square$$

7.7 EXERCISES

1–6 Find the numerical value of each expression.

1. (a) $\sinh 0$ (b) $\cosh 0$
2. (a) $\tanh 0$ (b) $\tanh 1$
3. (a) $\sinh(\ln 2)$ (b) $\sinh 2$
4. (a) $\cosh 3$ (b) $\cosh(\ln 3)$
5. (a) $\operatorname{sech} 0$ (b) $\cosh^{-1} 1$
6. (a) $\sinh 1$ (b) $\sinh^{-1} 1$

7–19 Prove the identity.

7. $\sinh(-x) = -\sinh x$
(This shows that sinh is an odd function.)
8. $\cosh(-x) = \cosh x$
(This shows that cosh is an even function.)
9. $\cosh x + \sinh x = e^x$
10. $\cosh x - \sinh x = e^{-x}$
11. $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
12. $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

13. $\coth^2 x - 1 = \operatorname{csch}^2 x$

14. $\tanh(x + y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

15. $\sinh 2x = 2 \sinh x \cosh x$

16. $\cosh 2x = \cosh^2 x + \sinh^2 x$

17. $\tanh(\ln x) = \dfrac{x^2 - 1}{x^2 + 1}$

18. $\dfrac{1 + \tanh x}{1 - \tanh x} = e^{2x}$

19. $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$
(n any real number)

20. If $\tanh x = \frac{12}{13}$, find the values of the other hyperbolic functions at x.

21. If $\cosh x = \frac{5}{3}$ and $x > 0$, find the values of the other hyperbolic functions at x.

22. (a) Use the graphs of sinh, cosh, and tanh in Figures 1–3 to draw the graphs of csch, sech, and coth.
(b) Check the graphs that you sketched in part (a) by using a graphing device to produce them.

23. Use the definitions of the hyperbolic functions to find each of the following limits.
(a) $\lim_{x \to \infty} \tanh x$ (b) $\lim_{x \to -\infty} \tanh x$
(c) $\lim_{x \to \infty} \sinh x$ (d) $\lim_{x \to -\infty} \sinh x$
(e) $\lim_{x \to \infty} \operatorname{sech} x$ (f) $\lim_{x \to \infty} \coth x$
(g) $\lim_{x \to 0^+} \coth x$ (h) $\lim_{x \to 0^-} \coth x$
(i) $\lim_{x \to -\infty} \operatorname{csch} x$

24. Prove the formulas given in Table 1 for the derivatives of the functions (a) cosh, (b) tanh, (c) csch, (d) sech, and (e) coth.

25. Give an alternative solution to Example 3 by letting $y = \sinh^{-1} x$ and then using Exercise 9 and Example 1(a) with x replaced by y.

26. Prove Equation 4.

27. Prove Equation 5 using (a) the method of Example 3 and (b) Exercise 18 with x replaced by y.

28. For each of the following functions (i) give a definition like those in (2), (ii) sketch the graph, and (iii) find a formula similar to Equation 3.
(a) csch^{-1} (b) sech^{-1} (c) $\coth^{-1}$

29. Prove the formulas given in Table 6 for the derivatives of the following functions.
(a) $\cosh^{-1}$ (b) $\tanh^{-1}$ (c) csch^{-1}
(d) sech^{-1} (e) $\coth^{-1}$

30–47 Find the derivative. Simplify where possible.

30. $f(x) = e^x \sinh x$

31. $f(x) = \tanh 3x$

32. $g(x) = \cosh^4 x$

33. $h(x) = \cosh(x^4)$

34. $y = x \coth(1 + x^2)$

35. $y = e^{\cosh 3x}$

36. $f(t) = \operatorname{csch} t(1 - \ln \operatorname{csch} t)$

37. $f(t) = \operatorname{sech}^2(e^t)$

38. $y = \sinh(\cosh x)$

39. $y = \arctan(\tanh x)$

40. $y = \sqrt[4]{\dfrac{1 + \tanh x}{1 - \tanh x}}$

41. $G(x) = \dfrac{1 - \cosh x}{1 + \cosh x}$

42. $y = x^2 \sinh^{-1}(2x)$

43. $y = \tanh^{-1}\sqrt{x}$

44. $y = x \tanh^{-1} x + \ln \sqrt{1 - x^2}$

45. $y = x \sinh^{-1}(x/3) - \sqrt{9 + x^2}$

46. $y = \operatorname{sech}^{-1}\sqrt{1 - x^2}, \quad x > 0$

47. $y = \coth^{-1}\sqrt{x^2 + 1}$

48. The Gateway Arch in St. Louis was designed by Eero Saarinen and was constructed using the equation

$$y = 211.49 - 20.96 \cosh 0.03291765x$$

for the central curve of the arch, where x and y are measured in meters and $|x| \leq 91.20$.
(a) Graph the central curve.
(b) What is the height of the arch at its center?
(c) At what points is the height 100 m?
(d) What is the slope of the arch at the points in part (c)?

49. If a water wave with length L moves with velocity v in a body of water with depth d, then

$$v = \sqrt{\frac{gL}{2\pi} \tanh\left(\frac{2\pi d}{L}\right)}$$

where g is the acceleration due to gravity. (See Figure 5.) Explain why the approximation

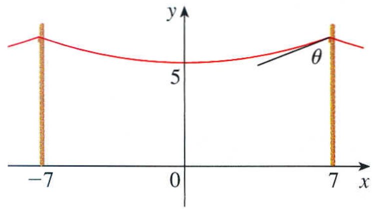
$$v \approx \sqrt{\frac{gL}{2\pi}}$$

is appropriate in deep water.

50. A flexible cable always hangs in the shape of a catenary $y = c + a \cosh(x/a)$, where c and a are constants and $a > 0$ (see Figure 4 and Exercise 52). Graph several members of the family of functions $y = a \cosh(x/a)$. How does the graph change as a varies?

51. A telephone line hangs between two poles 14 m apart in the shape of the catenary $y = 20 \cosh(x/20) - 15$, where x and y are measured in meters. (See the diagram on page 470.)
(a) Find the slope of this curve where it meets the right pole.

(b) Find the angle θ between the line and the pole.



52. Using principles from physics it can be shown that when a cable is hung between two poles, it takes the shape of a curve $y = f(x)$ that satisfies the differential equation

$$\frac{d^2y}{dx^2} = \frac{\rho g}{T}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

where ρ is the linear density of the cable, g is the acceleration due to gravity, and T is the tension in the cable at its lowest point, and the coordinate system is chosen appropriately. Verify that the function

$$y = f(x) = \frac{T}{\rho g}\cosh\left(\frac{\rho g x}{T}\right)$$

is a solution of this differential equation.

53. (a) Show that any function of the form

$$y = A \sinh mx + B \cosh mx$$

satisfies the differential equation $y'' = m^2 y$.

(b) Find $y = y(x)$ such that $y'' = 9y$, $y(0) = -4$, and $y'(0) = 6$.

54. Evaluate $\lim_{x \to \infty} \frac{\sinh x}{e^x}$.

55. At what point of the curve $y = \cosh x$ does the tangent have slope 1?

56. If $x = \ln(\sec\theta + \tan\theta)$, show that $\sec\theta = \cosh x$.

57–65 Evaluate the integral.

57. $\int \sinh x \cosh^2 x \, dx$

58. $\int \sinh(1 + 4x)\, dx$

59. $\int \frac{\sinh\sqrt{x}}{\sqrt{x}}\, dx$

60. $\int \tanh x \, dx$

61. $\int \frac{\cosh x}{\cosh^2 x - 1}\, dx$

62. $\int \frac{\operatorname{sech}^2 x}{2 + \tanh x}\, dx$

63. $\int_4^6 \frac{1}{\sqrt{t^2 - 9}}\, dt$

64. $\int_0^1 \frac{1}{\sqrt{16t^2 + 1}}\, dt$

65. $\int \frac{e^x}{1 - e^{2x}}\, dx$

66. Estimate the value of the number c such that the area under the curve $y = \sinh cx$ between $x = 0$ and $x = 1$ is equal to 1.

67. (a) Use Newton's method or a graphing device to find approximate solutions of the equation $\cosh 2x = 1 + \sinh x$.

(b) Estimate the area of the region bounded by the curves $y = \cosh 2x$ and $y = 1 + \sinh x$.

68. Show that the area of the shaded hyperbolic sector in Figure 7 is $A(t) = \frac{1}{2}t$. [*Hint:* First show that

$$A(t) = \tfrac{1}{2}\sinh t \cosh t - \int_1^{\cosh t} \sqrt{x^2 - 1}\, dx$$

and then verify that $A'(t) = \frac{1}{2}$.]

69. Show that if $a \neq 0$ and $b \neq 0$, then there exist numbers α and β such that $ae^x + be^{-x}$ equals either $\alpha \sinh(x + \beta)$ or $\alpha \cosh(x + \beta)$. In other words, almost every function of the form $f(x) = ae^x + be^{-x}$ is a shifted and stretched hyperbolic sine or cosine function.

7.8 INDETERMINATE FORMS AND L'HOSPITAL'S RULE

Suppose we are trying to analyze the behavior of the function

$$F(x) = \frac{\ln x}{x - 1}$$

Although F is not defined when $x = 1$, we need to know how F behaves *near* 1. In particular, we would like to know the value of the limit

$$\lim_{x \to 1} \frac{\ln x}{x - 1}$$

Likewise, G is continuous on I. Let $x \in I$ and $x > a$. Then F and G are continuous on $[a, x]$ and differentiable on (a, x) and $G' \neq 0$ there (since $F' = f'$ and $G' = g'$). Therefore, by Cauchy's Mean Value Theorem, there is a number y such that $a < y < x$ and

$$\frac{F'(y)}{G'(y)} = \frac{F(x) - F(a)}{G(x) - G(a)} = \frac{F(x)}{G(x)}$$

Here we have used the fact that, by definition, $F(a) = 0$ and $G(a) = 0$. Now, if we let $x \to a^+$, then $y \to a^+$ (since $a < y < x$), so

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{F(x)}{G(x)} = \lim_{y \to a^+} \frac{F'(y)}{G'(y)} = \lim_{y \to a^+} \frac{f'(y)}{g'(y)} = L$$

A similar argument shows that the left-hand limit is also L. Therefore

$$\lim_{x \to a} \frac{f(x)}{g(x)} = L$$

This proves l'Hospital's Rule for the case where a is finite.

If a is infinite, we let $t = 1/x$. Then $t \to 0^+$ as $x \to \infty$, so we have

$$\begin{aligned}
\lim_{x \to \infty} \frac{f(x)}{g(x)} &= \lim_{t \to 0^+} \frac{f(1/t)}{g(1/t)} \\
&= \lim_{t \to 0^+} \frac{f'(1/t)(-1/t^2)}{g'(1/t)(-1/t^2)} && \text{(by l'Hospital's Rule for finite } a\text{)} \\
&= \lim_{t \to 0^+} \frac{f'(1/t)}{g'(1/t)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}
\end{aligned}$$

□

7.8 EXERCISES

1–4 Given that

$$\lim_{x \to a} f(x) = 0 \qquad \lim_{x \to a} g(x) = 0 \qquad \lim_{x \to a} h(x) = 1$$

$$\lim_{x \to a} p(x) = \infty \qquad \lim_{x \to a} q(x) = \infty$$

which of the following limits are indeterminate forms? For those that are not an indeterminate form, evaluate the limit where possible.

1. (a) $\lim_{x \to a} \frac{f(x)}{g(x)}$ (b) $\lim_{x \to a} \frac{f(x)}{p(x)}$

(c) $\lim_{x \to a} \frac{h(x)}{p(x)}$ (d) $\lim_{x \to a} \frac{p(x)}{f(x)}$

(e) $\lim_{x \to a} \frac{p(x)}{q(x)}$

2. (a) $\lim_{x \to a} [f(x)p(x)]$ (b) $\lim_{x \to a} [h(x)p(x)]$

(c) $\lim_{x \to a} [p(x)q(x)]$

3. (a) $\lim_{x \to a} [f(x) - p(x)]$ (b) $\lim_{x \to a} [p(x) - q(x)]$

(c) $\lim_{x \to a} [p(x) + q(x)]$

4. (a) $\lim_{x \to a} [f(x)]^{g(x)}$ (b) $\lim_{x \to a} [f(x)]^{p(x)}$ (c) $\lim_{x \to a} [h(x)]^{p(x)}$

(d) $\lim_{x \to a} [p(x)]^{f(x)}$ (e) $\lim_{x \to a} [p(x)]^{q(x)}$ (f) $\lim_{x \to a} \sqrt[q(x)]{p(x)}$

5–64 Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule doesn't apply, explain why.

5. $\lim_{x \to -1} \frac{x^2 - 1}{x + 1}$

6. $\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2}$

7. $\lim_{x \to 1} \frac{x^9 - 1}{x^5 - 1}$

8. $\lim_{x \to 1} \frac{x^a - 1}{x^b - 1}$

9. $\lim_{x \to (\pi/2)^+} \frac{\cos x}{1 - \sin x}$

10. $\lim_{x \to 0} \frac{\sin 4x}{\tan 5x}$

11. $\lim_{x\to 0} \frac{\sin x}{x^3}$

12. $\lim_{t\to 0} \frac{e^{3t}-1}{t}$

13. $\lim_{x\to 0} \frac{\tan px}{\tan qx}$

14. $\lim_{x\to \pi} \frac{\tan x}{x}$

15. $\lim_{x\to \infty} \frac{\ln x}{\sqrt{x}}$

16. $\lim_{x\to \infty} \frac{x+x^2}{1-2x^2}$

17. $\lim_{x\to 0^+} \frac{\ln x}{x}$

18. $\lim_{x\to \infty} \frac{\ln \ln x}{x}$

19. $\lim_{x\to \infty} \frac{e^x}{x^3}$

20. $\lim_{x\to 1} \frac{\ln x}{\sin \pi x}$

21. $\lim_{x\to 0} \frac{e^x-1-x}{x^2}$

22. $\lim_{x\to 0} \frac{e^x-1-x-\frac{1}{2}x^2}{x^3}$

23. $\lim_{x\to 0} \frac{\tanh x}{\tan x}$

24. $\lim_{x\to 0} \frac{x-\sin x}{x-\tan x}$

25. $\lim_{t\to 0} \frac{5^t-3^t}{t}$

26. $\lim_{x\to 0} \frac{\sin x - x}{x^3}$

27. $\lim_{x\to 0} \frac{\sin^{-1}x}{x}$

28. $\lim_{x\to \infty} \frac{(\ln x)^2}{x}$

29. $\lim_{x\to 0} \frac{1-\cos x}{x^2}$

30. $\lim_{x\to 0} \frac{\cos mx - \cos nx}{x^2}$

31. $\lim_{x\to 0} \frac{x+\sin x}{x+\cos x}$

32. $\lim_{x\to 0} \frac{x}{\tan^{-1}(4x)}$

33. $\lim_{x\to 1} \frac{1-x+\ln x}{1+\cos \pi x}$

34. $\lim_{x\to \infty} \frac{\sqrt{x^2+2}}{\sqrt{2x^2+1}}$

35. $\lim_{x\to 1} \frac{x^a-ax+a-1}{(x-1)^2}$

36. $\lim_{x\to 0} \frac{e^x-e^{-x}-2x}{x-\sin x}$

37. $\lim_{x\to 0} \frac{\cos x - 1 + \frac{1}{2}x^2}{x^4}$

38. $\lim_{x\to a^+} \frac{\cos x \ln(x-a)}{\ln(e^x-e^a)}$

39. $\lim_{x\to \infty} x \sin(\pi/x)$

40. $\lim_{x\to -\infty} x^2 e^x$

41. $\lim_{x\to 0} \cot 2x \sin 6x$

42. $\lim_{x\to \infty} e^{-x} \ln x$

43. $\lim_{x\to \infty} x^3 e^{-x^2}$

44. $\lim_{x\to \pi/4} (1-\tan x)\sec x$

45. $\lim_{x\to 1^+} \ln x \tan(\pi x/2)$

46. $\lim_{x\to \infty} x \tan(1/x)$

47. $\lim_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$

48. $\lim_{x\to 0} (\csc x - \cot x)$

49. $\lim_{x\to \infty} \left(\sqrt{x^2+x} - x\right)$

50. $\lim_{x\to 0} \left(\cot x - \frac{1}{x}\right)$

51. $\lim_{x\to \infty} (x - \ln x)$

52. $\lim_{x\to \infty} (xe^{1/x} - x)$

53. $\lim_{x\to 0^+} x^{x^2}$

54. $\lim_{x\to 0^+} (\tan 2x)^x$

55. $\lim_{x\to 0} (1-2x)^{1/x}$

56. $\lim_{x\to \infty} \left(1+\frac{a}{x}\right)^{bx}$

57. $\lim_{x\to \infty} \left(1+\frac{3}{x}+\frac{5}{x^2}\right)^x$

58. $\lim_{x\to \infty} x^{(\ln 2)/(1+\ln x)}$

59. $\lim_{x\to \infty} x^{1/x}$

60. $\lim_{x\to \infty} (e^x + x)^{1/x}$

61. $\lim_{x\to 0^+} (4x+1)^{\cot x}$

62. $\lim_{x\to 1} (2-x)^{\tan(\pi x/2)}$

63. $\lim_{x\to 0^+} (\cos x)^{1/x^2}$

64. $\lim_{x\to \infty} \left(\frac{2x-3}{2x+5}\right)^{2x+1}$

65–66 Use a graph to estimate the value of the limit. Then use l'Hospital's Rule to find the exact value.

65. $\lim_{x\to \infty} \left(1+\frac{2}{x}\right)^x$

66. $\lim_{x\to 0} \frac{5^x-4^x}{3^x-2^x}$

67–68 Illustrate l'Hospital's Rule by graphing both $f(x)/g(x)$ and $f'(x)/g'(x)$ near $x=0$ to see that these ratios have the same limit as $x \to 0$. Also calculate the exact value of the limit.

67. $f(x) = e^x - 1, \quad g(x) = x^3 + 4x$

68. $f(x) = 2x \sin x, \quad g(x) = \sec x - 1$

69–74 Use l'Hospital's Rule to help sketch the curve. Use the guidelines of Section 4.5.

69. $y = xe^{-x}$

70. $y = \frac{\ln x}{x^2}$

71. $y = xe^{-x^2}$

72. $y = e^x/x$

73. $y = x - \ln(1+x)$

74. $y = (x^2-3)e^{-x}$

CAS **75–77**

(a) Graph the function.

(b) Use l'Hospital's Rule to explain the behavior as $x \to 0^+$ or as $x \to \infty$.

(c) Estimate the maximum and minimum values and then use calculus to find the exact values.

(d) Use a graph of f'' to estimate the x-coordinates of the inflection points.

75. $f(x) = x^{-x}$

76. $f(x) = (\sin x)^{\sin x}$

77. $f(x) = x^{1/x}$

78. Investigate the family of curves given by $f(x) = x^n e^{-x}$, where n is a positive integer. What features do these curves have in common? How do they differ from one another? In particular, what happens to the maximum and minimum points and inflection points as n increases? Illustrate by graphing several members of the family.

79. Investigate the family of curves given by $f(x) = xe^{-cx}$, where c is a real number. Start by computing the limits as $x \to \pm\infty$. Identify any transitional values of c where the basic shape changes. What happens to the maximum or minimum points and inflection points as c changes? Illustrate by graphing several members of the family.

80. The first appearance in print of l'Hospital's Rule was in the book *Analyse des Infiniment Petits* published by the Marquis de l'Hospital in 1696. This was the first calculus *textbook* ever published and the example that the Marquis used in that book to illustrate his rule was to find the limit of the function

$$y = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}}$$

as x approaches a, where $a > 0$. (At that time it was common to write aa instead of a^2.) Solve this problem.

81. What happens if you try to use l'Hospital's Rule to evaluate

$$\lim_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}}$$

Evaluate the limit using another method.

82. If a metal ball with mass m is projected in water and the force of resistance is proportional to the square of the velocity, then the distance the ball travels in time t is

$$s(t) = \frac{m}{c} \ln \cosh \sqrt{\frac{gc}{mt}}$$

where c is a positive constant. Find $\lim_{c \to 0^+} s(t)$.

83. If an electrostatic field E acts on a liquid or a gaseous polar dielectric, the net dipole moment P per unit volume is

$$P(E) = \frac{e^E + e^{-E}}{e^E - e^{-E}} - \frac{1}{E}$$

Show that $\lim_{E \to 0^+} P(E) = 0$.

84. A metal cable has radius r and is covered by insulation, so that the distance from the center of the cable to the exterior of the insulation is R. The velocity v of an electrical impulse in the cable is

$$v = -c\left(\frac{r}{R}\right)^2 \ln\left(\frac{r}{R}\right)$$

where c is a positive constant. Find the following limits and interpret your answers.

(a) $\lim_{R \to r^+} v$ (b) $\lim_{r \to 0^+} v$

85. If an initial amount A_0 of money is invested at an interest rate r compounded n times a year, the value of the investment after t years is

$$A = A_0\left(1 + \frac{r}{n}\right)^{nt}$$

If we let $n \to \infty$, we refer to the *continuous compounding* of interest. Use l'Hospital's Rule to show that if interest is compounded continuously, then the amount after t years is

$$A = A_0 e^{rt}$$

86. If an object with mass m is dropped from rest, one model for its speed v after t seconds, taking air resistance into account, is

$$v = \frac{mg}{c}(1 - e^{-ct/m})$$

where g is the acceleration due to gravity and c is a positive constant. (In Chapter 10 we will be able to deduce this equation from the assumption that the air resistance is proportional to the speed of the object; c is the proportionality constant.)

(a) Calculate $\lim_{t \to \infty} v$. What is the meaning of this limit?

(b) For fixed t, use l'Hospital's Rule to calculate $\lim_{c \to 0^+} v$. What can you conclude about the velocity of a falling object in a vacuum?

87. In Section 5.3 we investigated the Fresnel function $S(x) = \int_0^x \sin(\frac{1}{2}\pi t^2)\,dt$, which arises in the study of the diffraction of light waves. Evaluate

$$\lim_{x \to 0} \frac{S(x)}{x^3}$$

88. Suppose that the temperature in a long thin rod placed along the x-axis is initially $C/(2a)$ if $|x| \leqslant a$ and 0 if $|x| > a$. It can be shown that if the heat diffusivity of the rod is k, then the temperature of the rod at the point x at time t is

$$T(x, t) = \frac{C}{a\sqrt{4\pi kt}} \int_0^a e^{-(x-u)^2/(4kt)}\,du$$

To find the temperature distribution that results from an initial hot spot concentrated at the origin, we need to compute

$$\lim_{a \to 0} T(x, t)$$

Use l'Hospital's Rule to find this limit.

89. If f' is continuous, $f(2) = 0$, and $f'(2) = 7$, evaluate

$$\lim_{x \to 0} \frac{f(2 + 3x) + f(2 + 5x)}{x}$$

90. For what values of a and b is the following equation true?

$$\lim_{x \to 0} \left(\frac{\sin 2x}{x^3} + a + \frac{b}{x^2} \right) = 0$$

91. If f' is continuous, use l'Hospital's Rule to show that

$$\lim_{h \to 0} \frac{f(x + h) - f(x - h)}{2h} = f'(x)$$

Explain the meaning of this equation with the aid of a diagram.

92. If f'' is continuous, show that

$$\lim_{h \to 0} \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} = f''(x)$$

93. Prove that

$$\lim_{x \to \infty} \frac{e^x}{x^n} = \infty$$

for any positive integer n. This shows that the exponential function approaches infinity faster than any power of x.

94. Prove that

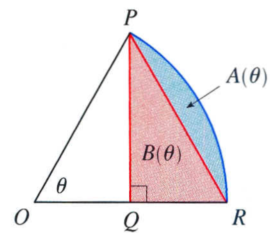
$$\lim_{x \to \infty} \frac{\ln x}{x^p} = 0$$

for any number $p > 0$. This shows that the logarithmic function approaches ∞ more slowly than any power of x.

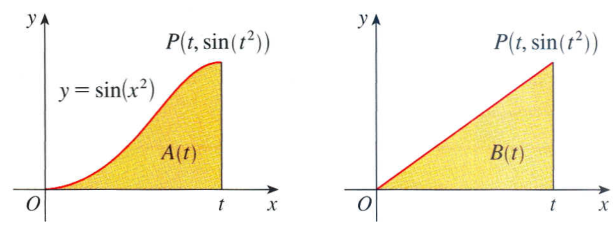
95. Prove that $\lim_{x \to 0^+} x^\alpha \ln x = 0$ for any $\alpha > 0$.

96. Evaluate $\displaystyle\lim_{x \to 0} \frac{1}{x^3} \int_0^x \sin(t^2)\,dt$.

97. The figure shows a sector of a circle with central angle θ. Let $A(\theta)$ be the area of the segment between the chord PR and the arc PR. Let $B(\theta)$ be the area of the triangle PQR. Find $\lim_{\theta \to 0^+} A(\theta)/B(\theta)$.



98. The figure shows two regions in the first quadrant: $A(t)$ is the area under the curve $y = \sin(x^2)$ from 0 to t, and $B(t)$ is the area of the triangle with vertices O, P, and $(t, 0)$. Find $\lim_{t \to 0^+} A(t)/B(t)$.



99. Let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(a) Use the definition of derivative to compute $f'(0)$.
(b) Show that f has derivatives of all orders that are defined on $\mathbb{R}$. [*Hint:* First show by induction that there is a polynomial $p_n(x)$ and a nonnegative integer k_n such that $f^{(n)}(x) = p_n(x)f(x)/x^{k_n}$ for $x \neq 0$.]

100. Let

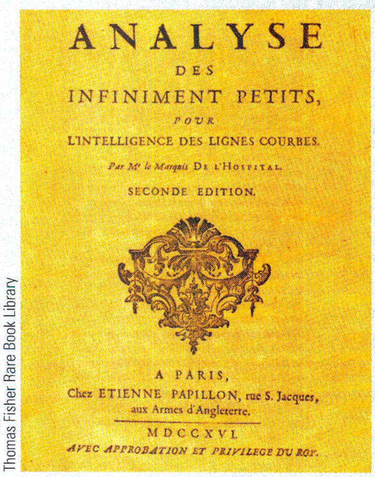
$$f(x) = \begin{cases} |x|^x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

(a) Show that f is continuous at 0.
(b) Investigate graphically whether f is differentiable at 0 by zooming in several times toward the point $(0, 1)$ on the graph of f.
(c) Show that f is not differentiable at 0. How can you reconcile this fact with the appearance of the graphs in part (b)?

WRITING PROJECT

THE ORIGINS OF L'HOSPITAL'S RULE

L'Hospital's Rule was first published in 1696 in the Marquis de l'Hospital's calculus textbook *Analyse des Infiniment Petits,* but the rule was discovered in 1694 by the Swiss mathematician John (Johann) Bernoulli. The explanation is that these two mathematicians had entered into a curious business arrangement whereby the Marquis de l'Hospital bought the rights to Bernoulli's



Thomas Fisher Rare Book Library

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The Internet is another source of information for this project. Click on *History of Mathematics* for a list of reliable websites.

mathematical discoveries. The details, including a translation of l'Hospital's letter to Bernoulli proposing the arrangement, can be found in the book by Eves [1].

Write a report on the historical and mathematical origins of l'Hospital's Rule. Start by providing brief biographical details of both men (the dictionary edited by Gillispie [2] is a good source) and outline the business deal between them. Then give l'Hospital's statement of his rule, which is found in Struik's sourcebook [4] and more briefly in the book of Katz [3]. Notice that l'Hospital and Bernoulli formulated the rule geometrically and gave the answer in terms of differentials. Compare their statement with the version of l'Hospital's Rule given in Section 7.8 and show that the two statements are essentially the same.

1. Howard Eves, *In Mathematical Circles (Volume 2: Quadrants III and IV)* (Boston: Prindle, Weber and Schmidt, 1969), pp. 20–22.
2. C. C. Gillispie, ed., *Dictionary of Scientific Biography* (New York: Scribner's, 1974). See the article on Johann Bernoulli by E. A. Fellmann and J. O. Fleckenstein in Volume II and the article on the Marquis de l'Hospital by Abraham Robinson in Volume VIII.
3. Victor Katz, *A History of Mathematics: An Introduction* (New York: HarperCollins, 1993), p. 484.
4. D. J. Struik, ed., *A Sourcebook in Mathematics, 1200–1800* (Princeton, NJ: Princeton University Press, 1969), pp. 315–316.

7 REVIEW

CONCEPT CHECK

1. (a) What is a one-to-one function? How can you tell if a function is one-to-one by looking at its graph?
   (b) If f is a one-to-one function, how is its inverse function f^{-1} defined? How do you obtain the graph of f^{-1} from the graph of f?
   (c) If f is a one-to-one function and $f'(f^{-1}(a)) \neq 0$, write a formula for $(f^{-1})'(a)$.
2. (a) What are the domain and range of the natural exponential function $f(x) = e^x$?
   (b) What are the domain and range of the natural logarithmic function $f(x) = \ln x$?
   (c) How are the graphs of these functions related? Sketch these graphs by hand, using the same axes.
   (d) If a is a positive number, $a \neq 1$, write an equation that expresses $\log_a x$ in terms of $\ln x$.
3. (a) How is the inverse sine function $f(x) = \sin^{-1}x$ defined? What are its domain and range?
   (b) How is the inverse cosine function $f(x) = \cos^{-1}x$ defined? What are its domain and range?
   (c) How is the inverse tangent function $f(x) = \tan^{-1}x$ defined? What are its domain and range? Sketch its graph.
4. Write the definitions of the hyperbolic functions $\sinh x$, $\cosh x$, and $\tanh x$.
5. State the derivative of each function.
   (a) $y = e^x$ (b) $y = a^x$ (c) $y = \ln x$
   (d) $y = \log_a x$ (e) $y = \sin^{-1}x$ (f) $y = \cos^{-1}x$
   (g) $y = \tan^{-1}x$ (h) $y = \sinh x$ (i) $y = \cosh x$
   (j) $y = \tanh x$ (k) $y = \sinh^{-1}x$ (l) $y = \cosh^{-1}x$
   (m) $y = \tanh^{-1}x$
6. (a) How is the number e defined?
   (b) Express e as a limit.
   (c) Why is the natural exponential function $y = e^x$ used more often in calculus than the other exponential functions $y = a^x$?
   (d) Why is the natural logarithmic function $y = \ln x$ used more often in calculus than the other logarithmic functions $y = \log_a x$?
7. (a) Write a differential equation that expresses the law of natural growth.
   (b) Under what circumstances is this an appropriate model for population growth?
   (c) What are the solutions of this equation?
8. (a) What does l'Hospital's Rule say?
   (b) How can you use l'Hospital's Rule if you have a product $f(x)g(x)$ where $f(x) \to 0$ and $g(x) \to \infty$ as $x \to a$?
   (c) How can you use l'Hospital's Rule if you have a difference $f(x) - g(x)$ where $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$?
   (d) How can you use l'Hospital's Rule if you have a power $[f(x)]^{g(x)}$ where $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If f is one-to-one, with domain $\mathbb{R}$, then $f^{-1}(f(6)) = 6$.

2. If f is one-to-one and differentiable, with domain $\mathbb{R}$, then $(f^{-1})'(6) = 1/f'(6)$.

3. The function $f(x) = \cos x$, $-\pi/2 \leqslant x \leqslant \pi/2$, is one-to-one.

4. $\tan^{-1}(-1) = 3\pi/4$

5. If $0 < a < b$, then $\ln a < \ln b$.

6. $\pi^{\sqrt{5}} = e^{\sqrt{5}\ln \pi}$

7. You can always divide by e^x.

8. If $a > 0$ and $b > 0$, then $\ln(a + b) = \ln a + \ln b$.

9. If $x > 0$, then $(\ln x)^6 = 6 \ln x$.

10. $\dfrac{d}{dx}(10^x) = x10^{x-1}$

11. $\dfrac{d}{dx}(\ln 10) = \dfrac{1}{10}$

12. The inverse function of $y = e^{3x}$ is $y = \frac{1}{3}\ln x$.

13. $\cos^{-1}x = \dfrac{1}{\cos x}$

14. $\tan^{-1}x = \dfrac{\sin^{-1}x}{\cos^{-1}x}$

15. $\cosh x \geqslant 1$ for all x

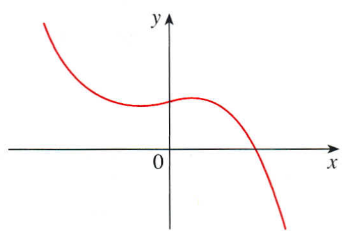
16. $\ln \dfrac{1}{10} = -\displaystyle\int_1^{10} \frac{dx}{x}$

17. $\displaystyle\int_2^{16} \frac{dx}{x} = 3 \ln 2$

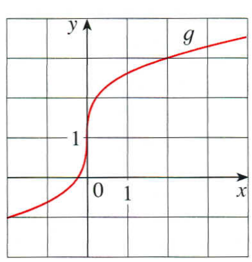
18. $\displaystyle\lim_{x\to\pi^-} \frac{\tan x}{1 - \cos x} = \lim_{x\to\pi^-} \frac{\sec^2 x}{\sin x} = \infty$

EXERCISES

1. The graph of f is shown. Is f one-to-one? Explain.



2. The graph of g is given.
(a) Why is g one-to-one?
(b) Estimate the value of $g^{-1}(2)$.
(c) Estimate the domain of g^{-1}.
(d) Sketch the graph of g^{-1}.



3. Suppose f is one-to-one, $f(7) = 3$, and $f'(7) = 8$. Find (a) $f^{-1}(3)$ and (b) $(f^{-1})'(3)$.

4. Find the inverse function of $f(x) = \dfrac{x + 1}{2x + 1}$.

5–9 Sketch a rough graph of the function without using a calculator.

5. $y = 5^x - 1$

6. $y = -e^{-x}$

7. $y = -\ln x$

8. $y = \ln(x - 1)$

9. $y = 2 \arctan x$

10. Let $a > 1$. For large values of x, which of the functions $y = x^a$, $y = a^x$, and $y = \log_a x$ has the largest values and which has the smallest values?

11–12 Find the exact value of each expression.

11. (a) $e^{2\ln 3}$ (b) $\log_{10} 25 + \log_{10} 4$

12. (a) $\ln e^{\pi}$ (b) $\tan\left(\arcsin \frac{1}{2}\right)$

13–20 Solve the equation for x.

13. $\ln x = \frac{1}{3}$

14. $e^x = \frac{1}{3}$

15. $e^{e^x} = 17$

16. $\ln(1 + e^{-x}) = 3$

17. $\ln(x + 1) + \ln(x - 1) = 1$

18. $\log_5(c^x) = d$

19. $\tan^{-1}x = 1$

20. $\sin x = 0.3$

21–47 Differentiate.

21. $f(t) = t^2 \ln t$

22. $g(t) = \dfrac{e^t}{1 + e^t}$

23. $h(\theta) = e^{\tan 2\theta}$

24. $h(u) = 10^{\sqrt{u}}$

25. $y = \ln|\sec 5x + \tan 5x|$

26. $y = e^{-t}(t^2 - 2t + 2)$

27. $y = e^{cx}(c \sin x - \cos x)$

28. $y = e^{mx} \cos nx$

29. $y = \ln(\sec^2 x)$

30. $y = \ln(x^2 e^x)$

31. $y = \dfrac{e^{1/x}}{x^2}$

32. $y = (\arcsin 2x)^2$

33. $y = 3^{x \ln x}$

34. $y = e^{\cos x} + \cos(e^x)$

35. $H(v) = v \tan^{-1} v$

36. $F(z) = \log_{10}(1 + z^2)$

37. $y = x \sinh(x^2)$

38. $y = (\cos x)^x$

39. $y = \ln \sin x - \frac{1}{2} \sin^2 x$

40. $y = \arctan\left(\arcsin \sqrt{x}\right)$

41. $y = \ln\left(\dfrac{1}{x}\right) + \dfrac{1}{\ln x}$

42. $xe^y = y - 1$

43. $y = \ln(\cosh 3x)$

44. $y = \dfrac{(x^2 + 1)^4}{(2x + 1)^3(3x - 1)^5}$

45. $y = \cosh^{-1}(\sinh x)$

46. $y = x \tanh^{-1}\sqrt{x}$

47. $y = \cos\left(e^{\sqrt{\tan 3x}}\right)$

48. Show that

$$\frac{d}{dx}\left(\frac{1}{2}\tan^{-1}x + \frac{1}{4}\ln\frac{(x+1)^2}{x^2+1}\right) = \frac{1}{(1+x)(1+x^2)}$$

49–52 Find f' in terms of g'.

49. $f(x) = e^{g(x)}$

50. $f(x) = g(e^x)$

51. $f(x) = \ln|g(x)|$

52. $f(x) = g(\ln x)$

53–54 Find $f^{(n)}(x)$.

53. $f(x) = 2^x$

54. $f(x) = \ln(2x)$

55. Use mathematical induction to show that if $f(x) = xe^x$, then $f^{(n)}(x) = (x + n)e^x$.

56. Find y' if $y = x + \arctan y$.

57–58 Find an equation of the tangent to the curve at the given point.

57. $y = (2 + x)e^{-x}$, $(0, 2)$

58. $y = x \ln x$, (e, e)

59. At what point on the curve $y = [\ln(x + 4)]^2$ is the tangent horizontal?

60. If $f(x) = xe^{\sin x}$, find $f'(x)$. Graph f and f' on the same screen and comment.

61. (a) Find an equation of the tangent to the curve $y = e^x$ that is parallel to the line $x - 4y = 1$.
(b) Find an equation of the tangent to the curve $y = e^x$ that passes through the origin.

62. The function $C(t) = K(e^{-at} - e^{-bt})$, where a, b, and K are positive constants and $b > a$, is used to model the concentration at time t of a drug injected into the bloodstream.
(a) Show that $\lim_{t\to\infty} C(t) = 0$.
(b) Find $C'(t)$, the rate at which the drug is cleared from circulation.
(c) When is this rate equal to 0?

63–78 Evaluate the limit.

63. $\lim_{x\to\infty} e^{-3x}$

64. $\lim_{x\to 10^-} \ln(100 - x^2)$

65. $\lim_{x\to 3^-} e^{2/(x-3)}$

66. $\lim_{x\to\infty} \arctan(x^3 - x)$

67. $\lim_{x\to 0^+} \ln(\sinh x)$

68. $\lim_{x\to\infty} e^{-x} \sin x$

69. $\lim_{x\to\infty} \dfrac{1 + 2^x}{1 - 2^x}$

70. $\lim_{x\to\infty} \left(1 + \dfrac{4}{x}\right)^x$

71. $\lim_{x\to 0} \dfrac{\tan \pi x}{\ln(1 + x)}$

72. $\lim_{x\to 0} \dfrac{1 - \cos x}{x^2 + x}$

73. $\lim_{x\to 0} \dfrac{e^{4x} - 1 - 4x}{x^2}$

74. $\lim_{x\to\infty} \dfrac{e^{4x} - 1 - 4x}{x^2}$

75. $\lim_{x\to\infty} x^3 e^{-x}$

76. $\lim_{x\to 0^+} x^2 \ln x$

77. $\lim_{x\to 1^+} \left(\dfrac{x}{x - 1} - \dfrac{1}{\ln x}\right)$

78. $\lim_{x\to(\pi/2)^-} (\tan x)^{\cos x}$

79–84 Sketch the curve using the guidelines of Section 4.5.

79. $y = \tan^{-1}(1/x)$

80. $y = \sin^{-1}(1/x)$

81. $y = x \ln x$

82. $y = e^{2x - x^2}$

83. $y = xe^{-2x}$

84. $y = x + \ln(x^2 + 1)$

85. Graph $f(x) = e^{-1/x^2}$ in a viewing rectangle that shows all the main aspects of this function. Estimate the inflection points. Then use calculus to find them exactly.

86. Investigate the family of functions $f(x) = cxe^{-cx^2}$. What happens to the maximum and minimum points and the inflection points as c changes? Illustrate your conclusions by graphing several members of the family.

87. An equation of motion of the form $s = Ae^{-ct}\cos(\omega t + \delta)$ represents damped oscillation of an object. Find the velocity and acceleration of the object.

88. (a) Show that there is exactly one root of the equation $\ln x = 3 - x$ and that it lies between 2 and e.
(b) Find the root of the equation in part (a) correct to four decimal places.

89. A bacteria culture contains 200 cells initially and grows at a rate proportional to its size. After half an hour the population has increased to 360 cells.
(a) Find the number of bacteria after t hours.
(b) Find the number of bacteria after 4 hours.
(c) Find the rate of growth after 4 hours.
(d) When will the population reach 10,000?

90. Cobalt-60 has a half-life of 5.24 years.
(a) Find the mass that remains from a 100-mg sample after 20 years.
(b) How long would it take for the mass to decay to 1 mg?

91. The biologist G. F. Gause conducted an experiment in the 1930s with the protozoan *Paramecium* and used the population function

$$P(t) = \frac{64}{1 + 31e^{-0.7944t}}$$

to model his data, where t was measured in days. Use this model to determine when the population was increasing most rapidly.

92–105 Evaluate the integral.

92. $\int_0^4 \frac{1}{16 + t^2}\,dt$

93. $\int_0^1 ye^{-2y^2}\,dy$

94. $\int_2^5 \frac{dr}{1 + 2r}$

95. $\int_0^1 \frac{e^x}{1 + e^{2x}}\,dx$

96. $\int_0^{\pi/2} \frac{\cos x}{1 + \sin^2 x}\,dx$

97. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

98. $\int \frac{\cos(\ln x)}{x}\,dx$

99. $\int \frac{x + 1}{x^2 + 2x}\,dx$

100. $\int \frac{\csc^2 x}{1 + \cot x}\,dx$

101. $\int \tan x \ln(\cos x)\,dx$

102. $\int \frac{x}{\sqrt{1 - x^4}}\,dx$

103. $\int 2^{\tan\theta} \sec^2\theta\,d\theta$

104. $\int \sinh au\,du$

105. $\int \left(\frac{1 - x}{x}\right)^2 dx$

106–108 Use properties of integrals to prove the inequality.

106. $\int_0^1 \sqrt{1 + e^{2x}}\,dx \geqslant e - 1$

107. $\int_0^1 e^x \cos x\,dx \leqslant e - 1$

108. $\int_0^1 x \sin^{-1} x\,dx \leqslant \pi/4$

109–110 Find $f'(x)$.

109. $f(x) = \int_1^{\sqrt{x}} \frac{e^s}{s}\,ds$

110. $f(x) = \int_{\ln x}^{2x} e^{-t^2}\,dt$

111. Find the average value of the function $f(x) = 1/x$ on the interval $[1, 4]$.

112. Find the area of the region bounded by the curves $y = e^x$, $y = e^{-x}$, $x = -2$, and $x = 1$.

113. Find the volume of the solid obtained by rotating about the y-axis the region under the curve $y = 1/(1 + x^4)$ from $x = 0$ to $x = 1$.

114. If $f(x) = x + x^2 + e^x$, find $(f^{-1})'(1)$.

115. If $f(x) = \ln x + \tan^{-1}x$, find $(f^{-1})'(\pi/4)$.

116. What is the area of the largest rectangle in the first quadrant with two sides on the axes and one vertex on the curve $y = e^{-x}$?

117. What is the area of the largest triangle in the first quadrant with two sides on the axes and the third side tangent to the curve $y = e^{-x}$?

118. Evaluate $\int_0^1 e^x\,dx$ without using the Fundamental Theorem of Calculus. [*Hint:* Use the definition of a definite integral with right endpoints, sum a geometric series, and then use l'Hospital's Rule.]

119. If $F(x) = \int_a^b t^x\,dt$, where $a, b > 0$, then, by the Fundamental Theorem,

$$F(x) = \frac{b^{x+1} - a^{x+1}}{x + 1} \qquad x \neq -1$$

$$F(-1) = \ln b - \ln a$$

Use l'Hospital's Rule to show that F is continuous at -1.

120. Show that

$$\cos\{\arctan[\sin(\operatorname{arccot} x)]\} = \sqrt{\frac{x^2 + 1}{x^2 + 2}}$$

121. If f is a continuous function such that

$$\int_0^x f(t)\,dt = xe^{2x} + \int_0^x e^{-t} f(t)\,dt$$

for all x, find an explicit formula for $f(x)$.

122. (a) Show that $\ln x < x - 1$ for $x > 0$, $x \neq 1$.

(b) Show that, for $x > 0$, $x \neq 1$,

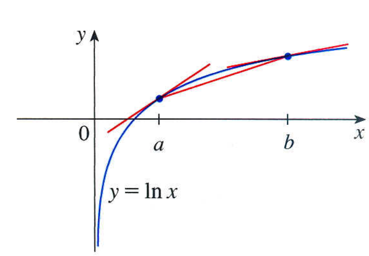
$$\frac{x - 1}{x} < \ln x$$

(c) Deduce *Napier's Inequality:*

$$\frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}$$

if $b > a > 0$.

(d) Give a geometric proof of Napier's Inequality by comparing the slopes of the three lines shown in the figure.



(e) Give another proof of Napier's Inequality by applying Property 8 of integrals (see Section 5.2) to $\int_a^b (1/x)\,dx$.

PROBLEMS

1. If a rectangle has its base on the x-axis and two vertices on the curve $y = e^{-x^2}$, show that the rectangle has the largest possible area when the two vertices are at the points of inflection of the curve.

2. Prove that $\log_2 5$ is an irrational number.

3. Show that

$$\frac{d^n}{dx^n}(e^{ax}\sin bx) = r^n e^{ax}\sin(bx + n\theta)$$

where a and b are positive numbers, $r^2 = a^2 + b^2$, and $\theta = \tan^{-1}(b/a)$.

4. Show that $\sin^{-1}(\tanh x) = \tan^{-1}(\sinh x)$.

5. Show that, for $x > 0$,

$$\frac{x}{1 + x^2} < \tan^{-1}x < x$$

6. Suppose f is continuous, $f(0) = 0$, $f(1) = 1$, $f'(x) > 0$, and $\int_0^1 f(x)\,dx = \frac{1}{3}$. Find the value of the integral $\int_0^1 f^{-1}(y)\,dy$.

7. Show that $f(x) = \int_1^x \sqrt{1 + t^3}\,dt$ is one-to-one and find $(f^{-1})'(0)$.

8. If

$$y = \frac{x}{\sqrt{a^2 - 1}} - \frac{2}{\sqrt{a^2 - 1}}\arctan\frac{\sin x}{a + \sqrt{a^2 - 1} + \cos x}$$

show that $y' = \dfrac{1}{a + \cos x}$.

9. For what value of a is the following equation true?

$$\lim_{x\to\infty}\left(\frac{x + a}{x - a}\right)^x = e$$

10. Sketch the set of all points (x, y) such that $|x + y| \leqslant e^x$.

11. Prove that $\cosh(\sinh x) < \sinh(\cosh x)$ for all x.

12. Show that, for all positive values of x and y,

$$\frac{e^{x+y}}{xy} \geqslant e^2$$

13. For what value of k does the equation $e^{2x} = k\sqrt{x}$ have exactly one solution?

14. For which positive numbers a is it true that $a^x \geqslant 1 + x$ for all x?

15. For which positive numbers a does the curve $y = a^x$ intersect the line $y = x$?

16. For what values of c does the curve $y = cx^3 + e^x$ have inflection points?