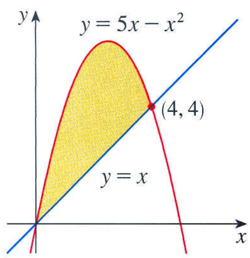
6.1 EXERCISES

1–4 Find the area of the shaded region.

1.



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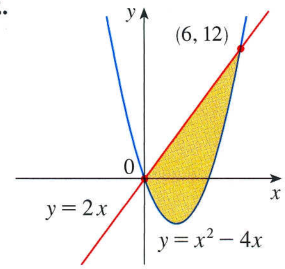

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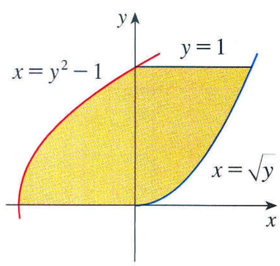

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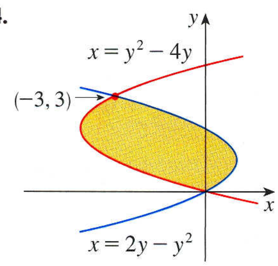


5–28 Sketch the region enclosed by the given curves. Decide whether to integrate with respect to x or y. Draw a typical approximating rectangle and label its height and width. Then find the area of the region.

5. $y = x + 1, \quad y = 9 - x^2, \quad x = -1, \quad x = 2$

6. $y = \sin x, \quad y = x, \quad x = \pi/2, \quad x = \pi$

7. $y = x, \quad y = x^2$

8. $y = x^2, \quad y = x^4$

9. $y = \sqrt{x + 3}, \quad y = (x + 3)/2$

10. $y = 1 + \sqrt{x}, \quad y = (3 + x)/3$

11. $y = x^2, \quad y^2 = x$

12. $y = x^2, \quad y = 4x - x^2$

13. $y = 12 - x^2, \quad y = x^2 - 6$

14. $y = \cos x, \quad y = 2 - \cos x, \quad 0 \leqslant x \leqslant 2\pi$

15. $y = \sec^2 x, \quad y = 8 \cos x, \quad -\pi/3 \leqslant x \leqslant \pi/3$

16. $y = x^3 - x, \quad y = 3x$

17. $y = \sqrt{x}, \quad y = \frac{1}{2}x, \quad x = 9$

18. $y = x^2 + 1, \quad y = 3 - x^2, \quad x = -2, \quad x = 2$

19. $x = 2y^2, \quad x = 4 + y^2$

20. $4x + y^2 = 12, \quad x = y$

21. $x = 1 - y^2, \quad x = y^2 - 1$

22. $y = \sin(\pi x/2), \quad y = x$

23. $y = \cos x, \quad y = \sin 2x, \quad x = 0, \quad x = \pi/2$

24. $y = \cos x, \quad y = 1 - \cos x, \quad 0 \leqslant x \leqslant \pi$

25. $y = \cos x, \quad y = 1 - 2x/\pi$

26. $y = |x|, \quad y = x^2 - 2$

27. $y = 1/x^2, \quad y = x, \quad y = \frac{1}{8}x$

28. $y = 3x^2, \quad y = 8x^2, \quad 4x + y = 4, \quad x \geqslant 0$

29–30 Use calculus to find the area of the triangle with the given vertices.

29. (0, 0), (2, 1), (−1, 6)

30. (0, 5), (2, −2), (5, 1)

31–32 Evaluate the integral and interpret it as the area of a region. Sketch the region.

31. $\displaystyle\int_0^{\pi/2} |\sin x - \cos 2x| \, dx$

32. $\displaystyle\int_0^4 |\sqrt{x + 2} - x| \, dx$

33–34 Use the Midpoint Rule with $n = 4$ to approximate the area of the region bounded by the given curves.

33. $y = \sin^2(\pi x/4), \quad y = \cos^2(\pi x/4), \quad 0 \leqslant x \leqslant 1$

34. $y = \sqrt[3]{16 - x^3}, \quad y = x, \quad x = 0$

35–38 Use a graph to find approximate x-coordinates of the points of intersection of the given curves. Then find (approximately) the area of the region bounded by the curves.

35. $y = x \sin(x^2), \quad y = x^4$

36. $y = x^4, \quad y = 3x - x^3$

37. $y = 3x^2 - 2x, \quad y = x^3 - 3x + 4$

38. $y = x \cos x, \quad y = x^{10}$

CAS **39.** Use a computer algebra system to find the exact area enclosed by the curves $y = x^5 - 6x^3 + 4x$ and $y = x$.

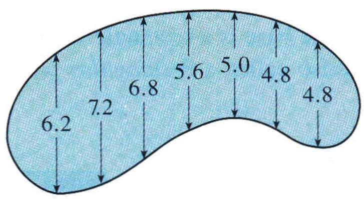
40. Sketch the region in the xy-plane defined by the inequalities $x - 2y^2 \geqslant 0$, $1 - x - |y| \geqslant 0$ and find its area.

41. Racing cars driven by Chris and Kelly are side by side at the start of a race. The table shows the velocities of each car (in kilometers per hour) during the first ten seconds of the race.

Use the Midpoint Rule to estimate how much farther Kelly travels than Chris does during the first ten seconds.

t	v_C	v_K	t	v_C	v_K
0	0	0	6	110	128
1	32	35	7	120	138
2	51	59	8	130	150
3	74	83	9	138	157
4	86	98	10	144	163
5	99	114			

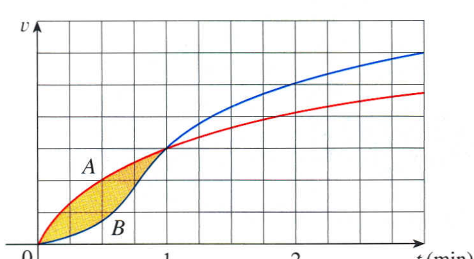
42. The widths (in meters) of a kidney-shaped swimming pool were measured at 2-meter intervals as indicated in the figure. Use the Midpoint Rule to estimate the area of the pool.



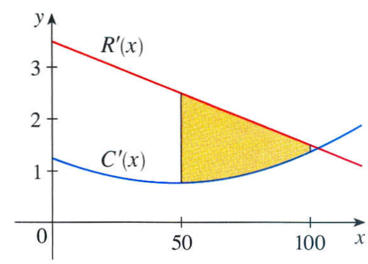
43. A cross-section of an airplane wing is shown. Measurements of the thickness of the wing, in centimeters, at 20-centimeter intervals are 5.8, 20.3, 26.7, 29.0, 27.6, 27.3, 23.8, 20.5, 15.1, 8.7, and 2.8. Use the Midpoint Rule to estimate the area of the wing's cross-section.

44. If the birth rate of a population is $b(t) = 2200 + 52.3t + 0.74t^2$ people per year and the death rate is $d(t) = 1460 + 28.8t$ people per year, find the area between these curves for $0 \leqslant t \leqslant 10$. What does this area represent?

45. Two cars, A and B, start side by side and accelerate from rest. The figure shows the graphs of their velocity functions.
(a) Which car is ahead after one minute? Explain.
(b) What is the meaning of the area of the shaded region?
(c) Which car is ahead after two minutes? Explain.
(d) Estimate the time at which the cars are again side by side.



46. The figure shows graphs of the marginal revenue function R' and the marginal cost function C' for a manufacturer. [Recall from Section 4.7 that $R(x)$ and $C(x)$ represent the revenue and cost when x units are manufactured. Assume that R and C are measured in thousands of dollars.] What is the meaning of the area of the shaded region? Use the Midpoint Rule to estimate the value of this quantity.



47. The curve with equation $y^2 = x^2(x + 3)$ is called **Tschirnhausen's cubic**. If you graph this curve you will see that part of the curve forms a loop. Find the area enclosed by the loop.

48. Find the area of the region bounded by the parabola $y = x^2$, the tangent line to this parabola at $(1, 1)$, and the x-axis.

49. Find the number b such that the line $y = b$ divides the region bounded by the curves $y = x^2$ and $y = 4$ into two regions with equal area.

50. (a) Find the number a such that the line $x = a$ bisects the area under the curve $y = 1/x^2$, $1 \leqslant x \leqslant 4$.
(b) Find the number b such that the line $y = b$ bisects the area in part (a).

51. Find the values of c such that the area of the region bounded by the parabolas $y = x^2 - c^2$ and $y = c^2 - x^2$ is 576.

52. Suppose that $0 < c < \pi/2$. For what value of c is the area of the region enclosed by the curves $y = \cos x$, $y = \cos(x - c)$, and $x = 0$ equal to the area of the region enclosed by the curves $y = \cos(x - c)$, $x = \pi$, and $y = 0$?

The following exercises are intended only for those who have already covered Chapter 7.

53–55 Sketch the region bounded by the given curves and find the area of the region.

53. $y = 1/x, \quad y = 1/x^2, \quad x = 2$

54. $y = \sin x, \quad y = e^x, \quad x = 0, \quad x = \pi/2$

55. $y = \tan x, \quad y = 2\sin x, \quad -\pi/3 \leqslant x \leqslant \pi/3$

56. For what values of m do the line $y = mx$ and the curve $y = x/(x^2 + 1)$ enclose a region? Find the area of the region.

we would have obtained the integral

$$V = \int_0^h \frac{L^2}{h^2}(h - y)^2\,dy = \frac{L^2 h}{3}$$

EXAMPLE 9 A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 30° along a diameter of the cylinder. Find the volume of the wedge.

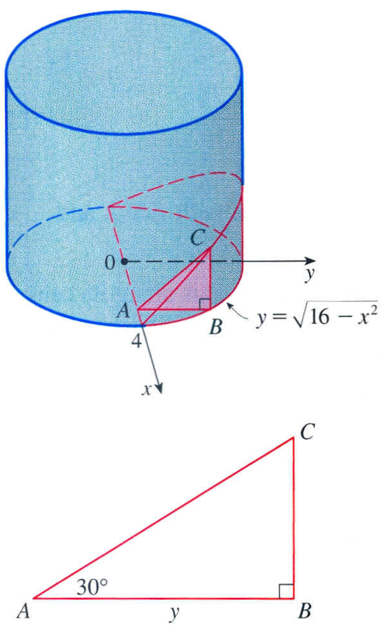


FIGURE 17

SOLUTION If we place the x-axis along the diameter where the planes meet, then the base of the solid is a semicircle with equation $y = \sqrt{16 - x^2}$, $-4 \leqslant x \leqslant 4$. A cross-section perpendicular to the x-axis at a distance x from the origin is a triangle ABC, as shown in Figure 17, whose base is $y = \sqrt{16 - x^2}$ and whose height is $|BC| = y \tan 30° = \sqrt{16 - x^2}/\sqrt{3}$. Thus the cross-sectional area is

$$A(x) = \tfrac{1}{2}\sqrt{16 - x^2} \cdot \frac{1}{\sqrt{3}}\sqrt{16 - x^2} = \frac{16 - x^2}{2\sqrt{3}}$$

and the volume is

$$\begin{aligned} V &= \int_{-4}^{4} A(x)\,dx = \int_{-4}^{4} \frac{16 - x^2}{2\sqrt{3}}\,dx \\ &= \frac{1}{\sqrt{3}}\int_0^4 (16 - x^2)\,dx = \frac{1}{\sqrt{3}}\left[16x - \frac{x^3}{3}\right]_0^4 \\ &= \frac{128}{3\sqrt{3}} \end{aligned}$$

For another method see Exercise 64. □

6.2 EXERCISES

1–18 Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region, the solid, and a typical disk or washer.

1. $y = 2 - \frac{1}{2}x$, $y = 0$, $x = 1$, $x = 2$; about the x-axis

2. $y = 1 - x^2$, $y = 0$; about the x-axis

3. $y = 1/x$, $x = 1$, $x = 2$, $y = 0$; about the x-axis

4. $y = \sqrt{25 - x^2}$, $y = 0$, $x = 2$, $x = 4$; about the x-axis

5. $x = 2\sqrt{y}$, $x = 0$, $y = 9$; about the y-axis

6. $x = y - y^2$, $x = 0$; about the y-axis

7. $y = x^3$, $y = x$, $x \geqslant 0$; about the x-axis

8. $y = \sec x$, $y = 1$, $x = -1$, $x = 1$; about the x-axis

9. $y^2 = x$, $x = 2y$; about the y-axis

10. $y = x^{2/3}$, $x = 1$, $y = 0$; about the y-axis

11. $y = x$, $y = \sqrt{x}$; about $y = 1$

12. $y = x^2$, $y = 4$; about $y = 4$

13. $y = 1 + \sec x$, $y = 3$; about $y = 1$

14. $y = 1/x^2$, $y = 0$, $x = 1$, $x = 3$; about $y = -1$

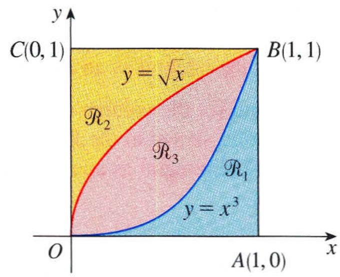
15. $x = y^2$, $x = 1$; about $x = 1$

16. $y = x$, $y = \sqrt{x}$; about $x = 2$

17. $y = x^2$, $x = y^2$; about $x = -1$

18. $y = x$, $y = 0$, $x = 2$, $x = 4$; about $x = 1$

19–30 Refer to the figure and find the volume generated by rotating the given region about the specified line.



19. $\mathcal{R}_1$ about OA

20. $\mathcal{R}_1$ about OC

21. $\mathcal{R}_1$ about AB

22. $\mathcal{R}_1$ about BC

23. $\mathcal{R}_2$ about OA

24. $\mathcal{R}_2$ about OC

25. $\mathcal{R}_2$ about AB

26. $\mathcal{R}_2$ about BC

27. $\mathcal{R}_3$ about OA

28. $\mathcal{R}_3$ about OC

29. $\mathcal{R}_3$ about AB

30. $\mathcal{R}_3$ about BC

31–36 Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

31. $y = \tan^3 x,\ y = 1,\ x = 0;$ about $y = 1$

32. $y = (x-2)^4,\ 8x - y = 16;$ about $x = 10$

33. $y = 0,\ y = \sin x,\ 0 \leqslant x \leqslant \pi;$ about $y = 1$

34. $y = 0,\ y = \sin x,\ 0 \leqslant x \leqslant \pi;$ about $y = -2$

35. $x^2 - y^2 = 1,\ x = 3;$ about $x = -2$

36. $y = \cos x,\ y = 2 - \cos x,\ 0 \leqslant x \leqslant 2\pi;$ about $y = 4$

37–38 Use a graph to find approximate x-coordinates of the points of intersection of the given curves. Then use your calculator to find (approximately) the volume of the solid obtained by rotating about the x-axis the region bounded by these curves.

37. $y = 2 + x^2 \cos x,\quad y = x^4 + x + 1$

38. $y = x^4,\quad y = 3x - x^3$

CAS **39–40** Use a computer algebra system to find the exact volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

39. $y = \sin^2 x,\ y = 0,\ 0 \leqslant x \leqslant \pi;$ about $y = -1$

40. $y = x^2 - 2x,\ y = x\cos(\pi x/4);$ about $y = 2$

41–44 Each integral represents the volume of a solid. Describe the solid.

41. $\pi \int_0^{\pi/2} \cos^2 x\, dx$

42. $\pi \int_2^5 y\, dy$

43. $\pi \int_0^1 (y^4 - y^8)\, dy$

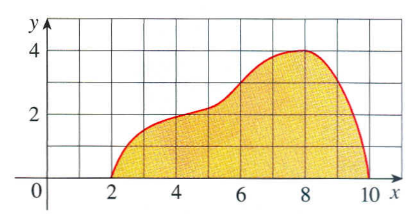
44. $\pi \int_0^{\pi/2} [(1+\cos x)^2 - 1^2]\, dx$

45. A CAT scan produces equally spaced cross-sectional views of a human organ that provide information about the organ otherwise obtained only by surgery. Suppose that a CAT scan of a human liver shows cross-sections spaced 1.5 cm apart. The liver is 15 cm long and the cross-sectional areas, in square centimeters, are 0, 18, 58, 79, 94, 106, 117, 128, 63, 39, and 0. Use the Midpoint Rule to estimate the volume of the liver.

46. A log 10 m long is cut at 1-meter intervals and its cross-sectional areas A (at a distance x from the end of the log) are listed in the table. Use the Midpoint Rule with $n = 5$ to estimate the volume of the log.

x (m)	A (m²)	x (m)	A (m²)
0	0.68	6	0.53
1	0.65	7	0.55
2	0.64	8	0.52
3	0.61	9	0.50
4	0.58	10	0.48
5	0.59		

47. (a) If the region shown in the figure is rotated about the x-axis to form a solid, use the Midpoint Rule with $n = 4$ to estimate the volume of the solid.



(b) Estimate the volume if the region is rotated about the y-axis. Again use the Midpoint Rule with $n = 4$.

CAS **48.** (a) A model for the shape of a bird's egg is obtained by rotating about the x-axis the region under the graph of

$$f(x) = (ax^3 + bx^2 + cx + d)\sqrt{1 - x^2}$$

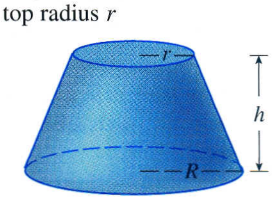
Use a CAS to find the volume of such an egg.

(b) For a Red-throated Loon, $a = -0.06$, $b = 0.04$, $c = 0.1$, and $d = 0.54$. Graph f and find the volume of an egg of this species.

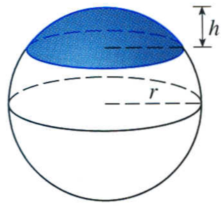
49–61 Find the volume of the described solid S.

49. A right circular cone with height h and base radius r

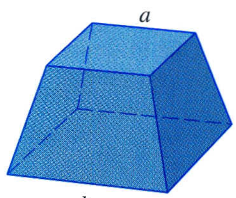
50. A frustum of a right circular cone with height h, lower base radius R, and top radius r



51. A cap of a sphere with radius r and height h



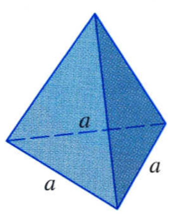
52. A frustum of a pyramid with square base of side b, square top of side a, and height h



What happens if $a = b$? What happens if $a = 0$?

53. A pyramid with height h and rectangular base with dimensions b and $2b$

54. A pyramid with height h and base an equilateral triangle with side a (a tetrahedron)



55. A tetrahedron with three mutually perpendicular faces and three mutually perpendicular edges with lengths 3 cm, 4 cm, and 5 cm

56. The base of S is a circular disk with radius r. Parallel cross-sections perpendicular to the base are squares.

57. The base of S is an elliptical region with boundary curve $9x^2 + 4y^2 = 36$. Cross-sections perpendicular to the x-axis are isosceles right triangles with hypotenuse in the base.

58. The base of S is the triangular region with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. Cross-sections perpendicular to the y-axis are equilateral triangles.

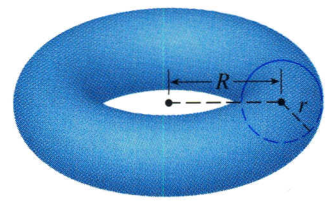
59. The base of S is the same base as in Exercise 58, but cross-sections perpendicular to the x-axis are squares.

60. The base of S is the region enclosed by the parabola $y = 1 - x^2$ and the x-axis. Cross-sections perpendicular to the y-axis are squares.

61. The base of S is the same base as in Exercise 60, but cross-sections perpendicular to the x-axis are isosceles triangles with height equal to the base.

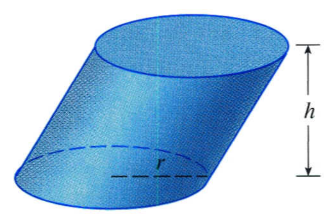
62. The base of S is a circular disk with radius r. Parallel cross-sections perpendicular to the base are isosceles triangles with height h and unequal side in the base.
    (a) Set up an integral for the volume of S.
    (b) By interpreting the integral as an area, find the volume of S.

63. (a) Set up an integral for the volume of a solid *torus* (the donut-shaped solid shown in the figure) with radii r and R.
    (b) By interpreting the integral as an area, find the volume of the torus.

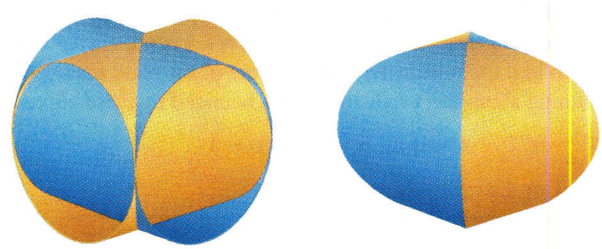


64. Solve Example 9 taking cross-sections to be parallel to the line of intersection of the two planes.

65. (a) Cavalieri's Principle states that if a family of parallel planes gives equal cross-sectional areas for two solids S_1 and S_2, then the volumes of S_1 and S_2 are equal. Prove this principle.
    (b) Use Cavalieri's Principle to find the volume of the oblique cylinder shown in the figure.



66. Find the volume common to two circular cylinders, each with radius r, if the axes of the cylinders intersect at right angles.



67. Find the volume common to two spheres, each with radius r, if the center of each sphere lies on the surface of the other sphere.

68. A bowl is shaped like a hemisphere with diameter 30 cm. A heavy ball with diameter 10 cm is placed in the bowl and water is poured into the bowl to a depth of h centimeters. Find the volume of water in the bowl.

69. A hole of radius r is bored through the middle of a cylinder of radius $R > r$ at right angles to the axis of the cylinder. Set up, but do not evaluate, an integral for the volume cut out.

70. A hole of radius r is bored through the center of a sphere of radius $R > r$. Find the volume of the remaining portion of the sphere.

71. Some of the pioneers of calculus, such as Kepler and Newton, were inspired by the problem of finding the volumes of wine barrels. (In fact Kepler published a book *Stereometria doliorum* in 1715 devoted to methods for finding the volumes of barrels.) They often approximated the shape of the sides by parabolas.

(a) A barrel with height h and maximum radius R is constructed by rotating about the x-axis the parabola $y = R - cx^2$, $-h/2 \leqslant x \leqslant h/2$, where c is a positive constant. Show that the radius of each end of the barrel is $r = R - d$, where $d = ch^2/4$.

(b) Show that the volume enclosed by the barrel is

$$V = \tfrac{1}{3}\pi h\left(2R^2 + r^2 - \tfrac{2}{5}d^2\right)$$

72. Suppose that a region $\mathscr{R}$ has area A and lies above the x-axis. When $\mathscr{R}$ is rotated about the x-axis, it sweeps out a solid with volume V_1. When $\mathscr{R}$ is rotated about the line $y = -k$ (where k is a positive number), it sweeps out a solid with volume V_2. Express V_2 in terms of V_1, k, and A.

6.3 VOLUMES BY CYLINDRICAL SHELLS

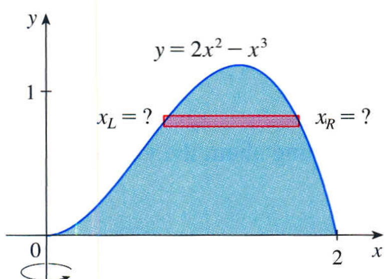


FIGURE 1

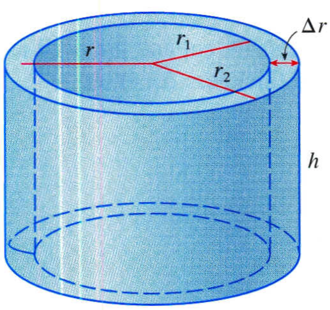


FIGURE 2

Some volume problems are very difficult to handle by the methods of the preceding section. For instance, let's consider the problem of finding the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$. (See Figure 1.) If we slice perpendicular to the y-axis, we get a washer. But to compute the inner radius and the outer radius of the washer, we would have to solve the cubic equation $y = 2x^2 - x^3$ for x in terms of y; that's not easy.

Fortunately, there is a method, called the **method of cylindrical shells**, that is easier to use in such a case. Figure 2 shows a cylindrical shell with inner radius r_1, outer radius r_2, and height h. Its volume V is calculated by subtracting the volume V_1 of the inner cylinder from the volume V_2 of the outer cylinder:

$$V = V_2 - V_1 = \pi r_2^2 h - \pi r_1^2 h = \pi(r_2^2 - r_1^2)h$$

$$= \pi(r_2 + r_1)(r_2 - r_1)h = 2\pi \frac{r_2 + r_1}{2} h(r_2 - r_1)$$

If we let $\Delta r = r_2 - r_1$ (the thickness of the shell) and $r = \frac{1}{2}(r_2 + r_1)$ (the average radius of the shell), then this formula for the volume of a cylindrical shell becomes

$$\boxed{1} \qquad \boxed{V = 2\pi r h \,\Delta r}$$

and it can be remembered as

$$V = [\text{circumference}][\text{height}][\text{thickness}]$$

Now let S be the solid obtained by rotating about the y-axis the region bounded by $y = f(x)$ [where $f(x) \geqslant 0$], $y = 0$, $x = a$, and $x = b$, where $b > a \geqslant 0$. (See Figure 3.)

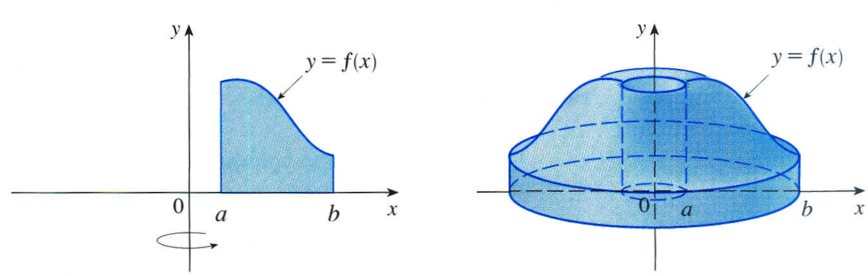


FIGURE 3

V EXAMPLE 4 Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

SOLUTION Figure 10 shows the region and a cylindrical shell formed by rotation about the line $x = 2$. It has radius $2 - x$, circumference $2\pi(2 - x)$, and height $x - x^2$.

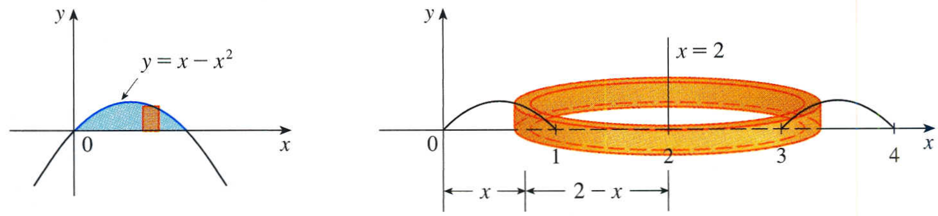


FIGURE 10

The volume of the given solid is

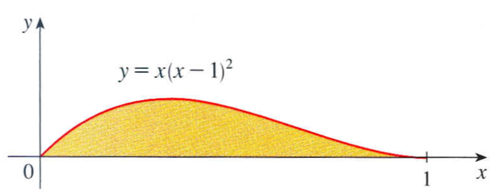
$$V = \int_0^1 2\pi(2 - x)(x - x^2)\,dx = 2\pi \int_0^1 (x^3 - 3x^2 + 2x)\,dx$$

$$= 2\pi \left[\frac{x^4}{4} - x^3 + x^2 \right]_0^1 = \frac{\pi}{2}$$

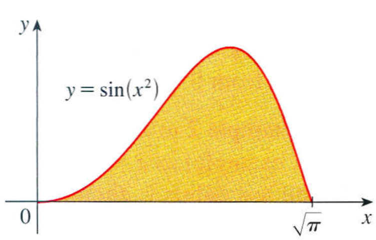
□

6.3 EXERCISES

1. Let S be the solid obtained by rotating the region shown in the figure about the y-axis. Explain why it is awkward to use slicing to find the volume V of S. Sketch a typical approximating shell. What are its circumference and height? Use shells to find V.



2. Let S be the solid obtained by rotating the region shown in the figure about the y-axis. Sketch a typical cylindrical shell and find its circumference and height. Use shells to find the volume of S. Do you think this method is preferable to slicing? Explain.



3–7 Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the y-axis. Sketch the region and a typical shell.

3. $y = 1/x, \quad y = 0, \quad x = 1, \quad x = 2$

4. $y = x^2, \quad y = 0, \quad x = 1$

5. $y = x^2, \quad 0 \leqslant x \leqslant 2, \quad y = 4, \quad x = 0$

6. $y = x^2 - 6x + 10, \quad y = -x^2 + 6x - 6$

7. $y^2 = x, \quad x = 2y$

8. Let V be the volume of the solid obtained by rotating about the y-axis the region bounded by $y = \sqrt{x}$ and $y = x^2$. Find V both by slicing and by cylindrical shells. In both cases draw a diagram to explain your method.

9–14 Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by the given curves about the x-axis. Sketch the region and a typical shell.

9. $x = 1 + y^2, \quad x = 0, \quad y = 1, \quad y = 2$

10. $x = \sqrt{y}, \quad x = 0, \quad y = 1$

11. $y = x^3, \quad y = 8, \quad x = 0$

12. $x = 4y^2 - y^3, \quad x = 0$

13. $x = 1 + (y - 2)^2, \quad x = 2$

14. $x + y = 3, \quad x = 4 - (y - 1)^2$

15–20 Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the specified axis. Sketch the region and a typical shell.

15. $y = x^2, \ y = 0, \ x = 1, \ x = 2; \quad$ about $x = 1$

16. $y = \sqrt{x}$, $y = 0$, $x = 1$; about $x = -1$

17. $y = 4x - x^2$, $y = 3$; about $x = 1$

18. $y = x^2$, $y = 2 - x^2$; about $x = 1$

19. $y = x^3$, $y = 0$, $x = 1$; about $y = 1$

20. $y = x^2$, $x = y^2$; about $y = -1$

21–26 Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis.

21. $y = \sin x$, $y = 0$, $x = 2\pi$, $x = 3\pi$; about the y-axis

22. $y = x$, $y = 4x - x^2$; about $x = 7$

23. $y = x^4$, $y = \sin(\pi x/2)$; about $x = -1$

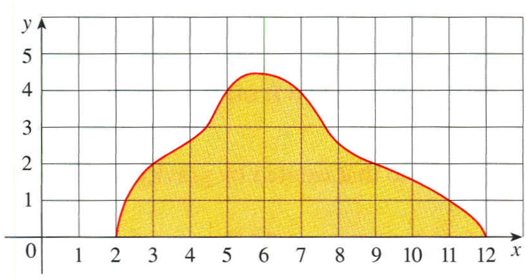
24. $y = 1/(1 + x^2)$, $y = 0$, $x = 0$, $x = 2$; about $x = 2$

25. $x = \sqrt{\sin y}$, $0 \leqslant y \leqslant \pi$, $x = 0$; about $y = 4$

26. $x^2 - y^2 = 7$, $x = 4$; about $y = 5$

27. Use the Midpoint Rule with $n = 5$ to estimate the volume obtained by rotating about the y-axis the region under the curve $y = \sqrt{1 + x^3}$, $0 \leqslant x \leqslant 1$.

28. If the region shown in the figure is rotated about the y-axis to form a solid, use the Midpoint Rule with $n = 5$ to estimate the volume of the solid.



29–32 Each integral represents the volume of a solid. Describe the solid.

29. $\int_0^3 2\pi x^5\,dx$

30. $2\pi \int_0^2 \dfrac{y}{1 + y^2}\,dy$

31. $\int_0^1 2\pi(3 - y)(1 - y^2)\,dy$

32. $\int_0^{\pi/4} 2\pi(\pi - x)(\cos x - \sin x)\,dx$

33–34 Use a graph to estimate the x-coordinates of the points of intersection of the given curves. Then use this information and your calculator to estimate the volume of the solid obtained by rotating about the y-axis the region enclosed by these curves.

33. $y = 0$, $y = x + x^2 - x^4$

34. $y = x^3 - x + 1$, $y = -x^4 + 4x - 1$

CAS **35–36** Use a computer algebra system to find the exact volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

35. $y = \sin^2 x$, $y = \sin^4 x$, $0 \leqslant x \leqslant \pi$; about $x = \pi/2$

36. $y = x^3 \sin x$, $y = 0$, $0 \leqslant x \leqslant \pi$; about $x = -1$

37–42 The region bounded by the given curves is rotated about the specified axis. Find the volume of the resulting solid by any method.

37. $y = -x^2 + 6x - 8$, $y = 0$; about the y-axis

38. $y = -x^2 + 6x - 8$, $y = 0$; about the x-axis

39. $y = 5$, $y = x^2 - 5x + 9$; about $x = -1$

40. $x = 1 - y^4$, $x = 0$; about $x = 2$

41. $x^2 + (y - 1)^2 = 1$; about the y-axis

42. $x = (y - 3)^2$, $x = 4$; about $y = 1$

43–45 Use cylindrical shells to find the volume of the solid.

43. A sphere of radius r

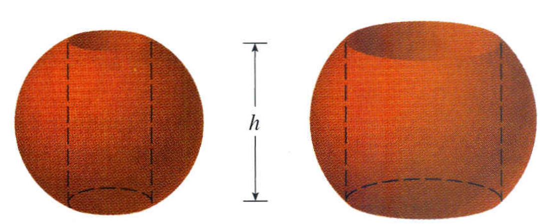
44. The solid torus of Exercise 63 in Section 6.2

45. A right circular cone with height h and base radius r

46. Suppose you make napkin rings by drilling holes with different diameters through two wooden balls (which also have different diameters). You discover that both napkin rings have the same height h, as shown in the figure.

(a) Guess which ring has more wood in it.

(b) Check your guess: Use cylindrical shells to compute the volume of a napkin ring created by drilling a hole with radius r through the center of a sphere of radius R and express the answer in terms of h.



To find the total work done in emptying the entire tank, we add the contributions of each of the n layers and then take the limit as $n \to \infty$:

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} 1570\pi x_i^*(10 - x_i^*)^2 \Delta x = \int_2^{10} 1570\pi x(10 - x)^2\, dx$$

$$= 1570\pi \int_2^{10} (100x - 20x^2 + x^3)\, dx = 1570\pi \left[50x^2 - \frac{20x^3}{3} + \frac{x^4}{4} \right]_2^{10}$$

$$= 1570\pi\left(\tfrac{2048}{3}\right) \approx 3.4 \times 10^6 \text{ J}$$

□

6.4 EXERCISES

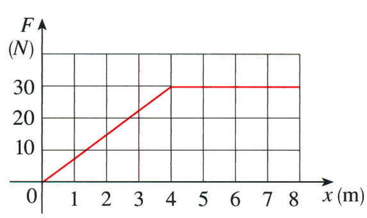
1. How much work is done in lifting a 40-kg sandbag to a height of 1.5 m?

2. Find the work done if a constant force of 100 lb is used to pull a cart a distance of 200 ft.

3. A particle is moved along the x-axis by a force that measures $10/(1 + x)^2$ newtons at a point x meters from the origin. Find the work done in moving the particle from the origin to a distance of 9 m.

4. When a particle is located a distance x meters from the origin, a force of $\cos(\pi x/3)$ newtons acts on it. How much work is done in moving the particle from $x = 1$ to $x = 2$? Interpret your answer by considering the work done from $x = 1$ to $x = 1.5$ and from $x = 1.5$ to $x = 2$.

5. Shown is the graph of a force function (in newtons) that increases to its maximum value and then remains constant. How much work is done by the force in moving an object a distance of 8 m?



6. The table shows values of a force function $f(x)$, where x is measured in meters and $f(x)$ in newtons. Use the Midpoint Rule to estimate the work done by the force in moving an object from $x = 4$ to $x = 20$.

x	4	6	8	10	12	14	16	18	20
$f(x)$	5	5.8	7.0	8.8	9.6	8.2	6.7	5.2	4.1

7. A force of 10 lb is required to hold a spring stretched 4 in. beyond its natural length. How much work is done in stretching it from its natural length to 6 in. beyond its natural length?

8. A spring has a natural length of 20 cm. If a 25-N force is required to keep it stretched to a length of 30 cm, how much work is required to stretch it from 20 cm to 25 cm?

9. Suppose that 2 J of work is needed to stretch a spring from its natural length of 30 cm to a length of 42 cm.
(a) How much work is needed to stretch the spring from 35 cm to 40 cm?
(b) How far beyond its natural length will a force of 30 N keep the spring stretched?

10. If the work required to stretch a spring 1 ft beyond its natural length is 12 ft-lb, how much work is needed to stretch it 9 in. beyond its natural length?

11. A spring has natural length 20 cm. Compare the work W_1 done in stretching the spring from 20 cm to 30 cm with the work W_2 done in stretching it from 30 cm to 40 cm. How are W_2 and W_1 related?

12. If 6 J of work is needed to stretch a spring from 10 cm to 12 cm and another 10 J is needed to stretch it from 12 cm to 14 cm, what is the natural length of the spring?

13–20 Show how to approximate the required work by a Riemann sum. Then express the work as an integral and evaluate it.

13. A heavy rope, 50 ft long, weighs 0.5 lb/ft and hangs over the edge of a building 120 ft high.
(a) How much work is done in pulling the rope to the top of the building?
(b) How much work is done in pulling half the rope to the top of the building?

14. A chain lying on the ground is 10 m long and its mass is 80 kg. How much work is required to raise one end of the chain to a height of 6 m?

15. A cable that weighs 2 lb/ft is used to lift 800 lb of coal up a mine shaft 500 ft deep. Find the work done.

16. A bucket that weighs 4 lb and a rope of negligible weight are used to draw water from a well that is 80 ft deep. The bucket is filled with 40 lb of water and is pulled up at a rate of 2 ft/s, but water leaks out of a hole in the bucket at a rate of 0.2 lb/s. Find the work done in pulling the bucket to the top of the well.

17. A leaky 10-kg bucket is lifted from the ground to a height of 12 m at a constant speed with a rope that weighs 0.8 kg/m. Initially the bucket contains 36 kg of water, but the water leaks at a constant rate and finishes draining just as the bucket reaches the 12 m level. How much work is done?

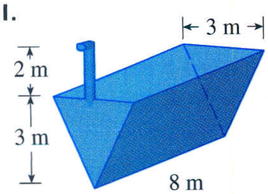
18. A 10-ft chain weighs 25 lb and hangs from a ceiling. Find the work done in lifting the lower end of the chain to the ceiling so that it's level with the upper end.

19. An aquarium 2 m long, 1 m wide, and 1 m deep is full of water. Find the work needed to pump half of the water out of the aquarium. (Use the fact that the density of water is 1000 kg/m^3.)

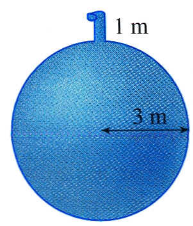
20. A circular swimming pool has a diameter of 10 m, the sides are 1.5 m high, and the depth of the water is 1.2 m. How much work is required to pump all of the water out over the side?

21–24 A tank is full of water. Find the work required to pump the water out of the spout. In Exercises 23 and 24 use the fact that water weighs 62.5 lb/ft^3.

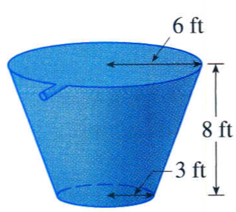
21.



22.

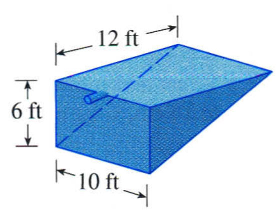


23.



frustum of a cone

24.

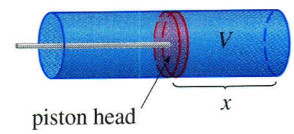


25. Suppose that for the tank in Exercise 21 the pump breaks down after 4.7×10^5 J of work has been done. What is the depth of the water remaining in the tank?

26. Solve Exercise 22 if the tank is half full of oil that has a density of 900 kg/m^3.

27. When gas expands in a cylinder with radius r, the pressure at any given time is a function of the volume: $P = P(V)$. The force exerted by the gas on the piston (see the figure) is the product of the pressure and the area: $F = \pi r^2 P$. Show that the work done by the gas when the volume expands from volume V_1 to volume V_2 is

$$W = \int_{V_1}^{V_2} P\,dV$$



28. In a steam engine the pressure P and volume V of steam satisfy the equation $PV^{1.4} = k$, where k is a constant. (This is true for adiabatic expansion, that is, expansion in which there is no heat transfer between the cylinder and its surroundings.) Use Exercise 27 to calculate the work done by the engine during a cycle when the steam starts at a pressure of 160 lb/in^2 and a volume of 100 in^3 and expands to a volume of 800 in^3.

29. Newton's Law of Gravitation states that two bodies with masses m_1 and m_2 attract each other with a force

$$F = G\frac{m_1 m_2}{r^2}$$

where r is the distance between the bodies and G is the gravitational constant. If one of the bodies is fixed, find the work needed to move the other from $r = a$ to $r = b$.

30. Use Newton's Law of Gravitation to compute the work required to launch a 1000-kg satellite vertically to an orbit 1000 km high. You may assume that the earth's mass is 5.98×10^{24} kg and is concentrated at its center. Take the radius of the earth to be 6.37×10^6 m and $G = 6.67 \times 10^{-11}$ N·m^2/kg^2.

6.5 AVERAGE VALUE OF A FUNCTION

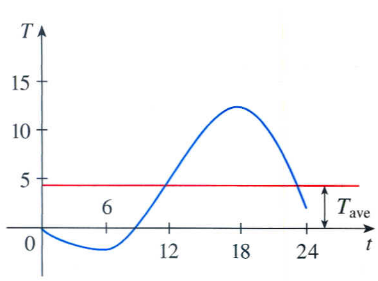


FIGURE 1

It is easy to calculate the average value of finitely many numbers $y_1, y_2, \ldots, y_n$:

$$y_{ave} = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

But how do we compute the average temperature during a day if infinitely many temperature readings are possible? Figure 1 shows the graph of a temperature function $T(t)$, where t is measured in hours and T in °C, and a guess at the average temperature, T_{ave}.

In general, let's try to compute the average value of a function $y = f(x)$, $a \leqslant x \leqslant b$. We start by dividing the interval $[a, b]$ into n equal subintervals, each with length $\Delta x = (b - a)/n$. Then we choose points $x_1^*, \ldots, x_n^*$ in successive subintervals and cal-

6.5 EXERCISES

1–8 Find the average value of the function on the given interval.

1. $f(x) = 4x - x^2$, $[0, 4]$
2. $f(x) = \sin 4x$, $[-\pi, \pi]$
3. $g(x) = \sqrt[3]{x}$, $[1, 8]$
4. $g(x) = x^2\sqrt{1 + x^3}$, $[0, 2]$
5. $f(t) = t\sqrt{1 + t^2}$, $[0, 5]$
6. $f(\theta) = \sec\theta \tan\theta$, $[0, \pi/4]$
7. $h(x) = \cos^4 x \sin x$, $[0, \pi]$
8. $h(r) = 3/(1 + r)^2$, $[1, 6]$

9–12

(a) Find the average value of f on the given interval.
(b) Find c such that $f_{ave} = f(c)$.
(c) Sketch the graph of f and a rectangle whose area is the same as the area under the graph of f.

9. $f(x) = (x - 3)^2$, $[2, 5]$
10. $f(x) = \sqrt{x}$, $[0, 4]$
11. $f(x) = 2\sin x - \sin 2x$, $[0, \pi]$
12. $f(x) = 2x/(1 + x^2)^2$, $[0, 2]$

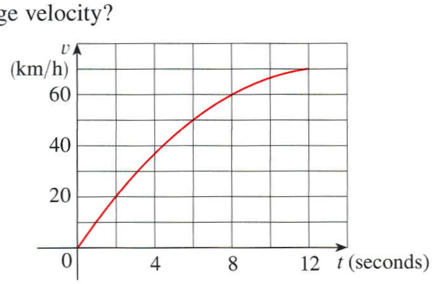
13. If f is continuous and $\int_1^3 f(x)\,dx = 8$, show that f takes on the value 4 at least once on the interval $[1, 3]$.

14. Find the numbers b such that the average value of $f(x) = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal to 3.

15. The table gives values of a continuous function. Use the Midpoint Rule to estimate the average value of f on $[20, 50]$.

x	20	25	30	35	40	45	50
$f(x)$	42	38	31	29	35	48	60

16. The velocity graph of an accelerating car is shown.
(a) Estimate the average velocity of the car during the first 12 seconds.
(b) At what time was the instantaneous velocity equal to the average velocity?



17. In a certain city the temperature (in °C) t hours after 9 AM was modeled by the function

$$T(t) = 20 + 6\sin\frac{\pi t}{12}$$

Find the average temperature during the period from 9 AM to 9 PM.

18. The temperature of a metal rod, 5 m long, is $4x$ (in °C) at a distance x meters from one end of the rod. What is the average temperature of the rod?

19. The linear density in a rod 8 m long is $12/\sqrt{x + 1}$ kg/m, where x is measured in meters from one end of the rod. Find the average density of the rod.

20. If a freely falling body starts from rest, then its displacement is given by $s = \frac{1}{2}gt^2$. Let the velocity after a time T be v_T. Show that if we compute the average of the velocities with respect to t we get $v_{ave} = \frac{1}{2}v_T$, but if we compute the average of the velocities with respect to s we get $v_{ave} = \frac{2}{3}v_T$.

21. Use the result of Exercise 55 in Section 5.5 to compute the average volume of inhaled air in the lungs in one respiratory cycle.

22. The velocity v of blood that flows in a blood vessel with radius R and length l at a distance r from the central axis is

$$v(r) = \frac{P}{4\eta l}(R^2 - r^2)$$

where P is the pressure difference between the ends of the vessel and η is the viscosity of the blood (see Example 7 in Section 3.7). Find the average velocity (with respect to r) over the interval $0 \leqslant r \leqslant R$. Compare the average velocity with the maximum velocity.

23. Prove the Mean Value Theorem for Integrals by applying the Mean Value Theorem for derivatives (see Section 4.2) to the function $F(x) = \int_a^x f(t)\,dt$.

24. If $f_{ave}[a, b]$ denotes the average value of f on the interval $[a, b]$ and $a < c < b$, show that

$$f_{ave}[a, b] = \frac{c - a}{b - a} f_{ave}[a, c] + \frac{b - c}{b - a} f_{ave}[c, b]$$

6 REVIEW

CONCEPT CHECK

1. (a) Draw two typical curves $y = f(x)$ and $y = g(x)$, where $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$. Show how to approximate the area between these curves by a Riemann sum and sketch the corresponding approximating rectangles. Then write an expression for the exact area.
   (b) Explain how the situation changes if the curves have equations $x = f(y)$ and $x = g(y)$, where $f(y) \geqslant g(y)$ for $c \leqslant y \leqslant d$.

2. Suppose that Sue runs faster than Kathy throughout a 1500-meter race. What is the physical meaning of the area between their velocity curves for the first minute of the race?

3. (a) Suppose S is a solid with known cross-sectional areas. Explain how to approximate the volume of S by a Riemann sum. Then write an expression for the exact volume.
   (b) If S is a solid of revolution, how do you find the cross-sectional areas?

4. (a) What is the volume of a cylindrical shell?
   (b) Explain how to use cylindrical shells to find the volume of a solid of revolution.
   (c) Why might you want to use the shell method instead of slicing?

5. Suppose that you push a book across a 6-meter-long table by exerting a force $f(x)$ at each point from $x = 0$ to $x = 6$. What does $\int_0^6 f(x)\,dx$ represent? If $f(x)$ is measured in newtons, what are the units for the integral?

6. (a) What is the average value of a function f on an interval $[a, b]$?
   (b) What does the Mean Value Theorem for Integrals say? What is its geometric interpretation?

EXERCISES

1–6 Find the area of the region bounded by the given curves.

1. $y = x^2, \quad y = 4x - x^2$
2. $y = 20 - x^2, \quad y = x^2 - 12$
3. $y = 1 - 2x^2, \quad y = |x|$
4. $x + y = 0, \quad x = y^2 + 3y$
5. $y = \sin(\pi x/2), \quad y = x^2 - 2x$
6. $y = \sqrt{x}, \quad y = x^2, \quad x = 2$

7–11 Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis.

7. $y = 2x,\ y = x^2$; about the x-axis
8. $x = 1 + y^2,\ y = x - 3$; about the y-axis
9. $x = 0,\ x = 9 - y^2$; about $x = -1$
10. $y = x^2 + 1,\ y = 9 - x^2$; about $y = -1$
11. $x^2 - y^2 = a^2,\ x = a + h$ (where $a > 0,\ h > 0$); about the y-axis

12–14 Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis.

12. $y = \tan x,\ y = x,\ x = \pi/3$; about the y-axis
13. $y = \cos^2 x,\ |x| \leqslant \pi/2,\ y = \frac{1}{4}$; about $x = \pi/2$
14. $y = \sqrt{x},\ y = x^2$; about $y = 2$

15. Find the volumes of the solids obtained by rotating the region bounded by the curves $y = x$ and $y = x^2$ about the following lines.
    (a) The x-axis (b) The y-axis (c) $y = 2$

16. Let $\mathcal{R}$ be the region in the first quadrant bounded by the curves $y = x^3$ and $y = 2x - x^2$. Calculate the following quantities.
    (a) The area of $\mathcal{R}$
    (b) The volume obtained by rotating $\mathcal{R}$ about the x-axis
    (c) The volume obtained by rotating $\mathcal{R}$ about the y-axis

17. Let $\mathcal{R}$ be the region bounded by the curves $y = \tan(x^2)$, $x = 1$, and $y = 0$. Use the Midpoint Rule with $n = 4$ to estimate the following quantities.
    (a) The area of $\mathcal{R}$
    (b) The volume obtained by rotating $\mathcal{R}$ about the x-axis

18. Let $\mathcal{R}$ be the region bounded by the curves $y = 1 - x^2$ and $y = x^6 - x + 1$. Estimate the following quantities.
    (a) The x-coordinates of the points of intersection of the curves
    (b) The area of $\mathcal{R}$
    (c) The volume generated when $\mathcal{R}$ is rotated about the x-axis
    (d) The volume generated when $\mathcal{R}$ is rotated about the y-axis

19–22 Each integral represents the volume of a solid. Describe the solid.

19. $\int_0^{\pi/2} 2\pi x \cos x\,dx$
20. $\int_0^{\pi/2} 2\pi \cos^2 x\,dx$
21. $\int_0^{\pi} \pi(2 - \sin x)^2\,dx$
22. $\int_0^{4} 2\pi(6 - y)(4y - y^2)\,dy$

23. The base of a solid is a circular disk with radius 3. Find the volume of the solid if parallel cross-sections perpendicular to the base are isosceles right triangles with hypotenuse lying along the base.

24. The base of a solid is the region bounded by the parabolas $y = x^2$ and $y = 2 - x^2$. Find the volume of the solid if the cross-sections perpendicular to the x-axis are squares with one side lying along the base.

25. The height of a monument is 20 m. A horizontal cross-section at a distance x meters from the top is an equilateral triangle with side $\frac{1}{4}x$ meters. Find the volume of the monument.

26. (a) The base of a solid is a square with vertices located at $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$. Each cross-section perpendicular to the x-axis is a semicircle. Find the volume of the solid.

(b) Show that by cutting the solid of part (a), we can rearrange it to form a cone. Thus compute its volume more simply.

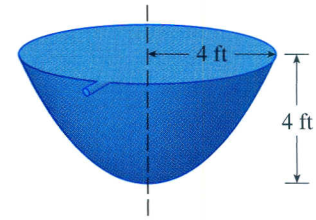
27. A force of 30 N is required to maintain a spring stretched from its natural length of 12 cm to a length of 15 cm. How much work is done in stretching the spring from 12 cm to 20 cm?

28. A 1600-lb elevator is suspended by a 200-ft cable that weighs 10 lb/ft. How much work is required to raise the elevator from the basement to the third floor, a distance of 30 ft?

29. A tank full of water has the shape of a paraboloid of revolution as shown in the figure; that is, its shape is obtained by rotating a parabola about a vertical axis.

(a) If its height is 4 ft and the radius at the top is 4 ft, find the work required to pump the water out of the tank.

(b) After 4000 ft-lb of work has been done, what is the depth of the water remaining in the tank?



30. Find the average value of the function $f(t) = t \sin(t^2)$ on the interval $[0, 10]$.

31. If f is a continuous function, what is the limit as $h \to 0$ of the average value of f on the interval $[x, x + h]$?

32. Let $\mathcal{R}_1$ be the region bounded by $y = x^2$, $y = 0$, and $x = b$, where $b > 0$. Let $\mathcal{R}_2$ be the region bounded by $y = x^2$, $x = 0$, and $y = b^2$.

(a) Is there a value of b such that $\mathcal{R}_1$ and $\mathcal{R}_2$ have the same area?

(b) Is there a value of b such that $\mathcal{R}_1$ sweeps out the same volume when rotated about the x-axis and the y-axis?

(c) Is there a value of b such that $\mathcal{R}_1$ and $\mathcal{R}_2$ sweep out the same volume when rotated about the x-axis?

(d) Is there a value of b such that $\mathcal{R}_1$ and $\mathcal{R}_2$ sweep out the same volume when rotated about the y-axis?

PROBLEMS PLUS

1. (a) Find a positive continuous function f such that the area under the graph of f from 0 to t is $A(t) = t^3$ for all $t > 0$.

(b) A solid is generated by rotating about the x-axis the region under the curve $y = f(x)$, where f is a positive function and $x \geqslant 0$. The volume generated by the part of the curve from $x = 0$ to $x = b$ is b^2 for all $b > 0$. Find the function f.

2. There is a line through the origin that divides the region bounded by the parabola $y = x - x^2$ and the x-axis into two regions with equal area. What is the slope of that line?

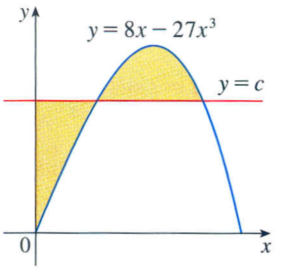


FIGURE FOR PROBLEM 3

3. The figure shows a horizontal line $y = c$ intersecting the curve $y = 8x - 27x^3$. Find the number c such that the areas of the shaded regions are equal.

4. A cylindrical glass of radius r and height L is filled with water and then tilted until the water remaining in the glass exactly covers its base.

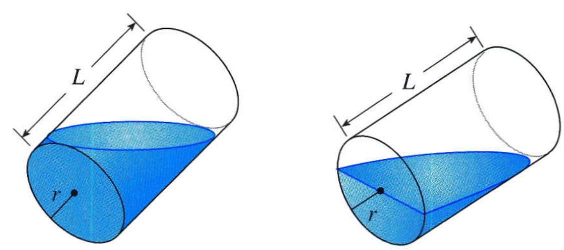
(a) Determine a way to "slice" the water into parallel rectangular cross-sections and then *set up* a definite integral for the volume of the water in the glass.

(b) Determine a way to "slice" the water into parallel cross-sections that are trapezoids and then *set up* a definite integral for the volume of the water.

(c) Find the volume of water in the glass by evaluating one of the integrals in part (a) or part (b).

(d) Find the volume of the water in the glass from purely geometric considerations.

(e) Suppose the glass is tilted until the water exactly covers half the base. In what direction can you "slice" the water into triangular cross-sections? Rectangular cross-sections? Cross-sections that are segments of circles? Find the volume of water in the glass.



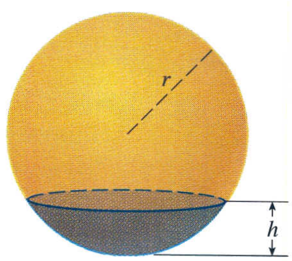


FIGURE FOR PROBLEM 5

5. (a) Show that the volume of a segment of height h of a sphere of radius r is

$$V = \tfrac{1}{3}\pi h^2(3r - h)$$

(b) Show that if a sphere of radius 1 is sliced by a plane at a distance x from the center in such a way that the volume of one segment is twice the volume of the other, then x is a solution of the equation

$$3x^3 - 9x + 2 = 0$$

where $0 < x < 1$. Use Newton's method to find x accurate to four decimal places.

(c) Using the formula for the volume of a segment of a sphere, it can be shown that the depth x to which a floating sphere of radius r sinks in water is a root of the equation

$$x^3 - 3rx^2 + 4r^3s = 0$$

where s is the specific gravity of the sphere. Suppose a wooden sphere of radius 0.5 m has specific gravity 0.75. Calculate, to four-decimal-place accuracy, the depth to which the sphere will sink.

(d) A hemispherical bowl has radius 12 cm and water is running into the bowl at the rate of $3\ \text{cm}^3/\text{s}$.
   (i) How fast is the water level in the bowl rising at the instant the water is 7 cm deep?
   (ii) At a certain instant, the water is 8 cm deep. How long will it take to fill the bowl?

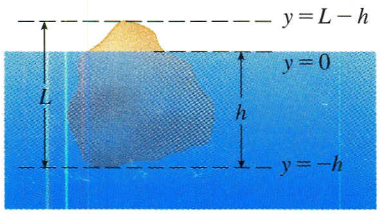


FIGURE FOR PROBLEM 6

6. Archimedes' Principle states that the buoyant force on an object partially or fully submerged in a fluid is equal to the weight of the fluid that the object displaces. Thus, for an object of density ρ_0 floating partly submerged in a fluid of density ρ_f, the buoyant force is given by $F = \rho_f g \int_{-h}^{0} A(y)\, dy$, where g is the acceleration due to gravity and $A(y)$ is the area of a typical cross-section of the object. The weight of the object is given by

$$W = \rho_0 g \int_{-h}^{L-h} A(y)\, dy$$

(a) Show that the percentage of the volume of the object above the surface of the liquid is

$$100\,\frac{\rho_f - \rho_0}{\rho_f}$$

(b) The density of ice is $917\ \text{kg/m}^3$ and the density of seawater is $1030\ \text{kg/m}^3$. What percentage of the volume of an iceberg is above water?

(c) An ice cube floats in a glass filled to the brim with water. Does the water overflow when the ice melts?

(d) A sphere of radius 0.4 m and having negligible weight is floating in a large freshwater lake. How much work is required to completely submerge the sphere? The density of the water is $1000\ \text{kg/m}^3$.

7. Water in an open bowl evaporates at a rate proportional to the area of the surface of the water. (This means that the rate of decrease of the volume is proportional to the area of the surface.) Show that the depth of the water decreases at a constant rate, regardless of the shape of the bowl.

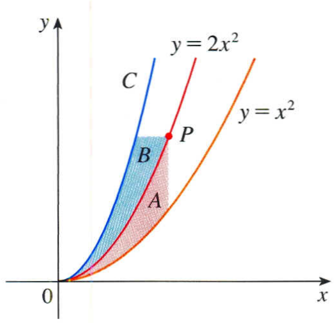
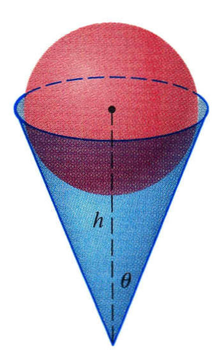


FIGURE FOR PROBLEM 9

8. A sphere of radius 1 overlaps a smaller sphere of radius r in such a way that their intersection is a circle of radius r. (In other words, they intersect in a great circle of the small sphere.) Find r so that the volume inside the small sphere and outside the large sphere is as large as possible.

9. The figure shows a curve C with the property that, for every point P on the middle curve $y = 2x^2$, the areas A and B are equal. Find an equation for C.

10. A paper drinking cup filled with water has the shape of a cone with height h and semivertical angle θ (see the figure). A ball is placed carefully in the cup, thereby displacing some of the water and making it overflow. What is the radius of the ball that causes the greatest volume of water to spill out of the cup?



11. A *clepsydra*, or water clock, is a glass container with a small hole in the bottom through which water can flow. The "clock" is calibrated for measuring time by placing markings on the container corresponding to water levels at equally spaced times. Let $x = f(y)$ be continuous on the interval $[0, b]$ and assume that the container is formed by rotating the graph of f about the y-axis. Let V denote the volume of water and h the height of the water level at time t.

(a) Determine V as a function of h.

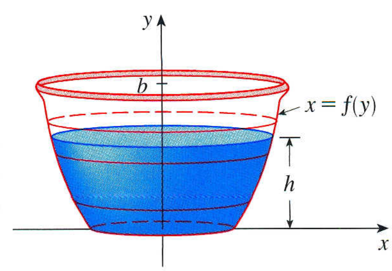
(b) Show that

$$\frac{dV}{dt} = \pi[f(h)]^2 \frac{dh}{dt}$$

(c) Suppose that A is the area of the hole in the bottom of the container. It follows from Torricelli's Law that the rate of change of the volume of the water is given by

$$\frac{dV}{dt} = kA\sqrt{h}$$

where k is a negative constant. Determine a formula for the function f such that dh/dt is a constant C. What is the advantage in having $dh/dt = C$?



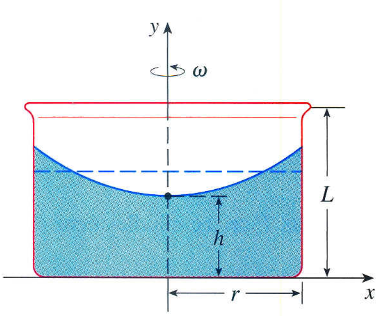


FIGURE FOR PROBLEM 12

12. A cylindrical container of radius r and height L is partially filled with a liquid whose volume is V. If the container is rotated about its axis of symmetry with constant angular speed ω, then the container will induce a rotational motion in the liquid around the same axis. Eventually, the liquid will be rotating at the same angular speed as the container. The surface of the liquid will be convex, as indicated in the figure, because the centrifugal force on the liquid particles increases with the distance from the axis of the container. It can be shown that the surface of the liquid is a paraboloid of revolution generated by rotating the parabola

$$y = h + \frac{\omega^2 x^2}{2g}$$

about the y-axis, where g is the acceleration due to gravity.

(a) Determine h as a function of ω.

(b) At what angular speed will the surface of the liquid touch the bottom? At what speed will it spill over the top?

(c) Suppose the radius of the container is 2 m, the height is 7 m, and the container and liquid are rotating at the same constant angular speed. The surface of the liquid is 5 m below the top of the tank at the central axis and 4 m below the top of the tank 1 m out from the central axis.

(i) Determine the angular speed of the container and the volume of the fluid.

(ii) How far below the top of the tank is the liquid at the wall of the container?

13. Suppose the graph of a cubic polynomial intersects the parabola $y = x^2$ when $x = 0$, $x = a$, and $x = b$, where $0 < a < b$. If the two regions between the curves have the same area, how is b related to a?

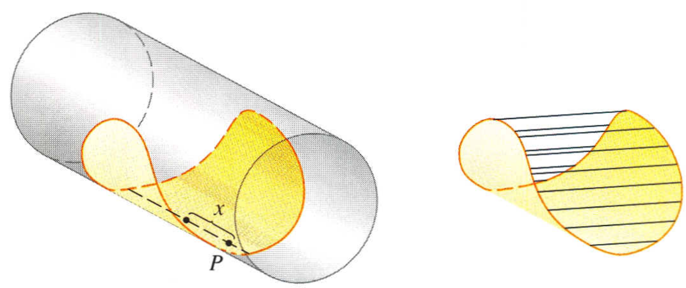
CAS **14.** Suppose we are planning to make a taco from a round tortilla with diameter 8 inches by bending the tortilla so that it is shaped as if it is partially wrapped around a circular cylinder. We will fill the tortilla to the edge (but no more) with meat, cheese, and other ingredients. Our problem is to decide how to curve the tortilla in order to maximize the volume of food it can hold.

(a) We start by placing a circular cylinder of radius r along a diameter of the tortilla and folding the tortilla around the cylinder. Let x represent the distance from the center of the tortilla to a point P on the diameter (see the figure). Show that the cross-sectional area of the filled taco in the plane through P perpendicular to the axis of the cylinder is

$$A(x) = r\sqrt{16 - x^2} - \tfrac{1}{2}r^2 \sin\left(\frac{2}{r}\sqrt{16 - x^2}\right)$$

and write an expression for the volume of the filled taco.

(b) Determine (approximately) the value of r that maximizes the volume of the taco. (Use a graphical approach with your CAS.)



15. If the tangent at a point P on the curve $y = x^3$ intersects the curve again at Q, let A be the area of the region bounded by the curve and the line segment PQ. Let B be the area of the region defined in the same way starting with Q instead of P. What is the relationship between A and B?