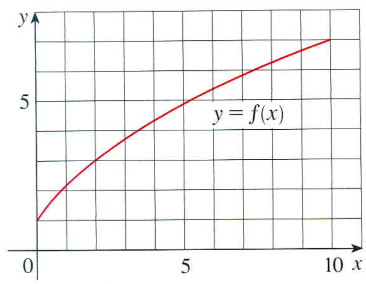
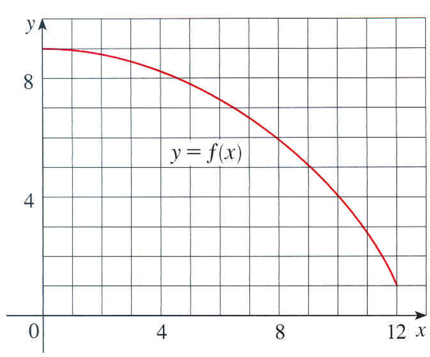
5.1 EXERCISES

1. (a) By reading values from the given graph of f, use five rectangles to find a lower estimate and an upper estimate for the area under the given graph of f from $x = 0$ to $x = 10$. In each case sketch the rectangles that you use.
(b) Find new estimates using ten rectangles in each case.



2. (a) Use six rectangles to find estimates of each type for the area under the given graph of f from $x = 0$ to $x = 12$.
  (i) L_6 (sample points are left endpoints)
  (ii) R_6 (sample points are right endpoints)
  (iii) M_6 (sample points are midpoints)
(b) Is L_6 an underestimate or overestimate of the true area?
(c) Is R_6 an underestimate or overestimate of the true area?
(d) Which of the numbers L_6, R_6, or M_6 gives the best estimate? Explain.



3. (a) Estimate the area under the graph of $f(x) = \cos x$ from $x = 0$ to $x = \pi/2$ using four approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate?
(b) Repeat part (a) using left endpoints.

4. (a) Estimate the area under the graph of $f(x) = \sqrt{x}$ from $x = 0$ to $x = 4$ using four approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate?
(b) Repeat part (a) using left endpoints.

5. (a) Estimate the area under the graph of $f(x) = 1 + x^2$ from $x = -1$ to $x = 2$ using three rectangles and right endpoints. Then improve your estimate by using six rectangles. Sketch the curve and the approximating rectangles.
(b) Repeat part (a) using left endpoints.
(c) Repeat part (a) using midpoints.
(d) From your sketches in parts (a)–(c), which appears to be the best estimate?

6. (a) Graph the function $f(x) = 1/(1 + x^2)$, $-2 \leqslant x \leqslant 2$.
(b) Estimate the area under the graph of f using four approximating rectangles and taking the sample points to be
  (i) right endpoints (ii) midpoints
  In each case sketch the curve and the rectangles.
(c) Improve your estimates in part (b) by using eight rectangles.

7–8 With a programmable calculator (or a computer), it is possible to evaluate the expressions for the sums of areas of approximating rectangles, even for large values of n, using looping. (On a TI use the Is> command or a For-EndFor loop, on a Casio use Isz, on an HP or in BASIC use a FOR-NEXT loop.) Compute the sum of the areas of approximating rectangles using equal subintervals and right endpoints for $n = 10$, 30, 50, and 100. Then guess the value of the exact area.

7. The region under $y = x^4$ from 0 to 1

8. The region under $y = \cos x$ from 0 to $\pi/2$

CAS **9.** Some computer algebra systems have commands that will draw approximating rectangles and evaluate the sums of their areas, at least if x_i^* is a left or right endpoint. (For instance, in Maple use `leftbox`, `rightbox`, `leftsum`, and `rightsum`.)
(a) If $f(x) = 1/(x^2 + 1)$, $0 \leqslant x \leqslant 1$, find the left and right sums for $n = 10$, 30, and 50.
(b) Illustrate by graphing the rectangles in part (a).
(c) Show that the exact area under f lies between 0.780 and 0.791.

CAS **10.** (a) If $f(x) = x/(x + 2)$, $1 \leqslant x \leqslant 4$, use the commands discussed in Exercise 9 to find the left and right sums for $n = 10$, 30, and 50.
(b) Illustrate by graphing the rectangles in part (a).
(c) Show that the exact area under f lies between 1.603 and 1.624.

11. The speed of a runner increased steadily during the first three seconds of a race. Her speed at half-second intervals is given in the table. Find lower and upper estimates for the distance that she traveled during these three seconds.

t (s)	0	0.5	1.0	1.5	2.0	2.5	3.0
v (m/s)	0	1.9	3.3	4.5	5.5	5.9	6.2

12. Speedometer readings for a motorcycle at 12-second intervals are given in the table.
(a) Estimate the distance traveled by the motorcycle during this time period using the velocities at the beginning of the time intervals.
(b) Give another estimate using the velocities at the end of the time periods.
(c) Are your estimates in parts (a) and (b) upper and lower estimates? Explain.

t (s)	0	12	24	36	48	60
v (m/s)	9.1	8.5	7.6	6.7	7.3	8.2

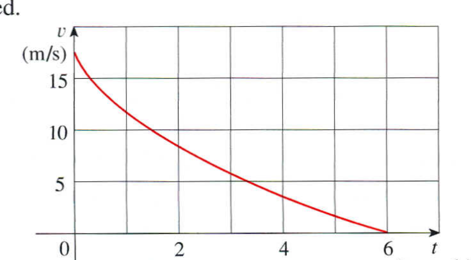
13. Oil leaked from a tank at a rate of $r(t)$ liters per hour. The rate decreased as time passed and values of the rate at two-hour time intervals are shown in the table. Find lower and upper estimates for the total amount of oil that leaked out.

t (h)	0	2	4	6	8	10
$r(t)$ (L/h)	8.7	7.6	6.8	6.2	5.7	5.3

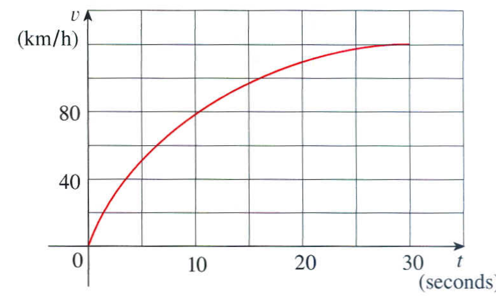
14. When we estimate distances from velocity data, it is sometimes necessary to use times $t_0, t_1, t_2, t_3, \ldots$ that are not equally spaced. We can still estimate distances using the time periods $\Delta t_i = t_i - t_{i-1}$. For example, on May 7, 1992, the space shuttle *Endeavour* was launched on mission STS-49, the purpose of which was to install a new perigee kick motor in an Intelsat communications satellite. The table, provided by NASA, gives the velocity data for the shuttle between liftoff and the jettisoning of the solid rocket boosters. Use these data to estimate the height above the earth's surface of the *Endeavour,* 62 seconds after liftoff.

Event	Time (s)	Velocity (m/s)
Launch	0	0
Begin roll maneuver	10	56
End roll maneuver	15	97
Throttle to 89%	20	136
Throttle to 67%	32	226
Throttle to 104%	59	404
Maximum dynamic pressure	62	440
Solid rocket booster separation	125	1265

15. The velocity graph of a braking car is shown. Use it to estimate the distance traveled by the car while the brakes are applied.



16. The velocity graph of a car accelerating from rest to a speed of 120 km/h over a period of 30 seconds is shown. Estimate the distance traveled during this period.



17–19 Use Definition 2 to find an expression for the area under the graph of f as a limit. Do not evaluate the limit.

17. $f(x) = \sqrt[4]{x}, \quad 1 \leqslant x \leqslant 16$

18. $f(x) = 1 + x^4, \quad 2 \leqslant x \leqslant 5$

19. $f(x) = x \cos x, \quad 0 \leqslant x \leqslant \pi/2$

20–21 Determine a region whose area is equal to the given limit. Do not evaluate the limit.

20. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(5 + \frac{2i}{n}\right)^{10}$

21. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan \frac{i\pi}{4n}$

22. (a) Use Definition 2 to find an expression for the area under the curve $y = x^3$ from 0 to 1 as a limit.
(b) The following formula for the sum of the cubes of the first n integers is proved in Appendix E. Use it to evaluate the limit in part (a).

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

CAS **23.** (a) Express the area under the curve $y = x^5$ from 0 to 2 as a limit.
(b) Use a computer algebra system to find the sum in your expression from part (a).
(c) Evaluate the limit in part (a).

CAS **24.** (a) Express the area under the curve $y = x^4 + 5x^2 + x$ from 2 to 7 as a limit.
(b) Use a computer algebra system to evaluate the sum in part (a).
(c) Use a computer algebra system to find the exact area by evaluating the limit of the expression in part (b).

CAS **25.** Find the exact area under the cosine curve $y = \cos x$ from $x = 0$ to $x = b$, where $0 \leqslant b \leqslant \pi/2$. (Use a computer algebra system both to evaluate the sum and compute the limit.) In particular, what is the area if $b = \pi/2$?

26. (a) Let A_n be the area of a polygon with n equal sides inscribed in a circle with radius r. By dividing the polygon into n congruent triangles with central angle $2\pi/n$, show that

$$A_n = \tfrac{1}{2}nr^2 \sin\left(\frac{2\pi}{n}\right)$$

(b) Show that $\lim_{n\to\infty} A_n = \pi r^2$. [*Hint:* Use Equation 3.4.2.]

5.2 THE DEFINITE INTEGRAL

We saw in Section 5.1 that a limit of the form

$$\boxed{1} \qquad \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n\to\infty} [f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x]$$

arises when we compute an area. We also saw that it arises when we try to find the distance traveled by an object. It turns out that this same type of limit occurs in a wide variety of situations even when f is not necessarily a positive function. In Chapters 6 and 9 we will see that limits of the form (1) also arise in finding lengths of curves, volumes of solids, centers of mass, force due to water pressure, and work, as well as other quantities. We therefore give this type of limit a special name and notation.

2 DEFINITION OF A DEFINITE INTEGRAL If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0\ (= a), x_1, x_2, \ldots, x_n\ (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of *f* from *a* to *b*** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists. If it does exist, we say that f is **integrable** on $[a, b]$.

The precise meaning of the limit that defines the integral is as follows:

For every number $\varepsilon > 0$ there is an integer N such that

$$\left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} f(x_i^*)\,\Delta x \right| < \varepsilon$$

for every integer $n > N$ and for every choice of x_i^* in $[x_{i-1}, x_i]$.

NOTE 1 The symbol $\int$ was introduced by Leibniz and is called an **integral sign**. It is an elongated S and was chosen because an integral is a limit of sums. In the notation $\int_a^b f(x)\,dx$, $f(x)$ is called the **integrand** and a and b are called the **limits of integration**; a is the **lower limit** and b is the **upper limit**. For now, the symbol dx has no meaning by itself; $\int_a^b f(x)\,dx$ is all one symbol. The dx simply indicates that the independent variable is x. The procedure of calculating an integral is called **integration**.

5.2 EXERCISES

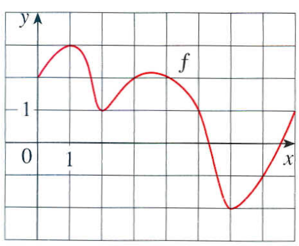
1. Evaluate the Riemann sum for $f(x) = 2 - x^2$, $0 \leqslant x \leqslant 2$, with four subintervals, taking the sample points to be right endpoints. Explain, with the aid of a diagram, what the Riemann sum represents.

2. If $f(x) = 3x - 7$, $0 \leqslant x \leqslant 3$, evaluate the Riemann sum with $n = 6$, taking the sample points to be left endpoints. What does the Riemann sum represent? Illustrate with a diagram.

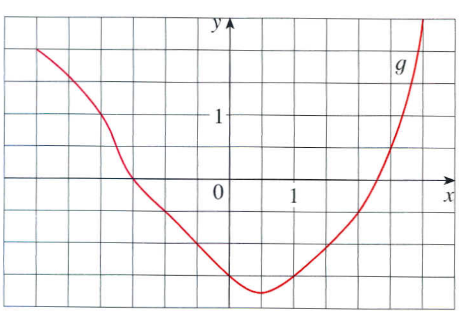
3. If $f(x) = \sqrt{x} - 2$, $1 \leqslant x \leqslant 6$, find the Riemann sum with $n = 5$ correct to six decimal places, taking the sample points to be midpoints. What does the Riemann sum represent? Illustrate with a diagram.

4. (a) Find the Riemann sum for $f(x) = \sin x$, $0 \leqslant x \leqslant 3\pi/2$, with six terms, taking the sample points to be right endpoints. (Give your answer correct to six decimal places.) Explain what the Riemann sum represents with the aid of a sketch.
(b) Repeat part (a) with midpoints as sample points.

5. The graph of a function f is given. Estimate $\int_0^8 f(x)\,dx$ using four subintervals with (a) right endpoints, (b) left endpoints, and (c) midpoints.



6. The graph of g is shown. Estimate $\int_{-3}^{3} g(x)\,dx$ with six subintervals using (a) right endpoints, (b) left endpoints, and (c) midpoints.



7. A table of values of an increasing function f is shown. Use the table to find lower and upper estimates for $\int_0^{25} f(x)\,dx$.

x	0	5	10	15	20	25
$f(x)$	-42	-37	-25	-6	15	36

8. The table gives the values of a function obtained from an experiment. Use them to estimate $\int_3^9 f(x)\,dx$ using three equal subintervals with (a) right endpoints, (b) left endpoints, and (c) midpoints. If the function is known to be an increasing function, can you say whether your estimates are less than or greater than the exact value of the integral?

x	3	4	5	6	7	8	9
$f(x)$	-3.4	-2.1	-0.6	0.3	0.9	1.4	1.8

9–12 Use the Midpoint Rule with the given value of n to approximate the integral. Round the answer to four decimal places.

9. $\displaystyle\int_2^{10} \sqrt{x^3 + 1}\,dx, \quad n = 4$

10. $\displaystyle\int_0^{\pi} \sec(x/3)\,dx, \quad n = 6$

11. $\displaystyle\int_0^1 \sin(x^2)\,dx, \quad n = 5$

12. $\displaystyle\int_1^5 \frac{x - 1}{x + 1}\,dx, \quad n = 4$

CAS **13.** If you have a CAS that evaluates midpoint approximations and graphs the corresponding rectangles (use `middlesum` and `middlebox` commands in Maple), check the answer to Exercise 11 and illustrate with a graph. Then repeat with $n = 10$ and $n = 20$.

14. With a programmable calculator or computer (see the instructions for Exercise 7 in Section 5.1), compute the left and right Riemann sums for the function $f(x) = \sin(x^2)$ on the interval $[0, 1]$ with $n = 100$. Explain why these estimates show that

$$0.306 < \int_0^1 \sin(x^2)\,dx < 0.315$$

Deduce that the approximation using the Midpoint Rule with $n = 5$ in Exercise 11 is accurate to two decimal places.

15. Use a calculator or computer to make a table of values of right Riemann sums R_n for the integral $\int_0^{\pi} \sin x\,dx$ with $n = 5$, 10, 50, and 100. What value do these numbers appear to be approaching?

16. Use a calculator or computer to make a table of values of left and right Riemann sums L_n and R_n for the integral $\int_0^2 \sqrt{1 + x^4}\,dx$ with $n = 5$, 10, 50, and 100. Between what two numbers must the value of the integral lie? Can you make a similar statement for the integral $\int_{-1}^{2} \sqrt{1 + x^4}\,dx$? Explain.

17–20 Express the limit as a definite integral on the given interval.

17. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1 - x_i^2}{4 + x_i^2} \Delta x, \quad [2, 6]$

18. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\cos x_i}{x_i} \Delta x, \quad [\pi, 2\pi]$

19. $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{2x_i^* + (x_i^*)^2}\, \Delta x, \quad [1, 8]$

20. $\lim_{n\to\infty} \sum_{i=1}^{n} [4 - 3(x_i^*)^2 + 6(x_i^*)^5]\, \Delta x, \quad [0, 2]$

21–25 Use the form of the definition of the integral given in Theorem 4 to evaluate the integral.

21. $\int_{-1}^{5} (1 + 3x)\, dx$

22. $\int_{1}^{4} (x^2 + 2x - 5)\, dx$

23. $\int_{0}^{2} (2 - x^2)\, dx$

24. $\int_{0}^{5} (1 + 2x^3)\, dx$

25. $\int_{1}^{2} x^3\, dx$

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26. (a) Find an approximation to the integral $\int_0^4 (x^2 - 3x)\, dx$ using a Riemann sum with right endpoints and $n = 8$.
(b) Draw a diagram like Figure 3 to illustrate the approximation in part (a).
(c) Use Theorem 4 to evaluate $\int_0^4 (x^2 - 3x)\, dx$.
(d) Interpret the integral in part (c) as a difference of areas and illustrate with a diagram like Figure 4.

27. Prove that $\int_a^b x\, dx = \frac{b^2 - a^2}{2}$.

28. Prove that $\int_a^b x^2\, dx = \frac{b^3 - a^3}{3}$.

29–30 Express the integral as a limit of Riemann sums. Do not evaluate the limit.

29. $\int_2^6 \frac{x}{1 + x^5}\, dx$

30. $\int_0^{2\pi} x^2 \sin x\, dx$

CAS **31–32** Express the integral as a limit of sums. Then evaluate, using a computer algebra system to find both the sum and the limit.

31. $\int_0^{\pi} \sin 5x\, dx$

32. $\int_2^{10} x^6\, dx$

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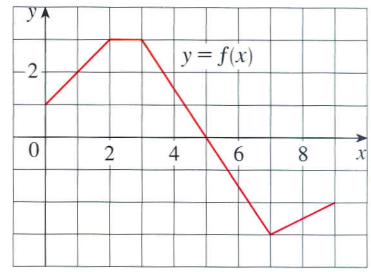
33. The graph of f is shown. Evaluate each integral by interpreting it in terms of areas.

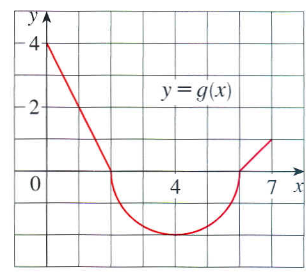
(a) $\int_0^2 f(x)\, dx$ (b) $\int_0^5 f(x)\, dx$

(c) $\int_5^7 f(x)\, dx$ (d) $\int_0^9 f(x)\, dx$



34. The graph of g consists of two straight lines and a semicircle. Use it to evaluate each integral.

(a) $\int_0^2 g(x)\, dx$ (b) $\int_2^6 g(x)\, dx$ (c) $\int_0^7 g(x)\, dx$



35–40 Evaluate the integral by interpreting it in terms of areas.

35. $\int_1^3 (1 + 2x)\, dx$

36. $\int_{-2}^{2} \sqrt{4 - x^2}\, dx$

37. $\int_{-3}^{0} \left(1 + \sqrt{9 - x^2}\right) dx$

38. $\int_{-1}^{3} (3 - 2x)\, dx$

39. $\int_{-1}^{2} |x|\, dx$

40. $\int_0^{10} |x - 5|\, dx$

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41. Evaluate $\int_{\pi}^{\pi} \sin^2 x \cos^4 x\, dx$.

42. Given that $\int_0^1 3x\sqrt{x^2 + 4}\, dx = 5\sqrt{5} - 8$, what is $\int_1^0 3u\sqrt{u^2 + 4}\, du$?

43. In Example 2 in Section 5.1 we showed that $\int_0^1 x^2\, dx = \frac{1}{3}$. Use this fact and the properties of integrals to evaluate $\int_0^1 (5 - 6x^2)\, dx$.

44. Use the properties of integrals and the result of Example 3 to evaluate $\int_2^5 (1 + 3x^4)\, dx$.

45. Use the results of Exercises 27 and 28 and the properties of integrals to evaluate $\int_1^4 (2x^2 - 3x + 1)\, dx$.

46. Use the result of Exercise 27 and the fact that $\int_0^{\pi/2} \cos x\, dx = 1$ (from Exercise 25 in Section 5.1), together with the properties of integrals, to evaluate $\int_0^{\pi/2} (2\cos x - 5x)\, dx$.

47. Write as a single integral in the form $\int_a^b f(x)\,dx$:

$$\int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx - \int_{-2}^{-1} f(x)\,dx$$

48. If $\int_1^5 f(x)\,dx = 12$ and $\int_4^5 f(x)\,dx = 3.6$, find $\int_1^4 f(x)\,dx$.

49. If $\int_0^9 f(x)\,dx = 37$ and $\int_0^9 g(x)\,dx = 16$, find $\int_0^9 [2f(x) + 3g(x)]\,dx$.

50. Find $\int_0^5 f(x)\,dx$ if

$$f(x) = \begin{cases} 3 & \text{for } x < 3 \\ x & \text{for } x \geqslant 3 \end{cases}$$

51. Suppose f has absolute minimum value m and absolute maximum value M. Between what two values must $\int_0^2 f(x)\,dx$ lie? Which property of integrals allows you to make your conclusion?

52–54 Use the properties of integrals to verify the inequality without evaluating the integrals.

52. $\displaystyle\int_0^1 \sqrt{1 + x^2}\,dx \leqslant \int_0^1 \sqrt{1 + x}\,dx$

53. $\displaystyle 2 \leqslant \int_{-1}^{1} \sqrt{1 + x^2}\,dx \leqslant 2\sqrt{2}$

54. $\displaystyle \frac{\sqrt{2}\,\pi}{24} \leqslant \int_{\pi/6}^{\pi/4} \cos x\,dx \leqslant \frac{\sqrt{3}\,\pi}{24}$

55–60 Use Property 8 to estimate the value of the integral.

55. $\displaystyle\int_1^4 \sqrt{x}\,dx$

56. $\displaystyle\int_0^2 \frac{1}{1 + x^2}\,dx$

57. $\displaystyle\int_{\pi/4}^{\pi/3} \tan x\,dx$

58. $\displaystyle\int_0^2 (x^3 - 3x + 3)\,dx$

59. $\displaystyle\int_{-1}^{1} \sqrt{1 + x^4}\,dx$

60. $\displaystyle\int_{\pi}^{2\pi} (x - 2\sin x)\,dx$

61–62 Use properties of integrals, together with Exercises 27 and 28, to prove the inequality.

61. $\displaystyle\int_1^3 \sqrt{x^4 + 1}\,dx \geqslant \frac{26}{3}$

62. $\displaystyle\int_0^{\pi/2} x \sin x\,dx \leqslant \frac{\pi^2}{8}$

63. Prove Property 3 of integrals.

64. Prove Property 6 of integrals.

65. If f is continuous on $[a, b]$, show that

$$\left| \int_a^b f(x)\,dx \right| \leqslant \int_a^b |f(x)|\,dx$$

[*Hint:* $-|f(x)| \leqslant f(x) \leqslant |f(x)|$.]

66. Use the result of Exercise 65 to show that

$$\left| \int_0^{2\pi} f(x) \sin 2x\,dx \right| \leqslant \int_0^{2\pi} |f(x)|\,dx$$

67. Let $f(x) = 0$ if x is any rational number and $f(x) = 1$ if x is any irrational number. Show that f is not integrable on $[0, 1]$.

68. Let $f(0) = 0$ and $f(x) = 1/x$ if $0 < x \leqslant 1$. Show that f is not integrable on $[0, 1]$. [*Hint:* Show that the first term in the Riemann sum, $f(x_i^*)\,\Delta x$, can be made arbitrarily large.]

69–70 Express the limit as a definite integral.

69. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^4}{n^5}$ [*Hint:* Consider $f(x) = x^4$.]

70. $\displaystyle\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1 + (i/n)^2}$

71. Find $\int_1^2 x^{-2}\,dx$. *Hint:* Choose x_i^* to be the geometric mean of x_{i-1} and x_i (that is, $x_i^* = \sqrt{x_{i-1}x_i}$) and use the identity

$$\frac{1}{m(m + 1)} = \frac{1}{m} - \frac{1}{m + 1}$$

DISCOVERY PROJECT — AREA FUNCTIONS

1. (a) Draw the line $y = 2t + 1$ and use geometry to find the area under this line, above the t-axis, and between the vertical lines $t = 1$ and $t = 3$.

(b) If $x > 1$, let $A(x)$ be the area of the region that lies under the line $y = 2t + 1$ between $t = 1$ and $t = x$. Sketch this region and use geometry to find an expression for $A(x)$.

(c) Differentiate the area function $A(x)$. What do you notice?

2. (a) If $x \geqslant -1$, let

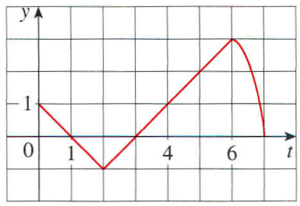
$$A(x) = \int_{-1}^{x} (1 + t^2)\,dt$$

$A(x)$ represents the area of a region. Sketch that region.

5.3 EXERCISES

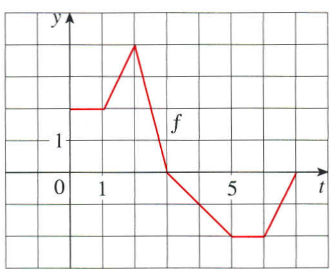
1. Explain exactly what is meant by the statement that "differentiation and integration are inverse processes."

2. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.

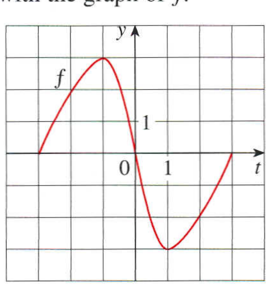


(a) Evaluate $g(x)$ for $x = 0, 1, 2, 3, 4, 5,$ and 6.
(b) Estimate $g(7)$.
(c) Where does g have a maximum value? Where does it have a minimum value?
(d) Sketch a rough graph of g.

3. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.
(a) Evaluate $g(0)$, $g(1)$, $g(2)$, $g(3)$, and $g(6)$.
(b) On what interval is g increasing?
(c) Where does g have a maximum value?
(d) Sketch a rough graph of g.



4. Let $g(x) = \int_{-3}^x f(t)\,dt$, where f is the function whose graph is shown.
(a) Evaluate $g(-3)$ and $g(3)$.
(b) Estimate $g(-2)$, $g(-1)$, and $g(0)$.
(c) On what interval is g increasing?
(d) Where does g have a maximum value?
(e) Sketch a rough graph of g.
(f) Use the graph in part (e) to sketch the graph of $g'(x)$. Compare with the graph of f.



5–6 Sketch the area represented by $g(x)$. Then find $g'(x)$ in two ways: (a) by using Part 1 of the Fundamental Theorem and (b) by evaluating the integral using Part 2 and then differentiating.

5. $g(x) = \displaystyle\int_1^x t^2\,dt$

6. $g(x) = \displaystyle\int_0^x \left(1 + \sqrt{t}\right) dt$

7–18 Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of the function.

7. $g(x) = \displaystyle\int_1^x \frac{1}{t^3 + 1}\,dt$

8. $g(x) = \displaystyle\int_1^x (2 + t^4)^5\,dt$

9. $g(y) = \displaystyle\int_2^y t^2 \sin t\,dt$

10. $g(u) = \displaystyle\int_3^u \frac{1}{x + x^2}\,dx$

11. $F(x) = \displaystyle\int_x^{\pi} \sqrt{1 + \sec t}\,dt$

$$\left[\textit{Hint: } \int_x^{\pi} \sqrt{1 + \sec t}\,dt = -\int_{\pi}^x \sqrt{1 + \sec t}\,dt\right]$$

12. $G(x) = \displaystyle\int_x^1 \cos\sqrt{t}\,dt$

13. $h(x) = \displaystyle\int_2^{1/x} \sin^4 t\,dt$

14. $h(x) = \displaystyle\int_0^{x^2} \sqrt{1 + r^3}\,dr$

15. $y = \displaystyle\int_0^{\tan x} \sqrt{t + \sqrt{t}}\,dt$

16. $y = \displaystyle\int_1^{\cos x} (1 + v^2)^{10}\,dv$

17. $y = \displaystyle\int_{1-3x}^1 \frac{u^3}{1 + u^2}\,du$

18. $y = \displaystyle\int_{1/x^2}^0 \sin^3 t\,dt$

19–36 Evaluate the integral.

19. $\displaystyle\int_{-1}^2 (x^3 - 2x)\,dx$

20. $\displaystyle\int_{-2}^5 6\,dx$

21. $\displaystyle\int_1^4 (5 - 2t + 3t^2)\,dt$

22. $\displaystyle\int_0^1 \left(1 + \tfrac{1}{2}u^4 - \tfrac{2}{5}u^9\right) du$

23. $\displaystyle\int_0^4 \sqrt{x}\,dx$

24. $\displaystyle\int_0^1 x^{3/7}\,dx$

25. $\displaystyle\int_1^2 \frac{3}{t^4}\,dt$

26. $\displaystyle\int_{\pi}^{2\pi} \cos\theta\,d\theta$

27. $\displaystyle\int_0^2 x(2 + x^5)\,dx$

28. $\displaystyle\int_0^1 \left(3 + x\sqrt{x}\right) dx$

29. $\displaystyle\int_1^9 \frac{x - 1}{\sqrt{x}}\,dx$

30. $\displaystyle\int_0^2 (y - 1)(2y + 1)\,dy$

31. $\displaystyle\int_0^{\pi/4} \sec^2 t\,dt$

32. $\displaystyle\int_0^{\pi/4} \sec\theta \tan\theta\,d\theta$

33. $\displaystyle\int_1^2 (1 + 2y)^2\,dy$

34. $\displaystyle\int_1^2 \frac{s^4 + 1}{s^2}\,ds$

35. $\int_0^\pi f(x)\,dx$ where $f(x) = \begin{cases} \sin x & \text{if } 0 \leqslant x < \pi/2 \\ \cos x & \text{if } \pi/2 \leqslant x \leqslant \pi \end{cases}$

36. $\int_{-2}^{2} f(x)\,dx$ where $f(x) = \begin{cases} 2 & \text{if } -2 \leqslant x \leqslant 0 \\ 4 - x^2 & \text{if } 0 < x \leqslant 2 \end{cases}$

37–40 What is wrong with the equation?

37. $\int_{-2}^{1} x^{-4}\,dx = \dfrac{x^{-3}}{-3}\Big]_{-2}^{1} = -\dfrac{3}{8}$

38. $\int_{-1}^{2} \dfrac{4}{x^3}\,dx = -\dfrac{2}{x^2}\Big]_{-1}^{2} = \dfrac{3}{2}$

39. $\int_{\pi/3}^{\pi} \sec\theta \tan\theta\,d\theta = \sec\theta\Big]_{\pi/3}^{\pi} = -3$

40. $\int_0^\pi \sec^2 x\,dx = \tan x\Big]_0^\pi = 0$

41–44 Use a graph to give a rough estimate of the area of the region that lies beneath the given curve. Then find the exact area.

41. $y = \sqrt[3]{x}, \quad 0 \leqslant x \leqslant 27$

42. $y = x^{-4}, \quad 1 \leqslant x \leqslant 6$

43. $y = \sin x, \; 0 \leqslant x \leqslant \pi$

44. $y = \sec^2 x, \; 0 \leqslant x \leqslant \pi/3$

45–46 Evaluate the integral and interpret it as a difference of areas. Illustrate with a sketch.

45. $\int_{-1}^{2} x^3\,dx$

46. $\int_{\pi/4}^{5\pi/2} \sin x\,dx$

47–50 Find the derivative of the function.

47. $g(x) = \int_{2x}^{3x} \dfrac{u^2 - 1}{u^2 + 1}\,du$

$$\left[\textit{Hint: } \int_{2x}^{3x} f(u)\,du = \int_{2x}^{0} f(u)\,du + \int_0^{3x} f(u)\,du\right]$$

48. $g(x) = \int_{\tan x}^{x^2} \dfrac{1}{\sqrt{2 + t^4}}\,dt$

49. $y = \int_{\sqrt{x}}^{x^3} \sqrt{t}\,\sin t\,dt$

50. $y = \int_{\cos x}^{5x} \cos(u^2)\,du$

51. If $F(x) = \int_1^x f(t)\,dt$, where $f(t) = \int_1^{t^2} \dfrac{\sqrt{1 + u^4}}{u}\,du$, find $F''(2)$.

52. Find the interval on which the curve $y = \int_0^x \dfrac{1}{1 + t + t^2}\,dt$ is concave upward.

53. The Fresnel function S was defined in Example 3 and graphed in Figures 7 and 8.

(a) At what values of x does this function have local maximum values?

(b) On what intervals is the function concave upward?

CAS (c) Use a graph to solve the following equation correct to two decimal places:

$$\int_0^x \sin(\pi t^2/2)\,dt = 0.2$$

CAS **54.** The **sine integral function**

$$\text{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt$$

is important in electrical engineering. [The integrand $f(t) = (\sin t)/t$ is not defined when $t = 0$, but we know that its limit is 1 when $t \to 0$. So we define $f(0) = 1$ and this makes f a continuous function everywhere.]

(a) Draw the graph of Si.

(b) At what values of x does this function have local maximum values?

(c) Find the coordinates of the first inflection point to the right of the origin.

(d) Does this function have horizontal asymptotes?

(e) Solve the following equation correct to one decimal place:

$$\int_0^x \frac{\sin t}{t}\,dt = 1$$

55–56 Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.

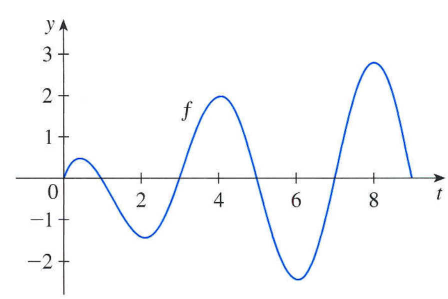
(a) At what values of x do the local maximum and minimum values of g occur?

(b) Where does g attain its absolute maximum value?

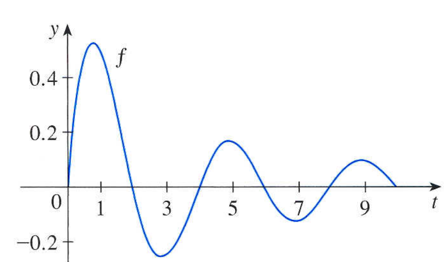
(c) On what intervals is g concave downward?

(d) Sketch the graph of g.

55.



56.



57–58 Evaluate the limit by first recognizing the sum as a Riemann sum for a function defined on [0, 1].

57. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^4}$

58. $\displaystyle \lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \sqrt{\frac{3}{n}} + \cdots + \sqrt{\frac{n}{n}}\right)$

59. Justify (3) for the case $h < 0$.

60. If f is continuous and g and h are differentiable functions, find a formula for

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt$$

61. (a) Show that $1 \leqslant \sqrt{1 + x^3} \leqslant 1 + x^3$ for $x \geqslant 0$.
(b) Show that $1 \leqslant \int_0^1 \sqrt{1 + x^3}\,dx \leqslant 1.25$.

62. (a) Show that $\cos(x^2) \geqslant \cos x$ for $0 \leqslant x \leqslant 1$.
(b) Deduce that $\int_0^{\pi/6} \cos(x^2)\,dx \geqslant \frac{1}{2}$.

63. Show that

$$0 \leqslant \int_5^{10} \frac{x^2}{x^4 + x^2 + 1}\,dx \leqslant 0.1$$

by comparing the integrand to a simpler function.

64. Let

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leqslant x \leqslant 1 \\ 2 - x & \text{if } 1 < x \leqslant 2 \\ 0 & \text{if } x > 2 \end{cases}$$

and $$g(x) = \int_0^x f(t)\,dt$$

(a) Find an expression for $g(x)$ similar to the one for $f(x)$.
(b) Sketch the graphs of f and g.
(c) Where is f differentiable? Where is g differentiable?

65. Find a function f and a number a such that

$$6 + \int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x} \qquad \text{for all } x > 0$$

66. Suppose h is a function such that $h(1) = -2$, $h'(1) = 2$, $h''(1) = 3$, $h(2) = 6$, $h'(2) = 5$, $h''(2) = 13$, and h'' is continuous everywhere. Evaluate $\int_1^2 h''(u)\,du$.

67. A manufacturing company owns a major piece of equipment that depreciates at the (continuous) rate $f = f(t)$, where t is the time measured in months since its last overhaul. Because a fixed cost A is incurred each time the machine is overhauled, the company wants to determine the optimal time T (in months) between overhauls.
(a) Explain why $\int_0^t f(s)\,ds$ represents the loss in value of the machine over the period of time t since the last overhaul.
(b) Let $C = C(t)$ be given by

$$C(t) = \frac{1}{t}\left[A + \int_0^t f(s)\,ds\right]$$

What does C represent and why would the company want to minimize C?
(c) Show that C has a minimum value at the numbers $t = T$ where $C(T) = f(T)$.

68. A high-tech company purchases a new computing system whose initial value is V. The system will depreciate at the rate $f = f(t)$ and will accumulate maintenance costs at the rate $g = g(t)$, where t is the time measured in months. The company wants to determine the optimal time to replace the system.
(a) Let

$$C(t) = \frac{1}{t}\int_0^t [f(s) + g(s)]\,ds$$

Show that the critical numbers of C occur at the numbers t where $C(t) = f(t) + g(t)$.
(b) Suppose that

$$f(t) = \begin{cases} \dfrac{V}{15} - \dfrac{V}{450}t & \text{if } 0 < t \leqslant 30 \\ 0 & \text{if } t > 30 \end{cases}$$

and $$g(t) = \frac{Vt^2}{12{,}900} \qquad t > 0$$

Determine the length of time T for the total depreciation $D(t) = \int_0^t f(s)\,ds$ to equal the initial value V.
(c) Determine the absolute minimum of C on $(0, T]$.
(d) Sketch the graphs of C and $f + g$ in the same coordinate system, and verify the result in part (a) in this case.

The following exercises are intended only for those who have already covered Chapter 7.

69–74 Evaluate the integral.

69. $\displaystyle \int_1^9 \frac{1}{2x}\,dx$

70. $\displaystyle \int_0^1 10^x\,dx$

71. $\displaystyle \int_{1/2}^{\sqrt{3}/2} \frac{6}{\sqrt{1 - t^2}}\,dt$

72. $\displaystyle \int_0^1 \frac{4}{t^2 + 1}\,dt$

73. $\displaystyle \int_{-1}^1 e^{u+1}\,du$

74. $\displaystyle \int_1^2 \frac{4 + u^2}{u^3}\,du$

EXAMPLE 7 Figure 4 shows the power consumption in the province of Ontario, Canada, for December 9, 2004 (P is measured in megawatts; t is measured in hours starting at midnight). Estimate the energy used on that day.

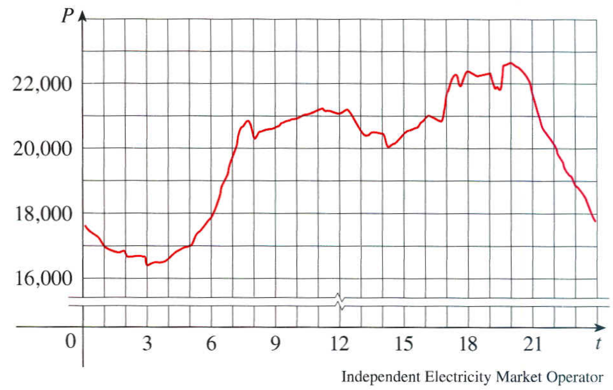


FIGURE 4

SOLUTION Power is the rate of change of energy: $P(t) = E'(t)$. So, by the Net Change Theorem,

$$\int_0^{24} P(t)\,dt = \int_0^{24} E'(t)\,dt = E(24) - E(0)$$

is the total amount of energy used on that day. We approximate the value of the integral using the Midpoint Rule with 12 subintervals and $\Delta t = 2$:

$$\begin{aligned}\int_0^{24} P(t)\,dt &\approx [P(1) + P(3) + P(5) + \cdots + P(21) + P(23)]\,\Delta t\\ &\approx (16{,}900 + 16{,}400 + 17{,}000 + 19{,}800 + 20{,}700 + 21{,}200 + 20{,}500\\ &\qquad + 20{,}500 + 21{,}700 + 22{,}300 + 21{,}700 + 18{,}900)(2)\\ &= 475{,}200\end{aligned}$$

The energy used was approximately 4.75×10^5 megawatt-hours. □

■ A note on units

How did we know what units to use for energy in Example 7? The integral $\int_0^{24} P(t)\,dt$ is defined as the limit of sums of terms of the form $P(t_i^*)\,\Delta t$. Now $P(t_i^*)$ is measured in megawatts and Δt is measured in hours, so their product is measured in megawatt-hours. The same is true of the limit. In general, the unit of measurement for $\int_a^b f(x)\,dx$ is the product of the unit for $f(x)$ and the unit for x.

5.4 EXERCISES

1–4 Verify by differentiation that the formula is correct.

1. $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\,dx = \sqrt{x^2+1} + C$

2. $\displaystyle\int x\cos x\,dx = x\sin x + \cos x + C$

3. $\displaystyle\int \cos^3 x\,dx = \sin x - \tfrac{1}{3}\sin^3 x + C$

4. $\displaystyle\int \frac{x}{\sqrt{a+bx}}\,dx = \frac{2}{3b^2}(bx - 2a)\sqrt{a+bx} + C$

5–16 Find the general indefinite integral.

5. $\displaystyle\int (x^2 + x^{-2})\,dx$

6. $\displaystyle\int \left(\sqrt{x^3} + \sqrt[3]{x^2}\right)dx$

7. $\displaystyle\int (x^3 + 6x + 1)\,dx$

8. $\displaystyle\int x(1 + 2x^4)\,dx$

9. $\displaystyle\int (1 - t)(2 + t^2)\,dt$

10. $\displaystyle\int v(v^2 + 2)^2\,dv$

11. $\displaystyle\int \frac{x^3 - 2\sqrt{x}}{x}\,dx$

12. $\displaystyle\int \left(u^2 + 1 + \frac{1}{u^2}\right)du$

13. $\displaystyle\int (\theta - \csc\theta \cot\theta)\, d\theta$

14. $\displaystyle\int \sec t\,(\sec t + \tan t)\, dt$

15. $\displaystyle\int (1 + \tan^2\alpha)\, d\alpha$

16. $\displaystyle\int \frac{\sin 2x}{\sin x}\, dx$

17–18 Find the general indefinite integral. Illustrate by graphing several members of the family on the same screen.

17. $\displaystyle\int \left(\cos x + \tfrac{1}{2}x\right) dx$

18. $\displaystyle\int (1 - x^2)^2\, dx$

19–42 Evaluate the integral.

19. $\displaystyle\int_0^2 (6x^2 - 4x + 5)\, dx$

20. $\displaystyle\int_1^3 (1 + 2x - 4x^3)\, dx$

21. $\displaystyle\int_{-3}^0 (5y^4 - 6y^2 + 14)\, dy$

22. $\displaystyle\int_{-2}^0 (u^5 - u^3 + u^2)\, du$

23. $\displaystyle\int_{-2}^2 (3u + 1)^2\, du$

24. $\displaystyle\int_0^4 (2v + 5)(3v - 1)\, dv$

25. $\displaystyle\int_1^4 \sqrt{t}\,(1 + t)\, dt$

26. $\displaystyle\int_0^9 \sqrt{2t}\, dt$

27. $\displaystyle\int_{-2}^{-1} \left(4y^3 + \frac{2}{y^3}\right) dy$

28. $\displaystyle\int_1^2 \frac{y + 5y^7}{y^3}\, dy$

29. $\displaystyle\int_0^1 x\left(\sqrt[3]{x} + \sqrt[4]{x}\right) dx$

30. $\displaystyle\int_1^2 \left(x + \frac{1}{x}\right)^2 dx$

31. $\displaystyle\int_1^4 \sqrt{\frac{5}{x}}\, dx$

32. $\displaystyle\int_1^9 \frac{3x - 2}{\sqrt{x}}\, dx$

33. $\displaystyle\int_0^\pi (4\sin\theta - 3\cos\theta)\, d\theta$

34. $\displaystyle\int_{\pi/4}^{\pi/3} \sec\theta\,\tan\theta\, d\theta$

35. $\displaystyle\int_0^{\pi/4} \frac{1 + \cos^2\theta}{\cos^2\theta}\, d\theta$

36. $\displaystyle\int_0^{\pi/3} \frac{\sin\theta + \sin\theta\,\tan^2\theta}{\sec^2\theta}\, d\theta$

37. $\displaystyle\int_1^{64} \frac{1 + \sqrt[3]{x}}{\sqrt{x}}\, dx$

38. $\displaystyle\int_0^1 (1 + x^2)^3\, dx$

39. $\displaystyle\int_0^1 \left(\sqrt[4]{x^5} + \sqrt[5]{x^4}\right) dx$

40. $\displaystyle\int_1^8 \frac{x - 1}{\sqrt[3]{x^2}}\, dx$

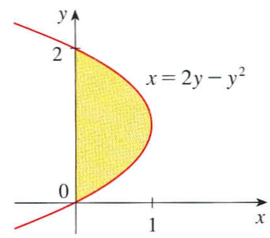
41. $\displaystyle\int_{-1}^2 (x - 2|x|)\, dx$

42. $\displaystyle\int_0^{3\pi/2} |\sin x|\, dx$

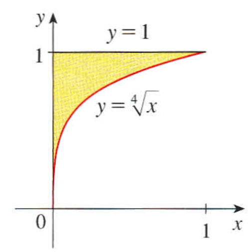
43. Use a graph to estimate the x-intercepts of the curve $y = x + x^2 - x^4$. Then use this information to estimate the area of the region that lies under the curve and above the x-axis.

44. Repeat Exercise 43 for the curve $y = 2x + 3x^4 - 2x^6$.

45. The area of the region that lies to the right of the y-axis and to the left of the parabola $x = 2y - y^2$ (the shaded region in the figure) is given by the integral $\int_0^2 (2y - y^2)\, dy$. (Turn your head clockwise and think of the region as lying below the curve $x = 2y - y^2$ from $y = 0$ to $y = 2$.) Find the area of the region.



46. The boundaries of the shaded region are the y-axis, the line $y = 1$, and the curve $y = \sqrt[4]{x}$. Find the area of this region by writing x as a function of y and integrating with respect to y (as in Exercise 45).



47. If $w'(t)$ is the rate of growth of a child in kilograms per year, what does $\int_5^{10} w'(t)\, dt$ represent?

48. The current in a wire is defined as the derivative of the charge: $I(t) = Q'(t)$. (See Example 3 in Section 3.7.) What does $\int_a^b I(t)\, dt$ represent?

49. If oil leaks from a tank at a rate of $r(t)$ liters per minute at time t, what does $\int_0^{120} r(t)\, dt$ represent?

50. A honeybee population starts with 100 bees and increases at a rate of $n'(t)$ bees per week. What does $100 + \int_0^{15} n'(t)\, dt$ represent?

51. In Section 4.7 we defined the marginal revenue function $R'(x)$ as the derivative of the revenue function $R(x)$, where x is the number of units sold. What does $\int_{1000}^{5000} R'(x)\, dx$ represent?

52. If $f(x)$ is the slope of a trail at a distance of x kilometers from the start of the trail, what does $\int_3^5 f(x)\, dx$ represent?

53. If x is measured in meters and $f(x)$ is measured in newtons, what are the units for $\int_0^{100} f(x)\, dx$?

54. If the units for x are meters and the units for $a(x)$ are kilograms per meter, what are the units for da/dx? What units does $\int_2^8 a(x)\, dx$ have?

55–56 The velocity function (in meters per second) is given for a particle moving along a line. Find (a) the displacement and (b) the distance traveled by the particle during the given time interval.

55. $v(t) = 3t - 5, \quad 0 \leqslant t \leqslant 3$

56. $v(t) = t^2 - 2t - 8, \quad 1 \leqslant t \leqslant 6$

57–58 The acceleration function (in $\mathrm{m/s^2}$) and the initial velocity are given for a particle moving along a line. Find (a) the velocity at time t and (b) the distance traveled during the given time interval.

57. $a(t) = t + 4, \quad v(0) = 5, \quad 0 \leqslant t \leqslant 10$

58. $a(t) = 2t + 3, \quad v(0) = -4, \quad 0 \leqslant t \leqslant 3$

59. The linear density of a rod of length 4 m is given by $\rho(x) = 9 + 2\sqrt{x}$ measured in kilograms per meter, where x is measured in meters from one end of the rod. Find the total mass of the rod.

60. Water flows from the bottom of a storage tank at a rate of $r(t) = 200 - 4t$ liters per minute, where $0 \leqslant t \leqslant 50$. Find the amount of water that flows from the tank during the first 10 minutes.

61. The velocity of a car was read from its speedometer at 10-second intervals and recorded in the table. Use the Midpoint Rule to estimate the distance traveled by the car.

t (s)	v (km/h)	t (s)	v (km/h)
0	0	60	90
10	61	70	85
20	83	80	80
30	93	90	75
40	88	100	72
50	82		

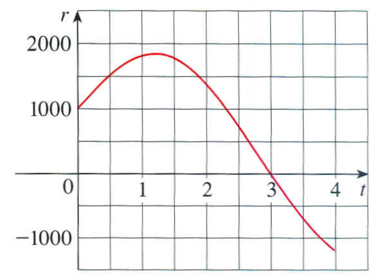
62. Suppose that a volcano is erupting and readings of the rate $r(t)$ at which solid materials are spewed into the atmosphere are given in the table. The time t is measured in seconds and the units for $r(t)$ are tonnes per second.

t	0	1	2	3	4	5	6
$r(t)$	2	10	24	36	46	54	60

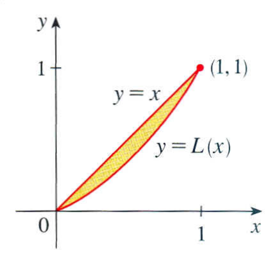
(a) Give upper and lower estimates for the total quantity $Q(6)$ of erupted materials after 6 seconds.
(b) Use the Midpoint Rule to estimate $Q(6)$.

63. The marginal cost of manufacturing x meters of a certain fabric is $C'(x) = 3 - 0.01x + 0.000006x^2$ (in dollars per meter). Find the increase in cost if the production level is raised from 2000 meters to 4000 meters.

64. Water flows into and out of a storage tank. A graph of the rate of change $r(t)$ of the volume of water in the tank, in liters per day, is shown. If the amount of water in the tank at time $t = 0$ is 25,000 L, use the Midpoint Rule to estimate the amount of water four days later.



65. Economists use a cumulative distribution called a *Lorenz curve* to describe the distribution of income between households in a given country. Typically, a Lorenz curve is defined on [0, 1] with endpoints (0, 0) and (1, 1), and is continuous, increasing, and concave upward. The points on this curve are determined by ranking all households by income and then computing the percentage of households whose income is less than or equal to a given percentage of the total income of the country. For example, the point $(a/100, b/100)$ is on the Lorenz curve if the bottom $a\%$ of the households receive less than or equal to $b\%$ of the total income. *Absolute equality* of income distribution would occur if the bottom $a\%$ of the households receive $a\%$ of the income, in which case the Lorenz curve would be the line $y = x$. The area between the Lorenz curve and the line $y = x$ measures how much the income distribution differs from absolute equality. The *coefficient of inequality* is the ratio of the area between the Lorenz curve and the line $y = x$ to the area under $y = x$.



(a) Show that the coefficient of inequality is twice the area between the Lorenz curve and the line $y = x$, that is, show that

$$\text{coefficient of inequality} = 2\int_0^1 [x - L(x)]\,dx$$

(b) The income distribution for a certain country is represented by the Lorenz curve defined by the equation

$$L(x) = \tfrac{5}{12}x^2 + \tfrac{7}{12}x$$

What is the percentage of total income received by the bottom 50% of the households? Find the coefficient of inequality.

66. On May 7, 1992, the space shuttle *Endeavour* was launched on mission STS-49, the purpose of which was to install a new perigee kick motor in an Intelsat communications satellite. The table gives the velocity data for the shuttle between liftoff and the jettisoning of the solid rocket boosters.

Event	Time (s)	Velocity (m/s)
Launch	0	0
Begin roll maneuver	10	56.4
End roll maneuver	15	97.2
Throttle to 89%	20	136.2
Throttle to 67%	32	226.2
Throttle to 104%	59	403.9
Maximum dynamic pressure	62	440.4
Solid rocket booster separation	125	1265.2

(a) Use a graphing calculator or computer to model these data by a third-degree polynomial.

(b) Use the model in part (a) to estimate the height reached by the *Endeavour*, 125 seconds after liftoff.

The following exercises are intended only for those who have already covered Chapter 7.

67–71 Evaluate the integral.

67. $\int (\sin x + \sinh x)\,dx$

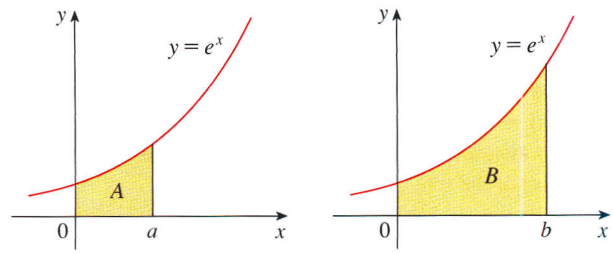
68. $\int_{-10}^{10} \frac{2e^x}{\sinh x + \cosh x}\,dx$

69. $\int \left(x^2 + 1 + \frac{1}{x^2+1}\right) dx$

70. $\int_1^2 \frac{(x-1)^3}{x^2}\,dx$

71. $\int_0^{1/\sqrt{3}} \frac{t^2-1}{t^4-1}\,dt$

72. The area labeled B is three times the area labeled A. Express b in terms of a.



WRITING PROJECT

NEWTON, LEIBNIZ, AND THE INVENTION OF CALCULUS

We sometimes read that the inventors of calculus were Sir Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716). But we know that the basic ideas behind integration were investigated 2500 years ago by ancient Greeks such as Eudoxus and Archimedes, and methods for finding tangents were pioneered by Pierre Fermat (1601–1665), Isaac Barrow (1630–1677), and others. Barrow—who taught at Cambridge and was a major influence on Newton—was the first to understand the inverse relationship between differentiation and integration. What Newton and Leibniz did was to use this relationship, in the form of the Fundamental Theorem of Calculus, in order to develop calculus into a systematic mathematical discipline. It is in this sense that Newton and Leibniz are credited with the invention of calculus.

Read about the contributions of these men in one or more of the given references and write a report on one of the following three topics. You can include biographical details, but the main thrust of your report should be a description, in some detail, of their methods and notations. In particular, you should consult one of the sourcebooks, which give excerpts from the original publications of Newton and Leibniz, translated from Latin to English.

- The Role of Newton in the Development of Calculus
- The Role of Leibniz in the Development of Calculus
- The Controversy between the Followers of Newton and Leibniz over Priority in the Invention of Calculus

References

1. Carl Boyer and Uta Merzbach, *A History of Mathematics* (New York: Wiley, 1987), Chapter 19.

PROOF We split the integral in two:

$$\boxed{7} \qquad \int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

In the first integral on the far right side we make the substitution $u = -x$. Then $du = -dx$ and when $x = -a$, $u = a$. Therefore

$$-\int_{0}^{-a} f(x)\,dx = -\int_{0}^{a} f(-u)(-du) = \int_{0}^{a} f(-u)\,du$$

and so Equation 7 becomes

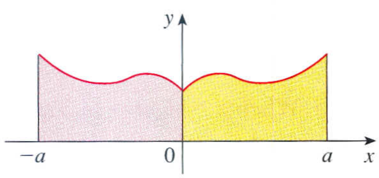
$$\boxed{8} \qquad \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(x)\,dx$$

(a) If f is even, then $f(-u) = f(u)$ so Equation 8 gives

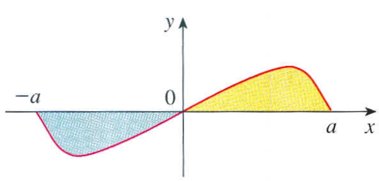
$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

(b) If f is odd, then $f(-u) = -f(u)$ and so Equation 8 gives

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx = 0 \qquad \square$$



(a) f even, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$



(b) f odd, $\int_{-a}^{a} f(x)\,dx = 0$

FIGURE 3

Theorem 6 is illustrated by Figure 3. For the case where f is positive and even, part (a) says that the area under $y = f(x)$ from $-a$ to a is twice the area from 0 to a because of symmetry. Recall that an integral $\int_a^b f(x)\,dx$ can be expressed as the area above the x-axis and below $y = f(x)$ minus the area below the axis and above the curve. Thus part (b) says the integral is 0 because the areas cancel.

EXAMPLE 8 Since $f(x) = x^6 + 1$ satisfies $f(-x) = f(x)$, it is even and so

$$\int_{-2}^{2} (x^6 + 1)\,dx = 2\int_{0}^{2} (x^6 + 1)\,dx$$
$$= 2\left[\tfrac{1}{7}x^7 + x\right]_0^2 = 2\left(\tfrac{128}{7} + 2\right) = \tfrac{284}{7} \qquad \square$$

EXAMPLE 9 Since $f(x) = (\tan x)/(1 + x^2 + x^4)$ satisfies $f(-x) = -f(x)$, it is odd and so

$$\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}\,dx = 0 \qquad \square$$

5.5 EXERCISES

1–6 Evaluate the integral by making the given substitution.

1. $\int \cos 3x\,dx, \quad u = 3x$

2. $\int x(4 + x^2)^{10}\,dx, \quad u = 4 + x^2$

3. $\int x^2\sqrt{x^3 + 1}\,dx, \quad u = x^3 + 1$

4. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx, \quad u = \sqrt{x}$

5. $\int \frac{4}{(1+2x)^3}\,dx, \quad u = 1 + 2x$

6. $\int \frac{\sec^2(1/x)}{x^2}\,dx, \quad u = 1/x$

7–30 Evaluate the indefinite integral.

7. $\int x \sin(x^2)\,dx$

8. $\int x^2(x^3+5)^9\,dx$

9. $\int (3x-2)^{20}\,dx$

10. $\int (3t+2)^{2.4}\,dt$

11. $\int (x+1)\sqrt{2x+x^2}\,dx$

12. $\int \frac{x}{(x^2+1)^2}\,dx$

13. $\int \sin \pi t\,dt$

14. $\int \frac{1}{(5t+4)^{2.7}}\,dt$

15. $\int \frac{a+bx^2}{\sqrt{3ax+bx^3}}\,dx$

16. $\int \sec 2\theta \tan 2\theta\,d\theta$

17. $\int \frac{\cos\sqrt{t}}{\sqrt{t}}\,dt$

18. $\int \sqrt{x}\sin(1+x^{3/2})\,dx$

19. $\int \cos\theta \sin^6\theta\,d\theta$

20. $\int (1+\tan\theta)^5 \sec^2\theta\,d\theta$

21. $\int \frac{z^2}{\sqrt[3]{1+z^3}}\,dz$

22. $\int \frac{\cos(\pi/x)}{x^2}\,dx$

23. $\int \sqrt{\cot x}\,\csc^2 x\,dx$

24. $\int \frac{dt}{\cos^2 t\sqrt{1+\tan t}}$

25. $\int \sec^3 x \tan x\,dx$

26. $\int \sin t \sec^2(\cos t)\,dt$

27. $\int \frac{\cos x}{\sin^2 x}\,dx$

28. $\int \frac{x^2}{\sqrt{1-x}}\,dx$

29. $\int \frac{x}{\sqrt[4]{x+2}}\,dx$

30. $\int x^3\sqrt{x^2+1}\,dx$

31–34 Evaluate the indefinite integral. Illustrate and check that your answer is reasonable by graphing both the function and its antiderivative (take $C = 0$).

31. $\int x(x^2-1)^3\,dx$

32. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

33. $\int \sin^3 x \cos x\,dx$

34. $\int \tan^2\theta \sec^2\theta\,d\theta$

35–50 Evaluate the definite integral.

35. $\int_0^2 (x-1)^{25}\,dx$

36. $\int_0^7 \sqrt{4+3x}\,dx$

37. $\int_0^1 x^2(1+2x^3)^5\,dx$

38. $\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx$

39. $\int_0^{\pi} \sec^2(t/4)\,dt$

40. $\int_{1/6}^{1/2} \csc \pi t \cot \pi t\,dt$

41. $\int_{-\pi/6}^{\pi/6} \tan^3\theta\,d\theta$

42. $\int_{-\pi/2}^{\pi/2} \frac{x^2\sin x}{1+x^6}\,dx$

43. $\int_0^{\pi/3} \frac{\sin\theta}{\cos^2\theta}\,d\theta$

44. $\int_0^{\pi/2} \cos x \sin(\sin x)\,dx$

45. $\int_0^a x\sqrt{x^2+a^2}\,dx \quad (a > 0)$

46. $\int_0^a x\sqrt{a^2-x^2}\,dx$

47. $\int_1^2 x\sqrt{x-1}\,dx$

48. $\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx$

49. $\int_{1/2}^1 \frac{\cos(x^{-2})}{x^3}\,dx$

50. $\int_0^{T/2} \sin(2\pi t/T - \alpha)\,dt$

51–52 Use a graph to give a rough estimate of the area of the region that lies under the given curve. Then find the exact area.

51. $y = \sqrt{2x+1}, \quad 0 \leqslant x \leqslant 1$

52. $y = 2\sin x - \sin 2x, \quad 0 \leqslant x \leqslant \pi$

53. Evaluate $\int_{-2}^{2} (x+3)\sqrt{4-x^2}\,dx$ by writing it as a sum of two integrals and interpreting one of those integrals in terms of an area.

54. Evaluate $\int_0^1 x\sqrt{1-x^4}\,dx$ by making a substitution and interpreting the resulting integral in terms of an area.

55. Breathing is cyclic and a full respiratory cycle from the beginning of inhalation to the end of exhalation takes about 5 s. The maximum rate of air flow into the lungs is about 0.5 L/s. This explains, in part, why the function $f(t) = \frac{1}{2}\sin(2\pi t/5)$ has often been used to model the rate of air flow into the lungs. Use this model to find the volume of inhaled air in the lungs at time t.

56. A model for the basal metabolism rate, in kcal/h, of a young man is $R(t) = 85 - 0.18\cos(\pi t/12)$, where t is the time in hours measured from 5:00 AM. What is the total basal metabolism of this man, $\int_0^{24} R(t)\,dt$, over a 24-hour time period?

57. If f is continuous and $\int_0^4 f(x)\,dx = 10$, find $\int_0^2 f(2x)\,dx$.

58. If f is continuous and $\int_0^9 f(x)\,dx = 4$, find $\int_0^3 xf(x^2)\,dx$.

59. If f is continuous on $\mathbb{R}$, prove that

$$\int_a^b f(-x)\,dx = \int_{-b}^{-a} f(x)\,dx$$

For the case where $f(x) \geqslant 0$ and $0 < a < b$, draw a diagram to interpret this equation geometrically as an equality of areas.

60. If f is continuous on $\mathbb{R}$, prove that

$$\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(x)\,dx$$

For the case where $f(x) \geqslant 0$, draw a diagram to interpret this equation geometrically as an equality of areas.

61. If a and b are positive numbers, show that

$$\int_0^1 x^a(1-x)^b\,dx = \int_0^1 x^b(1-x)^a\,dx$$

62. If f is continuous on $[0, \pi]$, use the substitution $u = \pi - x$ to show that

$$\int_0^\pi x f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$$

63. If f is continuous, prove that

$$\int_0^{\pi/2} f(\cos x)\,dx = \int_0^{\pi/2} f(\sin x)\,dx$$

64. Use Exercise 63 to evaluate $\int_0^{\pi/2} \cos^2 x\,dx$ and $\int_0^{\pi/2} \sin^2 x\,dx$.

The following exercises are intended only for those who have already covered Chapter 7.

65–82 Evaluate the integral.

65. $\displaystyle\int \frac{dx}{5-3x}$

66. $\displaystyle\int e^x \sin(e^x)\,dx$

67. $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$

68. $\displaystyle\int \frac{dx}{ax+b} \quad (a \neq 0)$

69. $\displaystyle\int e^x\sqrt{1+e^x}\,dx$

70. $\displaystyle\int e^{\cos t}\sin t\,dt$

71. $\displaystyle\int e^{\tan x}\sec^2 x\,dx$

72. $\displaystyle\int \frac{\tan^{-1}x}{1+x^2}\,dx$

73. $\displaystyle\int \frac{1+x}{1+x^2}\,dx$

74. $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx$

75. $\displaystyle\int \frac{\sin 2x}{1+\cos^2 x}\,dx$

76. $\displaystyle\int \frac{\sin x}{1+\cos^2 x}\,dx$

77. $\displaystyle\int \cot x\,dx$

78. $\displaystyle\int \frac{x}{1+x^4}\,dx$

79. $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

80. $\displaystyle\int_0^1 xe^{-x^2}\,dx$

81. $\displaystyle\int_0^1 \frac{e^z+1}{e^z+z}\,dz$

82. $\displaystyle\int_0^{1/2} \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

83. Use Exercise 62 to evaluate the integral

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx$$

5 REVIEW

CONCEPT CHECK

1. (a) Write an expression for a Riemann sum of a function f. Explain the meaning of the notation that you use.
(b) If $f(x) \geqslant 0$, what is the geometric interpretation of a Riemann sum? Illustrate with a diagram.
(c) If $f(x)$ takes on both positive and negative values, what is the geometric interpretation of a Riemann sum? Illustrate with a diagram.

2. (a) Write the definition of the definite integral of a function from a to b.
(b) What is the geometric interpretation of $\int_a^b f(x)\,dx$ if $f(x) \geqslant 0$?
(c) What is the geometric interpretation of $\int_a^b f(x)\,dx$ if $f(x)$ takes on both positive and negative values? Illustrate with a diagram.

3. State both parts of the Fundamental Theorem of Calculus.

4. (a) State the Net Change Theorem.
(b) If $r(t)$ is the rate at which water flows into a reservoir, what does $\int_{t_1}^{t_2} r(t)\,dt$ represent?

5. Suppose a particle moves back and forth along a straight line with velocity $v(t)$, measured in meters per second, and acceleration $a(t)$.
(a) What is the meaning of $\int_{60}^{120} v(t)\,dt$?
(b) What is the meaning of $\int_{60}^{120} |v(t)|\,dt$?
(c) What is the meaning of $\int_{60}^{120} a(t)\,dt$?

6. (a) Explain the meaning of the indefinite integral $\int f(x)\,dx$.
(b) What is the connection between the definite integral $\int_a^b f(x)\,dx$ and the indefinite integral $\int f(x)\,dx$?

7. Explain exactly what is meant by the statement that "differentiation and integration are inverse processes."

8. State the Substitution Rule. In practice, how do you use it?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If f and g are continuous on $[a, b]$, then

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

2. If f and g are continuous on $[a, b]$, then

$$\int_a^b [f(x)\,g(x)]\,dx = \left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$$

3. If f is continuous on $[a, b]$, then

$$\int_a^b 5f(x)\,dx = 5\int_a^b f(x)\,dx$$

4. If f is continuous on $[a, b]$, then

$$\int_a^b xf(x)\,dx = x\int_a^b f(x)\,dx$$

5. If f is continuous on $[a, b]$ and $f(x) \geqslant 0$, then

$$\int_a^b \sqrt{f(x)}\,dx = \sqrt{\int_a^b f(x)\,dx}$$

6. If f' is continuous on $[1, 3]$, then $\int_1^3 f'(v)\,dv = f(3) - f(1)$.

7. If f and g are continuous and $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then

$$\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$$

8. If f and g are differentiable and $f(x) \geqslant g(x)$ for $a < x < b$, then $f'(x) \geqslant g'(x)$ for $a < x < b$.

9. $\displaystyle\int_{-1}^{1}\left(x^5 - 6x^9 + \frac{\sin x}{(1 + x^4)^2}\right)dx = 0$

10. $\displaystyle\int_{-5}^{5}(ax^2 + bx + c)\,dx = 2\int_0^5 (ax^2 + c)\,dx$

11. $\displaystyle\int_{-2}^{1}\frac{1}{x^4}\,dx = -\frac{3}{8}$

12. $\int_0^2 (x - x^3)\,dx$ represents the area under the curve $y = x - x^3$ from 0 to 2.

13. All continuous functions have derivatives.

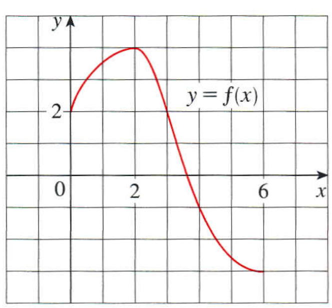
14. All continuous functions have antiderivatives.

15. If f is continuous on $[a, b]$, then

$$\frac{d}{dx}\left(\int_a^b f(x)\,dx\right) = f(x)$$

EXERCISES

1. Use the given graph of f to find the Riemann sum with six subintervals. Take the sample points to be (a) left endpoints and (b) midpoints. In each case draw a diagram and explain what the Riemann sum represents.



2. (a) Evaluate the Riemann sum for

$$f(x) = x^2 - x \qquad 0 \leqslant x \leqslant 2$$

with four subintervals, taking the sample points to be right endpoints. Explain, with the aid of a diagram, what the Riemann sum represents.

(b) Use the definition of a definite integral (with right endpoints) to calculate the value of the integral

$$\int_0^2 (x^2 - x)\,dx$$

(c) Use the Fundamental Theorem to check your answer to part (b).

(d) Draw a diagram to explain the geometric meaning of the integral in part (b).

3. Evaluate

$$\int_0^1 \left(x + \sqrt{1 - x^2}\right)dx$$

by interpreting it in terms of areas.

4. Express

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sin x_i\,\Delta x$$

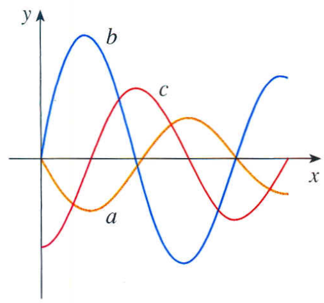
as a definite integral on the interval $[0, \pi]$ and then evaluate the integral.

5. If $\int_0^6 f(x)\,dx = 10$ and $\int_0^4 f(x)\,dx = 7$, find $\int_4^6 f(x)\,dx$.

CAS **6.** (a) Write $\int_1^5 (x + 2x^5)\,dx$ as a limit of Riemann sums, taking the sample points to be right endpoints. Use a computer algebra system to evaluate the sum and to compute the limit.

(b) Use the Fundamental Theorem to check your answer to part (a).

7. The following figure shows the graphs of f, f', and $\int_0^x f(t)\,dt$. Identify each graph, and explain your choices.



8. Evaluate:

(a) $\displaystyle\int_0^{\pi/2} \frac{d}{dx}\left(\sin\frac{x}{2}\cos\frac{x}{3}\right)dx$

(b) $\displaystyle\frac{d}{dx}\int_0^{\pi/2} \sin\frac{x}{2}\cos\frac{x}{3}\,dx$

(c) $\displaystyle\frac{d}{dx}\int_x^{\pi/2} \sin\frac{t}{2}\cos\frac{t}{3}\,dt$

9–28 Evaluate the integral, if it exists.

9. $\displaystyle\int_1^2 (8x^3 + 3x^2)\,dx$

10. $\displaystyle\int_0^T (x^4 - 8x + 7)\,dx$

11. $\displaystyle\int_0^1 (1 - x^9)\,dx$

12. $\displaystyle\int_0^1 (1 - x)^9\,dx$

13. $\displaystyle\int_1^9 \frac{\sqrt{u} - 2u^2}{u}\,du$

14. $\displaystyle\int_0^1 (\sqrt[4]{u} + 1)^2\,du$

15. $\displaystyle\int_0^1 y(y^2 + 1)^5\,dy$

16. $\displaystyle\int_0^2 y^2\sqrt{1 + y^3}\,dy$

17. $\displaystyle\int_1^5 \frac{dt}{(t - 4)^2}$

18. $\displaystyle\int_0^1 \sin(3\pi t)\,dt$

19. $\displaystyle\int_0^1 v^2\cos(v^3)\,dv$

20. $\displaystyle\int_{-1}^1 \frac{\sin x}{1 + x^2}\,dx$

21. $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{t^4\tan t}{2 + \cos t}\,dt$

22. $\displaystyle\int \frac{x + 2}{\sqrt{x^2 + 4x}}\,dx$

23. $\displaystyle\int \sin\pi t\,\cos\pi t\,dt$

24. $\displaystyle\int \sin x\,\cos(\cos x)\,dx$

25. $\displaystyle\int_0^{\pi/8} \sec 2\theta\tan 2\theta\,d\theta$

26. $\displaystyle\int_0^{\pi/4} (1 + \tan t)^3\sec^2 t\,dt$

27. $\displaystyle\int_0^3 |x^2 - 4|\,dx$

28. $\displaystyle\int_0^4 |\sqrt{x} - 1|\,dx$

29–30 Evaluate the indefinite integral. Illustrate and check that your answer is reasonable by graphing both the function and its antiderivative (take $C = 0$).

29. $\displaystyle\int \frac{\cos x}{\sqrt{1 + \sin x}}\,dx$

30. $\displaystyle\int \frac{x^3}{\sqrt{x^2 + 1}}\,dx$

31. Use a graph to give a rough estimate of the area of the region that lies under the curve $y = x\sqrt{x}$, $0 \leqslant x \leqslant 4$. Then find the exact area.

32. Graph the function $f(x) = \cos^2 x\,\sin^3 x$ and use the graph to guess the value of the integral $\int_0^{2\pi} f(x)\,dx$. Then evaluate the integral to confirm your guess.

33–38 Find the derivative of the function.

33. $\displaystyle F(x) = \int_0^x \frac{t^2}{1 + t^3}\,dt$

34. $\displaystyle F(x) = \int_x^1 \sqrt{t + \sin t}\,dt$

35. $\displaystyle g(x) = \int_0^{x^4} \cos(t^2)\,dt$

36. $\displaystyle g(x) = \int_1^{\sin x} \frac{1 - t^2}{1 + t^4}\,dt$

37. $\displaystyle y = \int_{\sqrt{x}}^x \frac{\cos\theta}{\theta}\,d\theta$

38. $\displaystyle y = \int_{2x}^{3x+1} \sin(t^4)\,dt$

39–40 Use Property 8 of integrals to estimate the value of the integral.

39. $\displaystyle\int_1^3 \sqrt{x^2 + 3}\,dx$

40. $\displaystyle\int_3^5 \frac{1}{x + 1}\,dx$

41–42 Use the properties of integrals to verify the inequality.

41. $\displaystyle\int_0^1 x^2\cos x\,dx \leqslant \frac{1}{3}$

42. $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\sin x}{x}\,dx \leqslant \frac{\sqrt{2}}{2}$

43. Use the Midpoint Rule with $n = 6$ to approximate $\int_0^3 \sin(x^3)\,dx$.

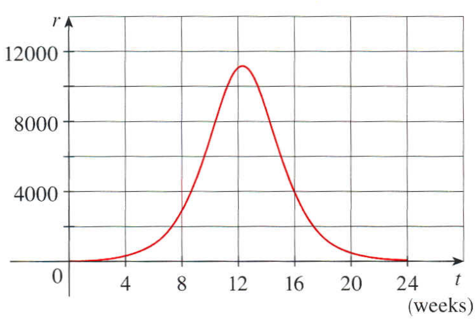
44. A particle moves along a line with velocity function $v(t) = t^2 - t$, where v is measured in meters per second. Find (a) the displacement and (b) the distance traveled by the particle during the time interval $[0, 5]$.

45. Let $r(t)$ be the rate at which the world's oil is consumed, where t is measured in years starting at $t = 0$ on January 1, 2000, and $r(t)$ is measured in barrels per year. What does $\int_0^8 r(t)\,dt$ represent?

46. A radar gun was used to record the speed of a runner at the times given in the table. Use the Midpoint Rule to estimate the distance the runner covered during those 5 seconds.

t (s)	v (m/s)	t (s)	v (m/s)
0	0	3.0	10.51
0.5	4.67	3.5	10.67
1.0	7.34	4.0	10.76
1.5	8.86	4.5	10.81
2.0	9.73	5.0	10.81
2.5	10.22		

47. A population of honeybees increased at a rate of $r(t)$ bees per week, where the graph of r is as shown. Use the Midpoint Rule with six subintervals to estimate the increase in the bee population during the first 24 weeks.



48. Let

$$f(x) = \begin{cases} -x - 1 & \text{if } -3 \leqslant x \leqslant 0 \\ -\sqrt{1 - x^2} & \text{if } 0 \leqslant x \leqslant 1 \end{cases}$$

Evaluate $\int_{-3}^1 f(x)\,dx$ by interpreting the integral as a difference of areas.

49. If f is continuous and $\int_0^2 f(x)\,dx = 6$, evaluate $\int_0^{\pi/2} f(2\sin\theta)\cos\theta\,d\theta$.

50. The Fresnel function $S(x) = \int_0^x \sin\left(\frac{1}{2}\pi t^2\right) dt$ was introduced in Section 5.3. Fresnel also used the function

$$C(x) = \int_0^x \cos\left(\tfrac{1}{2}\pi t^2\right) dt$$

in his theory of the diffraction of light waves.

(a) On what intervals is C increasing?

(b) On what intervals is C concave upward?

CAS (c) Use a graph to solve the following equation correct to two decimal places:

$$\int_0^x \cos\left(\tfrac{1}{2}\pi t^2\right) dt = 0.7$$

CAS (d) Plot the graphs of C and S on the same screen. How are these graphs related?

51. If f is a continuous function such that

$$\int_0^x f(t)\,dt = x\sin x + \int_0^x \frac{f(t)}{1 + t^2}\,dt$$

for all x, find an explicit formula for $f(x)$.

52. Find a function f and a value of the constant a such that

$$2\int_a^x f(t)\,dt = 2\sin x - 1$$

53. If f' is continuous on $[a, b]$, show that

$$2\int_a^b f(x)f'(x)\,dx = [f(b)]^2 - [f(a)]^2$$

54. Find $\displaystyle\lim_{h \to 0} \frac{1}{h}\int_2^{2+h} \sqrt{1 + t^3}\,dt$.

55. If f is continuous on $[0, 1]$, prove that

$$\int_0^1 f(x)\,dx = \int_0^1 f(1 - x)\,dx$$

56. Evaluate

$$\lim_{n \to \infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^9 + \left(\frac{2}{n}\right)^9 + \left(\frac{3}{n}\right)^9 + \cdots + \left(\frac{n}{n}\right)^9\right]$$

PROBLEMS PLUS

PROBLEMS

1. If $x \sin \pi x = \int_0^{x^2} f(t)\,dt$, where f is a continuous function, find $f(4)$.

2. Find the maximum value of the area of the region under the curve $y = 4x - x^3$ from $x = a$ to $x = a + 1$, for all $a > 0$.

3. If f is a differentiable function such that $f(x)$ is never 0 and $\int_0^x f(t)\,dt = [f(x)]^2$ for all x, find f.

4. (a) Graph several members of the family of functions $f(x) = (2cx - x^2)/c^3$ for $c > 0$ and look at the regions enclosed by these curves and the x-axis. Make a conjecture about how the areas of these regions are related.

(b) Prove your conjecture in part (a).

(c) Take another look at the graphs in part (a) and use them to sketch the curve traced out by the vertices (highest points) of the family of functions. Can you guess what kind of curve this is?

(d) Find an equation of the curve you sketched in part (c).

5. If $f(x) = \int_0^{g(x)} \frac{1}{\sqrt{1 + t^3}}\,dt$, where $g(x) = \int_0^{\cos x} [1 + \sin(t^2)]\,dt$, find $f'(\pi/2)$.

6. If $f(x) = \int_0^x x^2 \sin(t^2)\,dt$, find $f'(x)$.

7. Find the interval $[a, b]$ for which the value of the integral $\int_a^b (2 + x - x^2)\,dx$ is a maximum.

8. Use an integral to estimate the sum $\sum_{i=1}^{10000} \sqrt{i}$.

9. (a) Evaluate $\int_0^n [\![x]\!]\,dx$, where n is a positive integer.

(b) Evaluate $\int_a^b [\![x]\!]\,dx$, where a and b are real numbers with $0 \leqslant a < b$.

10. Find $\frac{d^2}{dx^2} \int_0^x \left(\int_1^{\sin t} \sqrt{1 + u^4}\,du \right) dt$.

11. Suppose the coefficients of the cubic polynomial $P(x) = a + bx + cx^2 + dx^3$ satisfy the equation

$$a + \frac{b}{2} + \frac{c}{3} + \frac{d}{4} = 0$$

Show that the equation $P(x) = 0$ has a root between 0 and 1. Can you generalize this result for an nth-degree polynomial?

12. A circular disk of radius r is used in an evaporator and is rotated in a vertical plane. If it is to be partially submerged in the liquid so as to maximize the exposed wetted area of the disk, show that the center of the disk should be positioned at a height $r/\sqrt{1 + \pi^2}$ above the surface of the liquid.

13. Prove that if f is continuous, then $\int_0^x f(u)(x - u)\,du = \int_0^x \left(\int_0^u f(t)\,dt \right) du$.

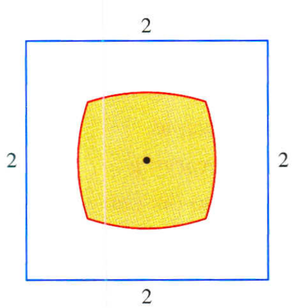


FIGURE FOR PROBLEM 14

14. The figure shows a region consisting of all points inside a square that are closer to the center than to the sides of the square. Find the area of the region.

15. Evaluate $\lim_{n \to \infty} \left(\frac{1}{\sqrt{n}\sqrt{n+1}} + \frac{1}{\sqrt{n}\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{n}\sqrt{n+n}} \right)$.

16. For any number c, we let $f_c(x)$ be the smaller of the two numbers $(x - c)^2$ and $(x - c - 2)^2$. Then we define $g(c) = \int_0^1 f_c(x)\,dx$. Find the maximum and minimum values of $g(c)$ if $-2 \leqslant c \leqslant 2$.