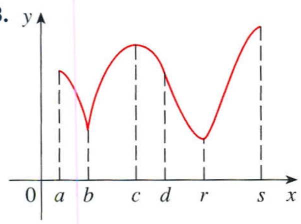
4.1 EXERCISES

1. Explain the difference between an absolute minimum and a local minimum.

2. Suppose f is a continuous function defined on a closed interval $[a, b]$.
(a) What theorem guarantees the existence of an absolute maximum value and an absolute minimum value for f?
(b) What steps would you take to find those maximum and minimum values?

3–4 For each of the numbers a, b, c, d, r, and s, state whether the function whose graph is shown has an absolute maximum or minimum, a local maximum or minimum, or neither a maximum nor a minimum.

3.



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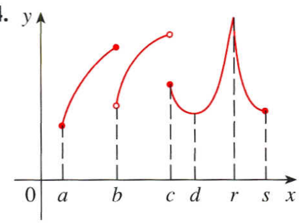
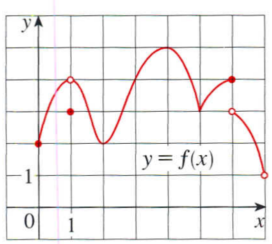


5–6 Use the graph to state the absolute and local maximum and minimum values of the function.

5.



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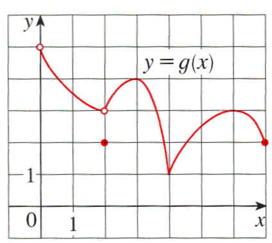


7–10 Sketch the graph of a function f that is continuous on $[1, 5]$ and has the given properties.

7. Absolute minimum at 2, absolute maximum at 3, local minimum at 4

8. Absolute minimum at 1, absolute maximum at 5, local maximum at 2, local minimum at 4

9. Absolute maximum at 5, absolute minimum at 2, local maximum at 3, local minima at 2 and 4

10. f has no local maximum or minimum, but 2 and 4 are critical numbers

11. (a) Sketch the graph of a function that has a local maximum at 2 and is differentiable at 2.
(b) Sketch the graph of a function that has a local maximum at 2 and is continuous but not differentiable at 2.
(c) Sketch the graph of a function that has a local maximum at 2 and is not continuous at 2.

12. (a) Sketch the graph of a function on $[-1, 2]$ that has an absolute maximum but no local maximum.
(b) Sketch the graph of a function on $[-1, 2]$ that has a local maximum but no absolute maximum.

13. (a) Sketch the graph of a function on $[-1, 2]$ that has an absolute maximum but no absolute minimum.
(b) Sketch the graph of a function on $[-1, 2]$ that is discontinuous but has both an absolute maximum and an absolute minimum.

14. (a) Sketch the graph of a function that has two local maxima, one local minimum, and no absolute minimum.
(b) Sketch the graph of a function that has three local minima, two local maxima, and seven critical numbers.

15–28 Sketch the graph of f by hand and use your sketch to find the absolute and local maximum and minimum values of f. (Use the graphs and transformations of Sections 1.2 and 1.3.)

15. $f(x) = 8 - 3x, \quad x \geqslant 1$

16. $f(x) = 3 - 2x, \quad x \leqslant 5$

17. $f(x) = x^2, \quad 0 < x < 2$

18. $f(x) = x^2, \quad 0 < x \leqslant 2$

19. $f(x) = x^2, \quad 0 \leqslant x < 2$

20. $f(x) = x^2, \quad 0 \leqslant x \leqslant 2$

21. $f(x) = x^2, \quad -3 \leqslant x \leqslant 2$

22. $f(x) = 1 + (x + 1)^2, \quad -2 \leqslant x < 5$

23. $f(t) = 1/t, \quad 0 < t < 1$

24. $f(\theta) = \tan \theta, \quad -\pi/4 \leqslant \theta < \pi/2$

25. $f(x) = 1 - \sqrt{x}$

26. $f(x) = 1 - x^3$

27. $f(x) = \begin{cases} 1 - x & \text{if } 0 \leqslant x < 2 \\ 2x - 4 & \text{if } 2 \leqslant x \leqslant 3 \end{cases}$

28. $f(x) = \begin{cases} 4 - x^2 & \text{if } -2 \leqslant x < 0 \\ 2x - 1 & \text{if } 0 \leqslant x \leqslant 2 \end{cases}$

29–42 Find the critical numbers of the function.

29. $f(x) = 5x^2 + 4x$

30. $f(x) = x^3 + x^2 - x$

31. $f(x) = x^3 + 3x^2 - 24x$

32. $f(x) = x^3 + x^2 + x$

33. $s(t) = 3t^4 + 4t^3 - 6t^2$

34. $g(t) = |3t - 4|$

35. $g(y) = \dfrac{y - 1}{y^2 - y + 1}$

36. $h(p) = \dfrac{p - 1}{p^2 + 4}$

37. $h(t) = t^{3/4} - 2t^{1/4}$

38. $g(x) = \sqrt{1 - x^2}$

39. $F(x) = x^{4/5}(x - 4)^2$

40. $g(x) = x^{1/3} - x^{-2/3}$

41. $f(\theta) = 2\cos\theta + \sin^2\theta$

42. $g(\theta) = 4\theta - \tan\theta$

43–44 A formula for the *derivative* of a function f is given. How many critical numbers does f have?

43. $f'(x) = 1 + \dfrac{210 \sin x}{x^2 - 6x + 10}$

44. $f'(x) = \dfrac{100 \cos^2 x}{10 + x^2} - 1$

45–56 Find the absolute maximum and absolute minimum values of f on the given interval.

45. $f(x) = 3x^2 - 12x + 5, \quad [0, 3]$

46. $f(x) = x^3 - 3x + 1, \quad [0, 3]$

47. $f(x) = 2x^3 - 3x^2 - 12x + 1, \quad [-2, 3]$

48. $f(x) = 18x + 15x^2 - 4x^3, \quad [-3, 4]$

49. $f(x) = x^4 - 4x^2 + 2, \quad [-3, 2]$

50. $f(x) = (x^2 - 1)^3, \quad [-1, 2]$

51. $f(x) = \dfrac{x}{x^2 + 1}, \quad [0, 2]$

52. $f(x) = \dfrac{x^2 - 4}{x^2 + 4}, \quad [-4, 4]$

53. $f(t) = t\sqrt{4 - t^2}, \quad [-1, 2]$

54. $f(t) = \sqrt[3]{t}\,(8 - t), \quad [0, 8]$

55. $f(t) = 2\cos t + \sin 2t, \quad [0, \pi/2]$

56. $f(t) = t + \cot(t/2), \quad [\pi/4, 7\pi/4]$

57. If a and b are positive numbers, find the maximum value of $f(x) = x^a(1 - x)^b$, $0 \leqslant x \leqslant 1$.

58. Use a graph to estimate the critical numbers of $f(x) = |x^3 - 3x^2 + 2|$ correct to one decimal place.

59–62

(a) Use a graph to estimate the absolute maximum and minimum values of the function to two decimal places.

(b) Use calculus to find the exact maximum and minimum values.

59. $f(x) = x^5 - x^3 + 2, \quad -1 \leqslant x \leqslant 1$

60. $f(x) = x^4 - 3x^3 + 3x^2 - x, \quad 0 \leqslant x \leqslant 2$

61. $f(x) = x\sqrt{x - x^2}$

62. $f(x) = x - 2\cos x, \quad -2 \leqslant x \leqslant 0$

63. Between 0°C and 30°C, the volume V (in cubic centimeters) of 1 kg of water at a temperature T is given approximately by the formula

$$V = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3$$

Find the temperature at which water has its maximum density.

64. An object with mass m is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is

$$F = \frac{\mu m g}{\mu \sin\theta + \cos\theta}$$

where μ is a positive constant called the *coefficient of friction* and where $0 \leqslant \theta \leqslant \pi/2$. Show that F is minimized when $\tan\theta = \mu$.

65. A model for the US average price of a pound of white sugar from 1993 to 2003 is given by the function

$$S(t) = -0.00003237t^5 + 0.0009037t^4 - 0.008956t^3 + 0.03629t^2 - 0.04458t + 0.4074$$

where t is measured in years since August of 1993. Estimate the times when sugar was cheapest and most expensive during the period 1993–2003.

66. On May 7, 1992, the space shuttle *Endeavour* was launched on mission STS-49, the purpose of which was to install a new perigee kick motor in an Intelsat communications satellite. The table gives the velocity data for the shuttle between liftoff and the jettisoning of the solid rocket boosters.

Event	Time (s)	Velocity (m/s)
Launch	0	0
Begin roll maneuver	10	56.4
End roll maneuver	15	97.2
Throttle to 89%	20	136.2
Throttle to 67%	32	226.2
Throttle to 104%	59	403.9
Maximum dynamic pressure	62	440.4
Solid rocket booster separation	125	1265.2

(a) Use a graphing calculator or computer to find the cubic polynomial that best models the velocity of the shuttle for the time interval $t \in [0, 125]$. Then graph this polynomial.

(b) Find a model for the acceleration of the shuttle and use it to estimate the maximum and minimum values of the acceleration during the first 125 seconds.

67. When a foreign object lodged in the trachea (windpipe) forces a person to cough, the diaphragm thrusts upward causing an increase in pressure in the lungs. This is accompanied by a contraction of the trachea, making a narrower channel for the expelled air to flow through. For a given amount of air to escape in a fixed time, it must move faster through the narrower channel than the wider one. The greater the velocity of the airstream, the greater the force on the foreign object. X rays show that the radius of the circular tracheal tube contracts to about two-thirds of its normal radius during a cough. According to a mathematical model of coughing, the velocity v of the airstream is related to the radius r of the trachea by the equation

$$v(r) = k(r_0 - r)r^2 \qquad \tfrac{1}{2}r_0 \leqslant r \leqslant r_0$$

where k is a constant and r_0 is the normal radius of the trachea. The restriction on r is due to the fact that the tracheal wall stiffens under pressure and a contraction greater than $\frac{1}{2}r_0$ is prevented (otherwise the person would suffocate).

(a) Determine the value of r in the interval $\left[\frac{1}{2}r_0, r_0\right]$ at which v has an absolute maximum. How does this compare with experimental evidence?

(b) What is the absolute maximum value of v on the interval?

(c) Sketch the graph of v on the interval $[0, r_0]$.

68. Show that 5 is a critical number of the function

$$g(x) = 2 + (x - 5)^3$$

but g does not have a local extreme value at 5.

69. Prove that the function

$$f(x) = x^{101} + x^{51} + x + 1$$

has neither a local maximum nor a local minimum.

70. If f has a minimum value at c, show that the function $g(x) = -f(x)$ has a maximum value at c.

71. Prove Fermat's Theorem for the case in which f has a local minimum at c.

72. A cubic function is a polynomial of degree 3; that is, it has the form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$.

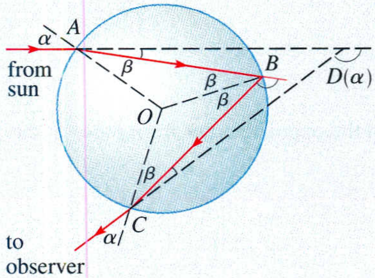
(a) Show that a cubic function can have two, one, or no critical number(s). Give examples and sketches to illustrate the three possibilities.

(b) How many local extreme values can a cubic function have?

APPLIED PROJECT

THE CALCULUS OF RAINBOWS

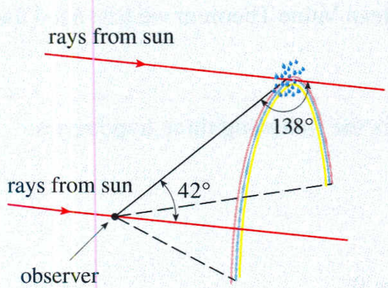
Rainbows are created when raindrops scatter sunlight. They have fascinated mankind since ancient times and have inspired attempts at scientific explanation since the time of Aristotle. In this project we use the ideas of Descartes and Newton to explain the shape, location, and colors of rainbows.



Formation of the primary rainbow

1. The figure shows a ray of sunlight entering a spherical raindrop at A. Some of the light is reflected, but the line AB shows the path of the part that enters the drop. Notice that the light is refracted toward the normal line AO and in fact Snell's Law says that $\sin\alpha = k\sin\beta$, where α is the angle of incidence, β is the angle of refraction, and $k \approx \frac{4}{3}$ is the index of refraction for water. At B some of the light passes through the drop and is refracted into the air, but the line BC shows the part that is reflected. (The angle of incidence equals the angle of reflection.) When the ray reaches C, part of it is reflected, but for the time being we are more interested in the part that leaves the raindrop at C. (Notice that it is refracted away from the normal line.) The *angle of deviation* $D(\alpha)$ is the amount of clockwise rotation that the ray has undergone during this three-stage process. Thus

$$D(\alpha) = (\alpha - \beta) + (\pi - 2\beta) + (\alpha - \beta) = \pi + 2\alpha - 4\beta$$



Show that the minimum value of the deviation is $D(\alpha) \approx 138°$ and occurs when $\alpha \approx 59.4°$.

The significance of the minimum deviation is that when $\alpha \approx 59.4°$ we have $D'(\alpha) \approx 0$, so $\Delta D/\Delta\alpha \approx 0$. This means that many rays with $\alpha \approx 59.4°$ become deviated by approximately the same amount. It is the *concentration* of rays coming from near the direction of minimum deviation that creates the brightness of the primary rainbow. The figure at the left shows that the angle of elevation from the observer up to the highest point on the rainbow is $180° - 138° = 42°$. (This angle is called the *rainbow angle*.)

2. Problem 1 explains the location of the primary rainbow, but how do we explain the colors? Sunlight comprises a range of wavelengths, from the red range through orange, yellow,

PROOF Let $F(x) = f(x) - g(x)$. Then

$$F'(x) = f'(x) - g'(x) = 0$$

for all x in (a, b). Thus, by Theorem 5, F is constant; that is, $f - g$ is constant. □

NOTE Care must be taken in applying Theorem 5. Let

$$f(x) = \frac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

The domain of f is $D = \{x \mid x \neq 0\}$ and $f'(x) = 0$ for all x in D. But f is obviously not a constant function. This does not contradict Theorem 5 because D is not an interval. Notice that f is constant on the interval $(0, \infty)$ and also on the interval $(-\infty, 0)$.

4.2 EXERCISES

1–4 Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

1. $f(x) = x^2 - 4x + 1, \quad [0, 4]$

2. $f(x) = x^3 - 3x^2 + 2x + 5, \quad [0, 2]$

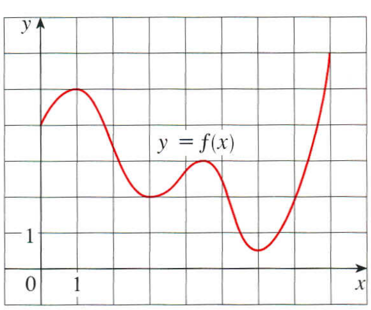
3. $f(x) = \sqrt{x} - \frac{1}{3}x, \quad [0, 9]$

4. $f(x) = \cos 2x, \quad [\pi/8, 7\pi/8]$

5. Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but there is no number c in $(-1, 1)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

6. Let $f(x) = \tan x$. Show that $f(0) = f(\pi)$ but there is no number c in $(0, \pi)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

7. Use the graph of f to estimate the values of c that satisfy the conclusion of the Mean Value Theorem for the interval $[0, 8]$.



8. Use the graph of f given in Exercise 7 to estimate the values of c that satisfy the conclusion of the Mean Value Theorem for the interval $[1, 7]$.

9. (a) Graph the function $f(x) = x + 4/x$ in the viewing rectangle $[0, 10]$ by $[0, 10]$.
(b) Graph the secant line that passes through the points $(1, 5)$ and $(8, 8.5)$ on the same screen with f.
(c) Find the number c that satisfies the conclusion of the Mean Value Theorem for this function f and the interval $[1, 8]$. Then graph the tangent line at the point $(c, f(c))$ and notice that it is parallel to the secant line.

10. (a) In the viewing rectangle $[-3, 3]$ by $[-5, 5]$, graph the function $f(x) = x^3 - 2x$ and its secant line through the points $(-2, -4)$ and $(2, 4)$. Use the graph to estimate the x-coordinates of the points where the tangent line is parallel to the secant line.
(b) Find the exact values of the numbers c that satisfy the conclusion of the Mean Value Theorem for the interval $[-2, 2]$ and compare with your answers to part (a).

11–14 Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

11. $f(x) = 3x^2 + 2x + 5, \quad [-1, 1]$

12. $f(x) = x^3 + x - 1, \quad [0, 2]$

13. $f(x) = \sqrt[3]{x}, \quad [0, 1]$

14. $f(x) = \dfrac{x}{x + 2}, \quad [1, 4]$

15. Let $f(x) = (x - 3)^{-2}$. Show that there is no value of c in $(1, 4)$ such that $f(4) - f(1) = f'(c)(4 - 1)$. Why does this not contradict the Mean Value Theorem?

16. Let $f(x) = 2 - |2x - 1|$. Show that there is no value of c such that $f(3) - f(0) = f'(c)(3 - 0)$. Why does this not contradict the Mean Value Theorem?

17. Show that the equation $1 + 2x + x^3 + 4x^5 = 0$ has exactly one real root.

18. Show that the equation $2x - 1 - \sin x = 0$ has exactly one real root.

19. Show that the equation $x^3 - 15x + c = 0$ has at most one root in the interval $[-2, 2]$.

20. Show that the equation $x^4 + 4x + c = 0$ has at most two real roots.

21. (a) Show that a polynomial of degree 3 has at most three real roots.
(b) Show that a polynomial of degree n has at most n real roots.

22. (a) Suppose that f is differentiable on $\mathbb{R}$ and has two roots. Show that f' has at least one root.
(b) Suppose f is twice differentiable on $\mathbb{R}$ and has three roots. Show that f'' has at least one real root.
(c) Can you generalize parts (a) and (b)?

23. If $f(1) = 10$ and $f'(x) \geqslant 2$ for $1 \leqslant x \leqslant 4$, how small can $f(4)$ possibly be?

24. Suppose that $3 \leqslant f'(x) \leqslant 5$ for all values of x. Show that $18 \leqslant f(8) - f(2) \leqslant 30$.

25. Does there exist a function f such that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leqslant 2$ for all x?

26. Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b). Suppose also that $f(a) = g(a)$ and $f'(x) < g'(x)$ for $a < x < b$. Prove that $f(b) < g(b)$. [*Hint*: Apply the Mean Value Theorem to the function $h = f - g$.]

27. Show that $\sqrt{1 + x} < 1 + \frac{1}{2}x$ if $x > 0$.

28. Suppose f is an odd function and is differentiable everywhere. Prove that for every positive number b, there exists a number c in $(-b, b)$ such that $f'(c) = f(b)/b$.

29. Use the Mean Value Theorem to prove the inequality

$$|\sin a - \sin b| \leqslant |a - b| \qquad \text{for all } a \text{ and } b$$

30. If $f'(x) = c$ (c a constant) for all x, use Corollary 7 to show that $f(x) = cx + d$ for some constant d.

31. Let $f(x) = 1/x$ and

$$g(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\[2mm] 1 + \dfrac{1}{x} & \text{if } x < 0 \end{cases}$$

Show that $f'(x) = g'(x)$ for all x in their domains. Can we conclude from Corollary 7 that $f - g$ is constant?

32. At 2:00 PM a car's speedometer reads 50 km/h. At 2:10 PM it reads 65 km/h. Show that at some time between 2:00 and 2:10 the acceleration is exactly 90 km/h^2.

33. Two runners start a race at the same time and finish in a tie. Prove that at some time during the race they have the same speed. [*Hint:* Consider $f(t) = g(t) - h(t)$, where g and h are the position functions of the two runners.]

34. A number a is called a **fixed point** of a function f if $f(a) = a$. Prove that if $f'(x) \neq 1$ for all real numbers x, then f has at most one fixed point.

4.3 HOW DERIVATIVES AFFECT THE SHAPE OF A GRAPH

Many of the applications of calculus depend on our ability to deduce facts about a function f from information concerning its derivatives. Because $f'(x)$ represents the slope of the curve $y = f(x)$ at the point $(x, f(x))$, it tells us the direction in which the curve proceeds at each point. So it is reasonable to expect that information about $f'(x)$ will provide us with information about $f(x)$.

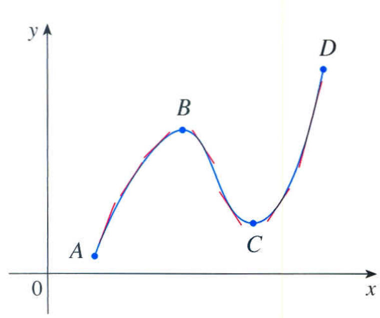


FIGURE 1

WHAT DOES f' SAY ABOUT f?

To see how the derivative of f can tell us where a function is increasing or decreasing, look at Figure 1. (Increasing functions and decreasing functions were defined in Section 1.1.) Between A and B and between C and D, the tangent lines have positive slope and so $f'(x) > 0$. Between B and C, the tangent lines have negative slope and so $f'(x) < 0$. Thus it appears that f increases when $f'(x)$ is positive and decreases when $f'(x)$ is negative. To prove that this is always the case, we use the Mean Value Theorem.

TEC In Module 4.3 you can practice using graphical information about f' to determine the shape of the graph of f.

at 6, so there is no minimum or maximum there. (The Second Derivative Test could be used at 4, but not at 0 or 6 since f'' does not exist at either of these numbers.)

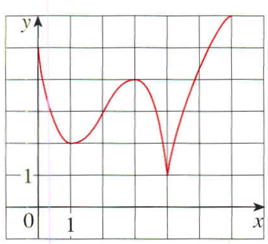
Looking at the expression for $f''(x)$ and noting that $x^{4/3} \geqslant 0$ for all x, we have $f''(x) < 0$ for $x < 0$ and for $0 < x < 6$ and $f''(x) > 0$ for $x > 6$. So f is concave downward on $(-\infty, 0)$ and $(0, 6)$ and concave upward on $(6, \infty)$, and the only inflection point is $(6, 0)$. The graph is sketched in Figure 12. Note that the curve has vertical tangents at $(0, 0)$ and $(6, 0)$ because $|f'(x)| \to \infty$ as $x \to 0$ and as $x \to 6$. □

4.3 EXERCISES

1–2 Use the given graph of f to find the following.

(a) The open intervals on which f is increasing.
(b) The open intervals on which f is decreasing.
(c) The open intervals on which f is concave upward.
(d) The open intervals on which f is concave downward.
(e) The coordinates of the points of inflection.

1.



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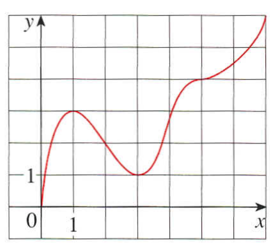


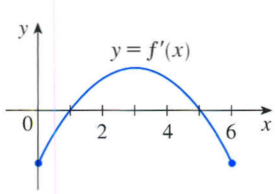
3. Suppose you are given a formula for a function f.
(a) How do you determine where f is increasing or decreasing?
(b) How do you determine where the graph of f is concave upward or concave downward?
(c) How do you locate inflection points?

4. (a) State the First Derivative Test.
(b) State the Second Derivative Test. Under what circumstances is it inconclusive? What do you do if it fails?

5–6 The graph of the *derivative* f' of a function f is shown.

(a) On what intervals is f increasing or decreasing?
(b) At what values of x does f have a local maximum or minimum?

5.



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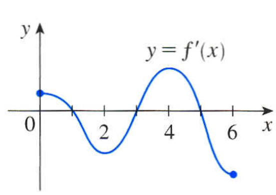
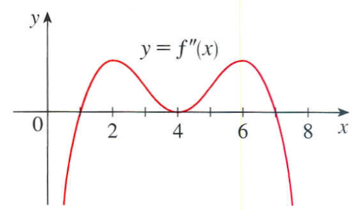
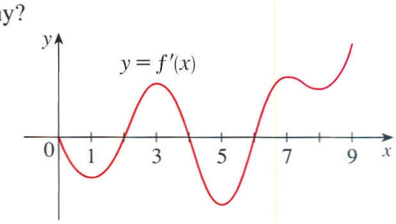


7. The graph of the second derivative f'' of a function f is shown. State the x-coordinates of the inflection points of f. Give reasons for your answers.



8. The graph of the first derivative f' of a function f is shown.
(a) On what intervals is f increasing? Explain.
(b) At what values of x does f have a local maximum or minimum? Explain.
(c) On what intervals is f concave upward or concave downward? Explain.
(d) What are the x-coordinates of the inflection points of f? Why?



9–14

(a) Find the intervals on which f is increasing or decreasing.
(b) Find the local maximum and minimum values of f.
(c) Find the intervals of concavity and the inflection points.

9. $f(x) = x^3 - 12x + 1$

10. $f(x) = 5 - 3x^2 + x^3$

11. $f(x) = x^4 - 2x^2 + 3$

12. $f(x) = \dfrac{x^2}{x^2 + 3}$

13. $f(x) = \sin x + \cos x, \quad 0 \leqslant x \leqslant 2\pi$

14. $f(x) = \cos^2 x - 2 \sin x, \quad 0 \leqslant x \leqslant 2\pi$

15–17 Find the local maximum and minimum values of f using both the First and Second Derivative Tests. Which method do you prefer?

15. $f(x) = x^5 - 5x + 3$

16. $f(x) = \dfrac{x}{x^2 + 4}$

17. $f(x) = x + \sqrt{1 - x}$

18. (a) Find the critical numbers of $f(x) = x^4(x - 1)^3$.
(b) What does the Second Derivative Test tell you about the behavior of f at these critical numbers?
(c) What does the First Derivative Test tell you?

19. Suppose f'' is continuous on $(-\infty, \infty)$.
(a) If $f'(2) = 0$ and $f''(2) = -5$, what can you say about f?
(b) If $f'(6) = 0$ and $f''(6) = 0$, what can you say about f?

20–25 Sketch the graph of a function that satisfies all of the given conditions.

20. $f'(x) > 0$ for all $x \neq 1$, vertical asymptote $x = 1$,
$f''(x) > 0$ if $x < 1$ or $x > 3$, $f''(x) < 0$ if $1 < x < 3$

21. $f'(0) = f'(2) = f'(4) = 0$,
$f'(x) > 0$ if $x < 0$ or $2 < x < 4$,
$f'(x) < 0$ if $0 < x < 2$ or $x > 4$,
$f''(x) > 0$ if $1 < x < 3$, $f''(x) < 0$ if $x < 1$ or $x > 3$

22. $f'(1) = f'(-1) = 0$, $f'(x) < 0$ if $|x| < 1$,
$f'(x) > 0$ if $1 < |x| < 2$, $f'(x) = -1$ if $|x| > 2$,
$f''(x) < 0$ if $-2 < x < 0$, inflection point $(0, 1)$

23. $f'(x) > 0$ if $|x| < 2$, $f'(x) < 0$ if $|x| > 2$,
$f'(-2) = 0$, $\lim_{x \to 2} |f'(x)| = \infty$, $f''(x) > 0$ if $x \neq 2$

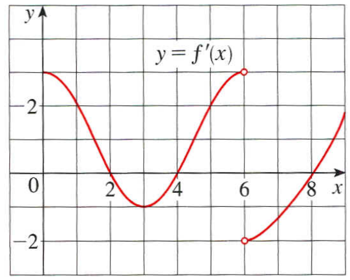
24. $f(0) = f'(0) = f'(2) = f'(4) = f'(6) = 0$,
$f'(x) > 0$ if $0 < x < 2$ or $4 < x < 6$,
$f'(x) < 0$ if $2 < x < 4$ or $x > 6$,
$f''(x) > 0$ if $0 < x < 1$ or $3 < x < 5$,
$f''(x) < 0$ if $1 < x < 3$ or $x > 5$, $f(-x) = f(x)$

25. $f'(x) < 0$ and $f''(x) < 0$ for all x

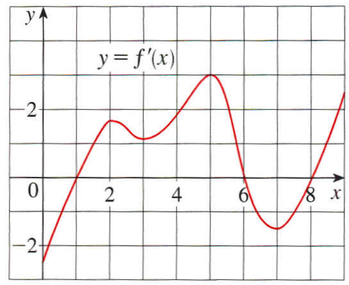
26. Suppose $f(3) = 2$, $f'(3) = \frac{1}{2}$, and $f'(x) > 0$ and $f''(x) < 0$ for all x.
(a) Sketch a possible graph for f.
(b) How many solutions does the equation $f(x) = 0$ have? Why?
(c) Is it possible that $f'(2) = \frac{1}{3}$? Why?

27–28 The graph of the derivative f' of a continuous function f is shown.
(a) On what intervals is f increasing or decreasing?
(b) At what values of x does f have a local maximum or minimum?
(c) On what intervals is f concave upward or downward?
(d) State the x-coordinate(s) of the point(s) of inflection.
(e) Assuming that $f(0) = 0$, sketch a graph of f.

27.



28.



29–40
(a) Find the intervals of increase or decrease.
(b) Find the local maximum and minimum values.
(c) Find the intervals of concavity and the inflection points.
(d) Use the information from parts (a)–(c) to sketch the graph. Check your work with a graphing device if you have one.

29. $f(x) = 2x^3 - 3x^2 - 12x$

30. $f(x) = 2 + 3x - x^3$

31. $f(x) = 2 + 2x^2 - x^4$

32. $g(x) = 200 + 8x^3 + x^4$

33. $h(x) = 3x^5 - 5x^3 + 3$

34. $h(x) = (x^2 - 1)^3$

35. $A(x) = x\sqrt{x + 3}$

36. $B(x) = 3x^{2/3} - x$

37. $C(x) = x^{1/3}(x + 4)$

38. $G(x) = x - 4\sqrt{x}$

39. $f(\theta) = 2\cos\theta + \cos^2\theta$, $0 \leqslant \theta \leqslant 2\pi$

40. $f(t) = t + \cos t$, $-2\pi \leqslant t \leqslant 2\pi$

41. Suppose the derivative of a function f is $f'(x) = (x + 1)^2(x - 3)^5(x - 6)^4$. On what interval is f increasing?

42. Use the methods of this section to sketch the curve $y = x^3 - 3a^2x + 2a^3$, where a is a positive constant. What do the members of this family of curves have in common? How do they differ from each other?

43–44
(a) Use a graph of f to estimate the maximum and minimum values. Then find the exact values.

(b) Estimate the value of x at which f increases most rapidly. Then find the exact value.

43. $f(x) = \dfrac{x+1}{\sqrt{x^2+1}}$

44. $f(x) = x + 2\cos x, \quad 0 \leqslant x \leqslant 2\pi$

45–46
(a) Use a graph of f to give a rough estimate of the intervals of concavity and the coordinates of the points of inflection.
(b) Use a graph of f'' to give better estimates.

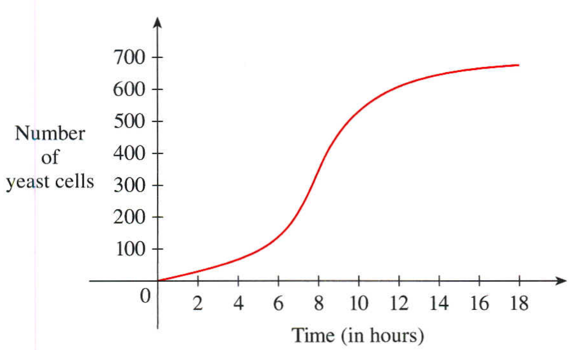
45. $f(x) = \cos x + \frac{1}{2}\cos 2x, \quad 0 \leqslant x \leqslant 2\pi$

46. $f(x) = x^3(x-2)^4$

CAS **47–48** Estimate the intervals of concavity to one decimal place by using a computer algebra system to compute and graph f''.

47. $f(x) = \dfrac{x^4 + x^3 + 1}{\sqrt{x^2 + x + 1}}$ **48.** $f(x) = \dfrac{(x+1)^3(x^2+5)}{(x^3+1)(x^2+4)}$

49. A graph of a population of yeast cells in a new laboratory culture as a function of time is shown.
(a) Describe how the rate of population increase varies.
(b) When is this rate highest?
(c) On what intervals is the population function concave upward or downward?
(d) Estimate the coordinates of the inflection point.



50. Let $f(t)$ be the temperature at time t where you live and suppose that at time $t = 3$ you feel uncomfortably hot. How do you feel about the given data in each case?
(a) $f'(3) = 2, \quad f''(3) = 4$ (b) $f'(3) = 2, \quad f''(3) = -4$
(c) $f'(3) = -2, \quad f''(3) = 4$ (d) $f'(3) = -2, \quad f''(3) = -4$

51. Let $K(t)$ be a measure of the knowledge you gain by studying for a test for t hours. Which do you think is larger, $K(8) - K(7)$ or $K(3) - K(2)$? Is the graph of K concave upward or concave downward? Why?

52. Coffee is being poured into the mug shown in the figure at a constant rate (measured in volume per unit time). Sketch a rough graph of the depth of the coffee in the mug as a function of time. Account for the shape of the graph in terms of concavity. What is the significance of the inflection point?

53. Find a cubic function $f(x) = ax^3 + bx^2 + cx + d$ that has a local maximum value of 3 at -2 and a local minimum value of 0 at 1.

54. Show that the curve $y = (1+x)/(1+x^2)$ has three points of inflection and they all lie on one straight line.

55. Suppose f is differentiable on an interval I and $f'(x) > 0$ for all numbers x in I except for a single number c. Prove that f is increasing on the entire interval I.

56–58 Assume that all of the functions are twice differentiable and the second derivatives are never 0.

56. (a) If f and g are concave upward on I, show that $f + g$ is concave upward on I.
(b) If f is positive and concave upward on I, show that the function $g(x) = [f(x)]^2$ is concave upward on I.

57. (a) If f and g are positive, increasing, concave upward functions on I, show that the product function fg is concave upward on I.
(b) Show that part (a) remains true if f and g are both decreasing.
(c) Suppose f is increasing and g is decreasing. Show, by giving three examples, that fg may be concave upward, concave downward, or linear. Why doesn't the argument in parts (a) and (b) work in this case?

58. Suppose f and g are both concave upward on $(-\infty, \infty)$. Under what condition on f will the composite function $h(x) = f(g(x))$ be concave upward?

59. Show that $\tan x > x$ for $0 < x < \pi/2$. [*Hint:* Show that $f(x) = \tan x - x$ is increasing on $(0, \pi/2)$.]

60. Prove that, for all $x > 1$,

$$2\sqrt{x} > 3 - \frac{1}{x}$$

61. Show that a cubic function (a third-degree polynomial) always has exactly one point of inflection. If its graph has three x-intercepts x_1, x_2, and x_3, show that the x-coordinate of the inflection point is $(x_1 + x_2 + x_3)/3$.

62. For what values of c does the polynomial $P(x) = x^4 + cx^3 + x^2$ have two inflection points? One inflection point? None? Illustrate by graphing P for several values of c. How does the graph change as c decreases?

63. Prove that if $(c, f(c))$ is a point of inflection of the graph of f and f'' exists in an open interval that contains c, then $f''(c) = 0$. [*Hint:* Apply the First Derivative Test and Fermat's Theorem to the function $g = f'$.]

64. Show that if $f(x) = x^4$, then $f''(0) = 0$, but $(0, 0)$ is not an inflection point of the graph of f.

65. Show that the function $g(x) = x|x|$ has an inflection point at $(0, 0)$ but $g''(0)$ does not exist.

66. Suppose that f''' is continuous and $f'(c) = f''(c) = 0$, but $f'''(c) > 0$. Does f have a local maximum or minimum at c? Does f have a point of inflection at c?

67. The three cases in the First Derivative Test cover the situations one commonly encounters but do not exhaust all possibilities. Consider the functions f, g, and h whose values at 0 are all 0 and, for $x \neq 0$,

$$f(x) = x^4 \sin\frac{1}{x} \qquad g(x) = x^4\left(2 + \sin\frac{1}{x}\right)$$

$$h(x) = x^4\left(-2 + \sin\frac{1}{x}\right)$$

(a) Show that 0 is a critical number of all three functions but their derivatives change sign infinitely often on both sides of 0.

(b) Show that f has neither a local maximum nor a local minimum at 0, g has a local minimum, and h has a local maximum.

4.4 LIMITS AT INFINITY; HORIZONTAL ASYMPTOTES

In Sections 2.2 and 2.4 we investigated infinite limits and vertical asymptotes. There we let x approach a number and the result was that the values of y became arbitrarily large (positive or negative). In this section we let x become arbitrarily large (positive or negative) and see what happens to y. We will find it very useful to consider this so-called *end behavior* when sketching graphs.

Let's begin by investigating the behavior of the function f defined by

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

as x becomes large. The table at the left gives values of this function correct to six decimal places, and the graph of f has been drawn by a computer in Figure 1.

x	$f(x)$
0	−1
±1	0
±2	0.600000
±3	0.800000
±4	0.882353
±5	0.923077
±10	0.980198
±50	0.999200
±100	0.999800
±1000	0.999998

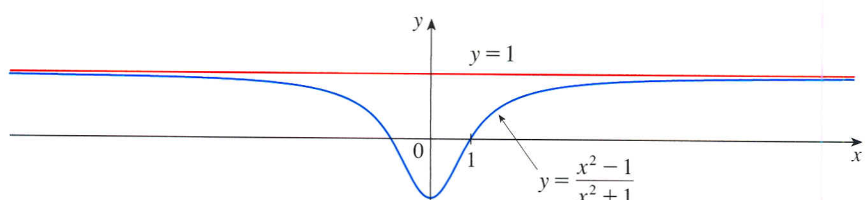


FIGURE 1

As x grows larger and larger you can see that the values of $f(x)$ get closer and closer to 1. In fact, it seems that we can make the values of $f(x)$ as close as we like to 1 by taking x sufficiently large. This situation is expressed symbolically by writing

$$\lim_{x\to\infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

In general, we use the notation

$$\lim_{x\to\infty} f(x) = L$$

to indicate that the values of $f(x)$ become closer and closer to L as x becomes larger and larger.

Finally we note that an infinite limit at infinity can be defined as follows. The geometric illustration is given in Figure 17.

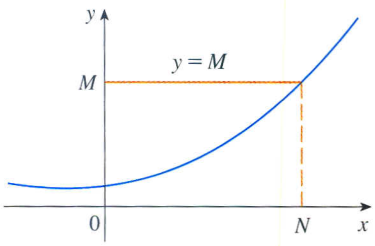


FIGURE 17
$\lim_{x \to \infty} f(x) = \infty$

7 DEFINITION Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = \infty$$

means that for every positive number M there is a corresponding positive number N such that

$$\text{if} \quad x > N \quad \text{then} \quad f(x) > M$$

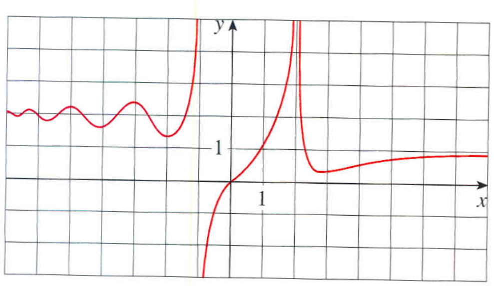
Similar definitions apply when the symbol ∞ is replaced by $-\infty$. (See Exercise 70.)

4.4 EXERCISES

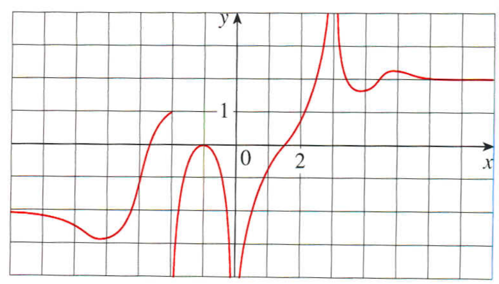
1. Explain in your own words the meaning of each of the following.
(a) $\lim_{x \to \infty} f(x) = 5$ (b) $\lim_{x \to -\infty} f(x) = 3$

2. (a) Can the graph of $y = f(x)$ intersect a vertical asymptote? Can it intersect a horizontal asymptote? Illustrate by sketching graphs.
(b) How many horizontal asymptotes can the graph of $y = f(x)$ have? Sketch graphs to illustrate the possibilities.

3. For the function f whose graph is given, state the following.
(a) $\lim_{x \to 2} f(x)$ (b) $\lim_{x \to -1^-} f(x)$
(c) $\lim_{x \to -1^+} f(x)$ (d) $\lim_{x \to \infty} f(x)$
(e) $\lim_{x \to -\infty} f(x)$ (f) The equations of the asymptotes



4. For the function g whose graph is given, state the following.
(a) $\lim_{x \to \infty} g(x)$ (b) $\lim_{x \to -\infty} g(x)$
(c) $\lim_{x \to 3} g(x)$ (d) $\lim_{x \to 0} g(x)$
(e) $\lim_{x \to -2^+} g(x)$ (f) The equations of the asymptotes



5. Guess the value of the limit

$$\lim_{x \to \infty} \frac{x^2}{2^x}$$

by evaluating the function $f(x) = x^2/2^x$ for $x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 50$, and 100. Then use a graph of f to support your guess.

6. (a) Use a graph of

$$f(x) = \left(1 - \frac{2}{x}\right)^x$$

to estimate the value of $\lim_{x \to \infty} f(x)$ correct to two decimal places.
(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

7–8 Evaluate the limit and justify each step by indicating the appropriate properties of limits.

7. $\lim_{x \to \infty} \dfrac{3x^2 - x + 4}{2x^2 + 5x - 8}$

8. $\lim_{x \to \infty} \sqrt{\dfrac{12x^3 - 5x + 2}{1 + 4x^2 + 3x^3}}$

9–30 Find the limit.

9. $\lim_{x\to\infty} \frac{1}{2x+3}$

10. $\lim_{x\to\infty} \frac{3x+5}{x-4}$

11. $\lim_{x\to-\infty} \frac{1-x-x^2}{2x^2-7}$

12. $\lim_{t\to-\infty} \frac{6t^2+5t}{(1-t)(2t-3)}$

13. $\lim_{x\to\infty} \frac{x^3+5x}{2x^3-x^2+4}$

14. $\lim_{t\to-\infty} \frac{t^2+2}{t^3+t^2-1}$

15. $\lim_{u\to\infty} \frac{4u^4+5}{(u^2-2)(2u^2-1)}$

16. $\lim_{x\to\infty} \frac{x+2}{\sqrt{9x^2+1}}$

17. $\lim_{x\to\infty} \frac{\sqrt{9x^6-x}}{x^3+1}$

18. $\lim_{x\to-\infty} \frac{\sqrt{9x^6-x}}{x^3+1}$

19. $\lim_{x\to\infty} \left(\sqrt{9x^2+x}-3x\right)$

20. $\lim_{x\to-\infty} \left(x+\sqrt{x^2+2x}\right)$

21. $\lim_{x\to\infty} \left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)$

22. $\lim_{x\to\infty} \cos x$

23. $\lim_{x\to\infty} \frac{x+x^3+x^5}{1-x^2+x^4}$

24. $\lim_{x\to\infty} \sqrt{x^2+1}$

25. $\lim_{x\to-\infty} (x^4+x^5)$

26. $\lim_{x\to\infty} \frac{x^3-2x+3}{5-2x^2}$

27. $\lim_{x\to\infty} \left(x-\sqrt{x}\right)$

28. $\lim_{x\to\infty} (x^2-x^4)$

29. $\lim_{x\to\infty} x\sin\frac{1}{x}$

30. $\lim_{x\to\infty} \sqrt{x}\sin\frac{1}{x}$

31. (a) Estimate the value of

$$\lim_{x\to-\infty} \left(\sqrt{x^2+x+1}+x\right)$$

by graphing the function $f(x)=\sqrt{x^2+x+1}+x$.

(b) Use a table of values of $f(x)$ to guess the value of the limit.

(c) Prove that your guess is correct.

32. (a) Use a graph of

$$f(x)=\sqrt{3x^2+8x+6}-\sqrt{3x^2+3x+1}$$

to estimate the value of $\lim_{x\to\infty} f(x)$ to one decimal place.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Find the exact value of the limit.

33–38 Find the horizontal and vertical asymptotes of each curve. If you have a graphing device, check your work by graphing the curve and estimating the asymptotes.

33. $y=\frac{x}{x+4}$

34. $y=\frac{x^2+4}{x^2-1}$

35. $y=\frac{2x^2+x-1}{x^2+x-2}$

36. $y=\frac{1+x^4}{x^2-x^4}$

37. $y=\frac{x^3-x}{x^2-6x+5}$

38. $F(x)=\frac{x-9}{\sqrt{4x^2+3x+2}}$

39. Estimate the horizontal asymptote of the function

$$f(x)=\frac{3x^3+500x^2}{x^3+500x^2+100x+2000}$$

by graphing f for $-10 \leqslant x \leqslant 10$. Then calculate the equation of the asymptote by evaluating the limit. How do you explain the discrepancy?

40. (a) Graph the function

$$f(x)=\frac{\sqrt{2x^2+1}}{3x-5}$$

How many horizontal and vertical asymptotes do you observe? Use the graph to estimate the values of the limits

$$\lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{3x-5} \quad\text{and}\quad \lim_{x\to-\infty}\frac{\sqrt{2x^2+1}}{3x-5}$$

(b) By calculating values of $f(x)$, give numerical estimates of the limits in part (a).

(c) Calculate the exact values of the limits in part (a). Did you get the same value or different values for these two limits? [In view of your answer to part (a), you might have to check your calculation for the second limit.]

41. Find a formula for a function f that satisfies the following conditions:

$$\lim_{x\to\pm\infty} f(x)=0, \quad \lim_{x\to 0} f(x)=-\infty, \quad f(2)=0,$$
$$\lim_{x\to 3^-} f(x)=\infty, \quad \lim_{x\to 3^+} f(x)=-\infty$$

42. Find a formula for a function that has vertical asymptotes $x=1$ and $x=3$ and horizontal asymptote $y=1$.

43–46 Find the horizontal asymptotes of the curve and use them, together with concavity and intervals of increase and decrease, to sketch the curve.

43. $y=\frac{1-x}{1+x}$

44. $y=\frac{1+2x^2}{1+x^2}$

45. $y=\frac{x}{x^2+1}$

46. $y=\frac{x}{\sqrt{x^2+1}}$

47–50 Find the limits as $x\to\infty$ and as $x\to-\infty$. Use this information, together with intercepts, to give a rough sketch of the graph as in Example 11.

47. $y=x^4-x^6$

48. $y=x^3(x+2)^2(x-1)$

49. $y = (3 - x)(1 + x)^2(1 - x)^4$

50. $y = x^2(x^2 - 1)^2(x + 2)$

51–54 Sketch the graph of a function that satisfies all of the given conditions.

51. $f'(2) = 0$, $f(2) = -1$, $f(0) = 0$, $f'(x) < 0$ if $0 < x < 2$, $f'(x) > 0$ if $x > 2$, $f''(x) < 0$ if $0 \leqslant x < 1$ or if $x > 4$, $f''(x) > 0$ if $1 < x < 4$, $\lim_{x\to\infty} f(x) = 1$, $f(-x) = f(x)$ for all x

52. $f'(2) = 0$, $f'(0) = 1$, $f'(x) > 0$ if $0 < x < 2$, $f'(x) < 0$ if $x > 2$, $f''(x) < 0$ if $0 < x < 4$, $f''(x) > 0$ if $x > 4$, $\lim_{x\to\infty} f(x) = 0$, $f(-x) = -f(x)$ for all x

53. $f(1) = f'(1) = 0$, $\lim_{x\to 2^+} f(x) = \infty$, $\lim_{x\to 2^-} f(x) = -\infty$, $\lim_{x\to 0} f(x) = -\infty$, $\lim_{x\to -\infty} f(x) = \infty$, $\lim_{x\to\infty} f(x) = 0$, $f''(x) > 0$ for $x > 2$, $f''(x) < 0$ for $x < 0$ and for $0 < x < 2$

54. $g(0) = 0$, $g''(x) < 0$ for $x \neq 0$, $\lim_{x\to -\infty} g(x) = \infty$, $\lim_{x\to\infty} g(x) = -\infty$, $\lim_{x\to 0^-} g'(x) = -\infty$, $\lim_{x\to 0^+} g'(x) = \infty$

55. (a) Use the Squeeze Theorem to evaluate $\lim_{x\to\infty} \dfrac{\sin x}{x}$.

(b) Graph $f(x) = (\sin x)/x$. How many times does the graph cross the asymptote?

56. By the *end behavior* of a function we mean the behavior of its values as $x \to \infty$ and as $x \to -\infty$.

(a) Describe and compare the end behavior of the functions

$$P(x) = 3x^5 - 5x^3 + 2x \qquad Q(x) = 3x^5$$

by graphing both functions in the viewing rectangles $[-2, 2]$ by $[-2, 2]$ and $[-10, 10]$ by $[-10{,}000, 10{,}000]$.

(b) Two functions are said to have the *same end behavior* if their ratio approaches 1 as $x \to \infty$. Show that P and Q have the same end behavior.

57. Let P and Q be polynomials. Find

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)}$$

if the degree of P is (a) less than the degree of Q and (b) greater than the degree of Q.

58. Make a rough sketch of the curve $y = x^n$ (n an integer) for the following five cases:

(i) $n = 0$ (ii) $n > 0$, n odd
(iii) $n > 0$, n even (iv) $n < 0$, n odd
(v) $n < 0$, n even

Then use these sketches to find the following limits.

(a) $\lim_{x\to 0^+} x^n$ (b) $\lim_{x\to 0^-} x^n$
(c) $\lim_{x\to\infty} x^n$ (d) $\lim_{x\to -\infty} x^n$

59. Find $\lim_{x\to\infty} f(x)$ if

$$\frac{4x - 1}{x} < f(x) < \frac{4x^2 + 3x}{x^2}$$

for all $x > 5$.

60. (a) A tank contains 5000 L of pure water. Brine that contains 30 g of salt per liter of water is pumped into the tank at a rate of 25 L/min. Show that the concentration of salt after t minutes (in grams per liter) is

$$C(t) = \frac{30t}{200 + t}$$

(b) What happens to the concentration as $t \to \infty$?

61. Use a graph to find a number N such that

$$\text{if} \quad x > N \quad \text{then} \quad \left| \frac{3x^2 + 1}{2x^2 + x + 1} - 1.5 \right| < 0.05$$

62. For the limit

$$\lim_{x\to\infty} \frac{\sqrt{4x^2 + 1}}{x + 1} = 2$$

illustrate Definition 5 by finding values of N that correspond to $\varepsilon = 0.5$ and $\varepsilon = 0.1$.

63. For the limit

$$\lim_{x\to -\infty} \frac{\sqrt{4x^2 + 1}}{x + 1} = -2$$

illustrate Definition 6 by finding values of N that correspond to $\varepsilon = 0.5$ and $\varepsilon = 0.1$.

64. For the limit

$$\lim_{x\to\infty} \frac{2x + 1}{\sqrt{x + 1}} = \infty$$

illustrate Definition 7 by finding a value of N that corresponds to $M = 100$.

65. (a) How large do we have to take x so that $1/x^2 < 0.0001$?

(b) Taking $r = 2$ in Theorem 4, we have the statement

$$\lim_{x\to\infty} \frac{1}{x^2} = 0$$

Prove this directly using Definition 5.

66. (a) How large do we have to take x so that $1/\sqrt{x} < 0.0001$?
(b) Taking $r = \frac{1}{2}$ in Theorem 4, we have the statement

$$\lim_{x \to \infty} \frac{1}{\sqrt{x}} = 0$$

Prove this directly using Definition 5.

67. Use Definition 6 to prove that $\lim_{x \to -\infty} \frac{1}{x} = 0$.

68. Prove, using Definition 7, that $\lim_{x \to \infty} x^3 = \infty$.

69. Prove that

$$\lim_{x \to \infty} f(x) = \lim_{t \to 0^+} f(1/t)$$

and $$\lim_{x \to -\infty} f(x) = \lim_{t \to 0^-} f(1/t)$$

if these limits exist.

70. Formulate a precise definition of

$$\lim_{x \to -\infty} f(x) = -\infty$$

Then use your definition to prove that

$$\lim_{x \to -\infty} (1 + x^3) = -\infty$$

4.5 SUMMARY OF CURVE SKETCHING

So far we have been concerned with some particular aspects of curve sketching: domain, range, and symmetry in Chapter 1; limits, continuity, and vertical asymptotes in Chapter 2; derivatives and tangents in Chapter 3; and extreme values, intervals of increase and decrease, concavity, points of inflection, and horizontal asymptotes in this chapter. It is now time to put all of this information together to sketch graphs that reveal the important features of functions.

You might ask: Why don't we just use a graphing calculator or computer to graph a curve? Why do we need to use calculus?

It's true that modern technology is capable of producing very accurate graphs. But even the best graphing devices have to be used intelligently. We saw in Section 1.4 that it is extremely important to choose an appropriate viewing rectangle to avoid getting a misleading graph. (See especially Examples 1, 3, 4, and 5 in that section.) The use of calculus enables us to discover the most interesting aspects of graphs and in many cases to calculate maximum and minimum points and inflection points *exactly* instead of approximately.

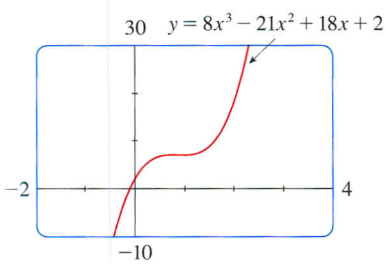


FIGURE 1

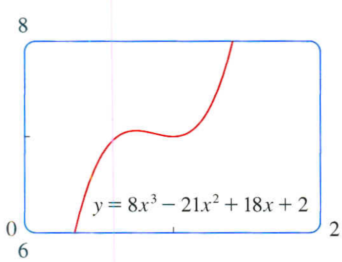


FIGURE 2

For instance, Figure 1 shows the graph of $f(x) = 8x^3 - 21x^2 + 18x + 2$. At first glance it seems reasonable: It has the same shape as cubic curves like $y = x^3$, and it appears to have no maximum or minimum point. But if you compute the derivative, you will see that there is a maximum when $x = 0.75$ and a minimum when $x = 1$. Indeed, if we zoom in to this portion of the graph, we see that behavior exhibited in Figure 2. Without calculus, we could easily have overlooked it.

In the next section we will graph functions by using the interaction between calculus and graphing devices. In this section we draw graphs by first considering the following information. We don't assume that you have a graphing device, but if you do have one you should use it as a check on your work.

GUIDELINES FOR SKETCHING A CURVE

The following checklist is intended as a guide to sketching a curve $y = f(x)$ by hand. Not every item is relevant to every function. (For instance, a given curve might not have an asymptote or possess symmetry.) But the guidelines provide all the information you need to make a sketch that displays the most important aspects of the function.

A. Domain It's often useful to start by determining the domain D of f, that is, the set of values of x for which $f(x)$ is defined.

D. Since $x^2 + 1$ is never 0, there is no vertical asymptote. Since $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$, there is no horizontal asymptote. But long division gives

$$f(x) = \frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}$$

$$f(x) - x = -\frac{x}{x^2+1} = -\frac{\dfrac{1}{x}}{1 + \dfrac{1}{x^2}} \to 0 \quad \text{as} \quad x \to \pm\infty$$

So the line $y = x$ is a slant asymptote.

E.
$$f'(x) = \frac{3x^2(x^2+1) - x^3 \cdot 2x}{(x^2+1)^2} = \frac{x^2(x^2+3)}{(x^2+1)^2}$$

Since $f'(x) > 0$ for all x (except 0), f is increasing on $(-\infty, \infty)$.

F. Although $f'(0) = 0$, f' does not change sign at 0, so there is no local maximum or minimum.

G.
$$f''(x) = \frac{(4x^3 + 6x)(x^2+1)^2 - (x^4 + 3x^2) \cdot 2(x^2+1)2x}{(x^2+1)^4} = \frac{2x(3 - x^2)}{(x^2+1)^3}$$

Since $f''(x) = 0$ when $x = 0$ or $x = \pm\sqrt{3}$, we set up the following chart:

Interval	x	$3 - x^2$	$(x^2+1)^3$	$f''(x)$	f
$x < -\sqrt{3}$	$-$	$-$	$+$	$+$	CU on $(-\infty, -\sqrt{3})$
$-\sqrt{3} < x < 0$	$-$	$+$	$+$	$-$	CD on $(-\sqrt{3}, 0)$
$0 < x < \sqrt{3}$	$+$	$+$	$+$	$+$	CU on $(0, \sqrt{3})$
$x > \sqrt{3}$	$+$	$-$	$+$	$-$	CD on $(\sqrt{3}, \infty)$

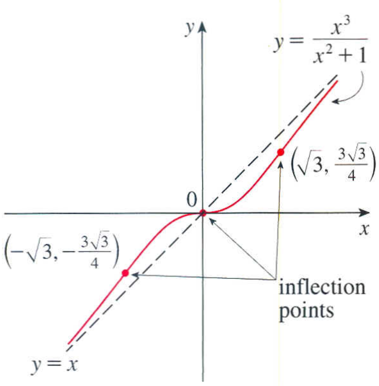


FIGURE 11

The points of inflection are $(-\sqrt{3}, -\frac{3}{4}\sqrt{3})$, $(0, 0)$, and $(\sqrt{3}, \frac{3}{4}\sqrt{3})$.

H. The graph of f is sketched in Figure 11. □

4.5 EXERCISES

1–38 Use the guidelines of this section to sketch the curve.

1. $y = x^3 + x$

2. $y = x^3 + 6x^2 + 9x$

3. $y = 2 - 15x + 9x^2 - x^3$

4. $y = 8x^2 - x^4$

5. $y = x^4 + 4x^3$

6. $y = x(x + 2)^3$

7. $y = 2x^5 - 5x^2 + 1$

8. $y = 20x^3 - 3x^5$

9. $y = \dfrac{x}{x - 1}$

10. $y = \dfrac{x}{(x - 1)^2}$

11. $y = \dfrac{1}{x^2 - 9}$

12. $y = \dfrac{x}{x^2 - 9}$

13. $y = \dfrac{x}{x^2 + 9}$

14. $y = \dfrac{x^2}{x^2 + 9}$

15. $y = \dfrac{x - 1}{x^2}$

16. $y = 1 + \dfrac{1}{x} + \dfrac{1}{x^2}$

17. $y = \dfrac{x^2}{x^2 + 3}$

18. $y = \dfrac{x}{x^3 - 1}$

19. $y = x\sqrt{5 - x}$

20. $y = \sqrt{x} - \sqrt{x - 1}$

21. $y = \sqrt{x^2 + x - 2}$

22. $y = \sqrt{x^2 + x} - x$

23. $y = \dfrac{x}{\sqrt{x^2 + 1}}$

24. $y = x\sqrt{2 - x^2}$

25. $y = \dfrac{\sqrt{1 - x^2}}{x}$

26. $y = \dfrac{x}{\sqrt{x^2 - 1}}$

27. $y = x + 3x^{2/3}$

28. $y = x^{5/3} - 5x^{2/3}$

29. $y = \sqrt[3]{x^2 - 1}$

30. $y = \sqrt[3]{x^3 + 1}$

31. $y = 3 \sin x - \sin^3 x$

32. $y = x + \cos x$

33. $y = x \tan x, \quad -\pi/2 < x < \pi/2$

34. $y = 2x - \tan x, \quad -\pi/2 < x < \pi/2$

35. $y = \frac{1}{2}x - \sin x, \quad 0 < x < 3\pi$

36. $y = \sec x + \tan x, \quad 0 < x < \pi/2$

37. $y = \dfrac{\sin x}{1 + \cos x}$

38. $y = \dfrac{\sin x}{2 + \cos x}$

39. In the theory of relativity, the mass of a particle is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle, m is the mass when the particle moves with speed v relative to the observer, and c is the speed of light. Sketch the graph of m as a function of v.

40. In the theory of relativity, the energy of a particle is

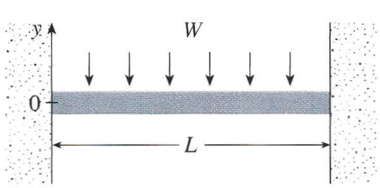
$$E = \sqrt{m_0^2 c^4 + h^2 c^2/\lambda^2}$$

where m_0 is the rest mass of the particle, λ is its wave length, and h is Planck's constant. Sketch the graph of E as a function of λ. What does the graph say about the energy?

41. The figure shows a beam of length L embedded in concrete walls. If a constant load W is distributed evenly along its length, the beam takes the shape of the deflection curve

$$y = -\frac{W}{24EI}x^4 + \frac{WL}{12EI}x^3 - \frac{WL^2}{24EI}x^2$$

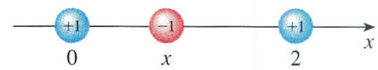
where E and I are positive constants. (E is Young's modulus of elasticity and I is the moment of inertia of a cross-section of the beam.) Sketch the graph of the deflection curve.



42. Coulomb's Law states that the force of attraction between two charged particles is directly proportional to the product of the charges and inversely proportional to the square of the distance between them. The figure shows particles with charge 1 located at positions 0 and 2 on a coordinate line and a particle with charge -1 at a position x between them. It follows from Coulomb's Law that the net force acting on the middle particle is

$$F(x) = -\frac{k}{x^2} + \frac{k}{(x - 2)^2} \qquad 0 < x < 2$$

where k is a positive constant. Sketch the graph of the net force function. What does the graph say about the force?



43–46 Find an equation of the slant asymptote. Do not sketch the curve.

43. $y = \dfrac{x^2 + 1}{x + 1}$

44. $y = \dfrac{2x^3 + x^2 + x + 3}{x^2 + 2x}$

45. $y = \dfrac{4x^3 - 2x^2 + 5}{2x^2 + x - 3}$

46. $y = \dfrac{5x^4 + x^2 + x}{x^3 - x^2 + 2}$

47–52 Use the guidelines of this section to sketch the curve. In guideline D find an equation of the slant asymptote.

47. $y = \dfrac{-2x^2 + 5x - 1}{2x - 1}$

48. $y = \dfrac{x^2 + 12}{x - 2}$

49. $xy = x^2 + 4$

50. $xy = x^2 + x + 1$

51. $y = \dfrac{2x^3 + x^2 + 1}{x^2 + 1}$

52. $y = \dfrac{(x + 1)^3}{(x - 1)^2}$

53. Show that the curve $y = \sqrt{4x^2 + 9}$ has two slant asymptotes: $y = 2x$ and $y = -2x$. Use this fact to help sketch the curve.

54. Show that the curve $y = \sqrt{x^2 + 4x}$ has two slant asymptotes: $y = x + 2$ and $y = -x - 2$. Use this fact to help sketch the curve.

55. Show that the lines $y = (b/a)x$ and $y = -(b/a)x$ are slant asymptotes of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$.

56. Let $f(x) = (x^3 + 1)/x$. Show that

$$\lim_{x \to \pm\infty} [f(x) - x^2] = 0$$

This shows that the graph of f approaches the graph of $y = x^2$, and we say that the curve $y = f(x)$ is *asymptotic* to the parabola $y = x^2$. Use this fact to help sketch the graph of f.

57. Discuss the asymptotic behavior of $f(x) = (x^4 + 1)/x$ in the same manner as in Exercise 56. Then use your results to help sketch the graph of f.

58. Use the asymptotic behavior of $f(x) = \cos x + 1/x^2$ to sketch its graph without going through the curve-sketching procedure of this section.

TEC See an animation of Figure 21 in Visual 4.4.

as $c \to -\infty$. Again, the maximum point approaches the x-axis because $1/(c-1) \to 0$ as $c \to -\infty$.

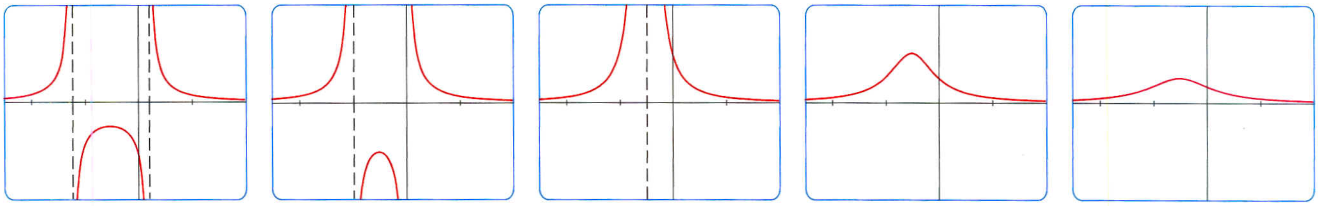


FIGURE 21 The family of functions $f(x) = 1/(x^2 + 2x + c)$

There is clearly no inflection point when $c \leqslant 1$. For $c > 1$ we calculate that

$$f''(x) = \frac{2(3x^2 + 6x + 4 - c)}{(x^2 + 2x + c)^3}$$

and deduce that inflection points occur when $x = -1 \pm \sqrt{3(c-1)}/3$. So the inflection points become more spread out as c increases and this seems plausible from the last two parts of Figure 21. □

4.6 EXERCISES

1–8 Produce graphs of f that reveal all the important aspects of the curve. In particular, you should use graphs of f' and f'' to estimate the intervals of increase and decrease, extreme values, intervals of concavity, and inflection points.

1. $f(x) = 4x^4 - 7x^2 + 4x + 6$

2. $f(x) = 8x^5 + 45x^4 + 80x^3 + 90x^2 + 200x$

3. $f(x) = x^6 - 10x^5 - 400x^4 + 2500x^3$

4. $f(x) = \dfrac{x^2 - 1}{40x^3 + x + 1}$

5. $f(x) = \dfrac{x}{x^3 - x^2 - 4x + 1}$

6. $f(x) = \tan x + 5 \cos x$

7. $f(x) = x^2 - 4x + 7 \cos x, \quad -4 \leqslant x \leqslant 4$

8. $f(x) = \dfrac{\sin x}{x}, \quad -2\pi \leqslant x \leqslant 2\pi$

9–10 Produce graphs of f that reveal all the important aspects of the curve. Estimate the intervals of increase and decrease and intervals of concavity, and use calculus to find these intervals exactly.

9. $f(x) = 1 + \dfrac{1}{x} + \dfrac{8}{x^2} + \dfrac{1}{x^3}$

10. $f(x) = \dfrac{1}{x^8} - \dfrac{2 \times 10^8}{x^4}$

11–12 Sketch the graph by hand using asymptotes and intercepts, but not derivatives. Then use your sketch as a guide to producing graphs (with a graphing device) that display the major features of the curve. Use these graphs to estimate the maximum and minimum values.

11. $f(x) = \dfrac{(x+4)(x-3)^2}{x^4(x-1)}$

12. $f(x) = \dfrac{(2x+3)^2(x-2)^5}{x^3(x-5)^2}$

CAS **13.** If f is the function considered in Example 3, use a computer algebra system to calculate f' and then graph it to confirm that all the maximum and minimum values are as given in the example. Calculate f'' and use it to estimate the intervals of concavity and inflection points.

CAS **14.** If f is the function of Exercise 12, find f' and f'' and use their graphs to estimate the intervals of increase and decrease and concavity of f.

CAS **15–18** Use a computer algebra system to graph f and to find f' and f''. Use graphs of these derivatives to estimate the intervals of increase and decrease, extreme values, intervals of concavity, and inflection points of f.

15. $f(x) = \dfrac{\sqrt{x}}{x^2 + x + 1}$

16. $f(x) = \dfrac{x^{2/3}}{1 + x + x^4}$

17. $f(x) = \sqrt{x + 5 \sin x}, \quad x \leqslant 20$

18. $f(x) = \dfrac{2x - 1}{\sqrt[4]{x^4 + x + 1}}$

19. In Example 4 we considered a member of the family of functions $f(x) = \sin(x + \sin cx)$ that occur in FM synthesis. Here we investigate the function with $c = 3$. Start by graphing f in the viewing rectangle $[0, \pi]$ by $[-1.2, 1.2]$. How many local maximum points do you see? The graph has more than are visible to the naked eye. To discover the hidden maximum and minimum points you will need to examine the graph of f' very carefully. In fact, it helps to look at the graph of f'' at the same time. Find all the maximum and minimum values and inflection points. Then graph f in the viewing rectangle $[-2\pi, 2\pi]$ by $[-1.2, 1.2]$ and comment on symmetry.

20–25 Describe how the graph of f varies as c varies. Graph several members of the family to illustrate the trends that you discover. In particular, you should investigate how maximum and minimum points and inflection points move when c changes. You should also identify any transitional values of c at which the basic shape of the curve changes.

20. $f(x) = x^3 + cx$

21. $f(x) = x^4 + cx^2$

22. $f(x) = x\sqrt{c^2 - x^2}$

23. $f(x) = \dfrac{cx}{1 + c^2x^2}$

24. $f(x) = \dfrac{1}{(1 - x^2)^2 + cx^2}$

25. $f(x) = cx + \sin x$

26. Investigate the family of curves given by the equation $f(x) = x^4 + cx^2 + x$. Start by determining the transitional value of c at which the number of inflection points changes. Then graph several members of the family to see what shapes are possible. There is another transitional value of c at which the number of critical numbers changes. Try to discover it graphically. Then prove what you have discovered.

27. (a) Investigate the family of polynomials given by the equation $f(x) = cx^4 - 2x^2 + 1$. For what values of c does the curve have minimum points?
(b) Show that the minimum and maximum points of every curve in the family lie on the parabola $y = 1 - x^2$. Illustrate by graphing this parabola and several members of the family.

28. (a) Investigate the family of polynomials given by the equation $f(x) = 2x^3 + cx^2 + 2x$. For what values of c does the curve have maximum and minimum points?
(b) Show that the minimum and maximum points of every curve in the family lie on the curve $y = x - x^3$. Illustrate by graphing this curve and several members of the family.

4.7 OPTIMIZATION PROBLEMS

The methods we have learned in this chapter for finding extreme values have practical applications in many areas of life. A businessperson wants to minimize costs and maximize profits. A traveler wants to minimize transportation time. Fermat's Principle in optics states that light follows the path that takes the least time. In this section and the next we solve such problems as maximizing areas, volumes, and profits and minimizing distances, times, and costs.

In solving such practical problems the greatest challenge is often to convert the word problem into a mathematical optimization problem by setting up the function that is to be maximized or minimized. Let's recall the problem-solving principles discussed on page 54 and adapt them to this situation:

STEPS IN SOLVING OPTIMIZATION PROBLEMS

1. **Understand the Problem** The first step is to read the problem carefully until it is clearly understood. Ask yourself: What is the unknown? What are the given quantities? What are the given conditions?
2. **Draw a Diagram** In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram.
3. **Introduce Notation** Assign a symbol to the quantity that is to be maximized or minimized (let's call it Q for now). Also select symbols $(a, b, c, \ldots, x, y)$ for other unknown quantities and label the diagram with these symbols. It may help to use initials as suggestive symbols—for example, A for area, h for height, t for time.

If x units are sold, then the total profit is

$$P(x) = R(x) - C(x)$$

and P is called the **profit function**. The **marginal profit function** is P', the derivative of the profit function. In Exercises 53–58 you are asked to use the marginal cost, revenue, and profit functions to minimize costs and maximize revenues and profits.

V EXAMPLE 6 A store has been selling 200 DVD burners a week at \$350 each. A market survey indicates that for each \$10 rebate offered to buyers, the number of units sold will increase by 20 a week. Find the demand function and the revenue function. How large a rebate should the store offer to maximize its revenue?

SOLUTION If x is the number of DVD burners sold per week, then the weekly increase in sales is $x - 200$. For each increase of 20 units sold, the price is decreased by \$10. So for each additional unit sold, the decrease in price will be $\frac{1}{20} \times 10$ and the demand function is

$$p(x) = 350 - \tfrac{10}{20}(x - 200) = 450 - \tfrac{1}{2}x$$

The revenue function is

$$R(x) = xp(x) = 450x - \tfrac{1}{2}x^2$$

Since $R'(x) = 450 - x$, we see that $R'(x) = 0$ when $x = 450$. This value of x gives an absolute maximum by the First Derivative Test (or simply by observing that the graph of R is a parabola that opens downward). The corresponding price is

$$p(450) = 450 - \tfrac{1}{2}(450) = 225$$

and the rebate is $350 - 225 = 125$. Therefore, to maximize revenue, the store should offer a rebate of \$125. □

4.7 EXERCISES

1. Consider the following problem: Find two numbers whose sum is 23 and whose product is a maximum.
   (a) Make a table of values, like the following one, so that the sum of the numbers in the first two columns is always 23. On the basis of the evidence in your table, estimate the answer to the problem.

First number	Second number	Product
1	22	22
2	21	42
3	20	60
⋮	⋮	⋮

   (b) Use calculus to solve the problem and compare with your answer to part (a).

2. Find two numbers whose difference is 100 and whose product is a minimum.

3. Find two positive numbers whose product is 100 and whose sum is a minimum.

4. Find a positive number such that the sum of the number and its reciprocal is as small as possible.

5. Find the dimensions of a rectangle with perimeter 100 m whose area is as large as possible.

6. Find the dimensions of a rectangle with area 1000 m^2 whose perimeter is as small as possible.

7. A model used for the yield Y of an agricultural crop as a function of the nitrogen level N in the soil (measured in appropriate units) is

$$Y = \frac{kN}{1 + N^2}$$

where k is a positive constant. What nitrogen level gives the best yield?

8. The rate (in mg carbon/m^3/h) at which photosynthesis takes place for a species of phytoplankton is modeled by the function

$$P = \frac{100I}{I^2 + I + 4}$$

where I is the light intensity (measured in thousands of foot-candles). For what light intensity is P a maximum?

9. Consider the following problem: A farmer with 300 m of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?
(a) Draw several diagrams illustrating the situation, some with shallow, wide pens and some with deep, narrow pens. Find the total areas of these configurations. Does it appear that there is a maximum area? If so, estimate it.
(b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.
(c) Write an expression for the total area.
(d) Use the given information to write an equation that relates the variables.
(e) Use part (d) to write the total area as a function of one variable.
(f) Finish solving the problem and compare the answer with your estimate in part (a).

10. Consider the following problem: A box with an open top is to be constructed from a square piece of cardboard, 3 m wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.
(a) Draw several diagrams to illustrate the situation, some short boxes with large bases and some tall boxes with small bases. Find the volumes of several such boxes. Does it appear that there is a maximum volume? If so, estimate it.
(b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.
(c) Write an expression for the volume.
(d) Use the given information to write an equation that relates the variables.
(e) Use part (d) to write the volume as a function of one variable.
(f) Finish solving the problem and compare the answer with your estimate in part (a).

11. A farmer wants to fence an area of 15,000 m^2 in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can he do this so as to minimize the cost of the fence?

12. A box with a square base and open top must have a volume of 32,000 cm^3. Find the dimensions of the box that minimize the amount of material used.

13. If 1200 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

14. A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest such container.

15. Do Exercise 14 assuming the container has a lid that is made from the same material as the sides.

16. (a) Show that of all the rectangles with a given area, the one with smallest perimeter is a square.
(b) Show that of all the rectangles with a given perimeter, the one with greatest area is a square.

17. Find the point on the line $y = 4x + 7$ that is closest to the origin.

18. Find the point on the line $6x + y = 9$ that is closest to the point $(-3, 1)$.

19. Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point $(1, 0)$.

20. Find, correct to two decimal places, the coordinates of the point on the curve $y = \tan x$ that is closest to the point $(1, 1)$.

21. Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius r.

22. Find the area of the largest rectangle that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$.

23. Find the dimensions of the rectangle of largest area that can be inscribed in an equilateral triangle of side L if one side of the rectangle lies on the base of the triangle.

24. Find the dimensions of the rectangle of largest area that has its base on the x-axis and its other two vertices above the x-axis and lying on the parabola $y = 8 - x^2$.

25. Find the dimensions of the isosceles triangle of largest area that can be inscribed in a circle of radius r.

26. Find the area of the largest rectangle that can be inscribed in a right triangle with legs of lengths 3 cm and 4 cm if two sides of the rectangle lie along the legs.

27. A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible volume of such a cylinder.

28. A right circular cylinder is inscribed in a cone with height h and base radius r. Find the largest possible volume of such a cylinder.

29. A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible surface area of such a cylinder.

30. A Norman window has the shape of a rectangle surmounted by a semicircle. (Thus the diameter of the semicircle is equal to the width of the rectangle. See Exercise 56 on page 23.) If the perimeter of the window is 10 m, find the dimensions of the window so that the greatest possible amount of light is admitted.

31. The top and bottom margins of a poster are each 6 cm and the side margins are each 4 cm. If the area of printed material on the poster is fixed at 384 cm^2, find the dimensions of the poster with the smallest area.

32. A poster is to have an area of 900 cm^2 with 3-cm margins at the bottom and sides and a 5-cm margin at the top. What dimensions will give the largest printed area?

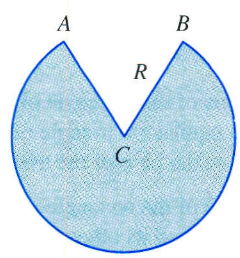
33. A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is (a) a maximum? (b) A minimum?

34. Answer Exercise 33 if one piece is bent into a square and the other into a circle.

35. A cylindrical can without a top is made to contain V cm^3 of liquid. Find the dimensions that will minimize the cost of the metal to make the can.

36. A fence 2 m tall runs parallel to a tall building at a distance of 1 m from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?

37. A cone-shaped drinking cup is made from a circular piece of paper of radius R by cutting out a sector and joining the edges CA and CB. Find the maximum capacity of such a cup.



38. A cone-shaped paper drinking cup is to be made to hold 27 cm^3 of water. Find the height and radius of the cup that will use the smallest amount of paper.

39. A cone with height h is inscribed in a larger cone with height H so that its vertex is at the center of the base of the larger cone. Show that the inner cone has maximum volume when $h = \frac{1}{3}H$.

40. An object with mass m is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with a plane, then the magnitude of the force is

$$F = \frac{\mu m g}{\mu \sin \theta + \cos \theta}$$

where μ is a constant called the coefficient of friction. For what value of θ is F smallest?

41. If a resistor of R ohms is connected across a battery of E volts with internal resistance r ohms, then the power (in watts) in the external resistor is

$$P = \frac{E^2 R}{(R + r)^2}$$

If E and r are fixed but R varies, what is the maximum value of the power?

42. For a fish swimming at a speed v relative to the water, the energy expenditure per unit time is proportional to v^3. It is believed that migrating fish try to minimize the total energy required to swim a fixed distance. If the fish are swimming against a current u ($u < v$), then the time required to swim a distance L is $L/(v - u)$ and the total energy E required to swim the distance is given by

$$E(v) = av^3 \cdot \frac{L}{v - u}$$

where a is the proportionality constant.
(a) Determine the value of v that minimizes E.
(b) Sketch the graph of E.

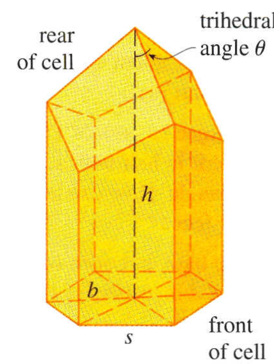
Note: This result has been verified experimentally; migrating fish swim against a current at a speed 50% greater than the current speed.

43. In a beehive, each cell is a regular hexagonal prism, open at one end with a trihedral angle at the other end as in the figure. It is believed that bees form their cells in such a way as to minimize the surface area for a given volume, thus using the least amount of wax in cell construction. Examination of these cells has shown that the measure of the apex angle θ is amazingly consistent. Based on the geometry of the cell, it can be shown that the surface area S is given by

$$S = 6sh - \tfrac{3}{2}s^2 \cot \theta + \left(3s^2\sqrt{3}/2\right) \csc \theta$$

where s, the length of the sides of the hexagon, and h, the height, are constants.
(a) Calculate $dS/d\theta$.
(b) What angle should the bees prefer?
(c) Determine the minimum surface area of the cell (in terms of s and h).

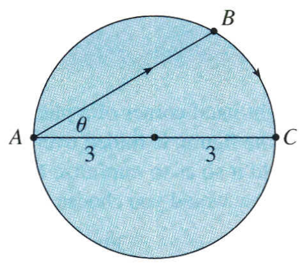
Note: Actual measurements of the angle θ in beehives have been made, and the measures of these angles seldom differ from the calculated value by more than 2°.



44. A boat leaves a dock at 2:00 PM and travels due south at a speed of 20 km/h. Another boat has been heading due east at 15 km/h and reaches the same dock at 3:00 PM. At what time were the two boats closest together?

45. Solve the problem in Example 4 if the river is 5 km wide and point B is only 5 km downstream from A.

46. A woman at a point A on the shore of a circular lake with radius 3 km wants to arrive at the point C diametrically opposite A on the other side of the lake in the shortest possible time. She can walk at the rate of 6 km/h and row a boat at 3 km/h. How should she proceed?



47. An oil refinery is located on the north bank of a straight river that is 2 km wide. A pipeline is to be constructed from the refinery to storage tanks located on the south bank of the river 6 km east of the refinery. The cost of laying pipe is \$400,000/km over land to a point P on the north bank and \$800,000/km under the river to the tanks. To minimize the cost of the pipeline, where should P be located?

48. Suppose the refinery in Exercise 47 is located 1 km north of the river. Where should P be located?

49. The illumination of an object by a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. If two light sources, one three times as strong as the other, are placed 4 m apart, where should an object be placed on the line between the sources so as to receive the least illumination?

50. Find an equation of the line through the point (3, 5) that cuts off the least area from the first quadrant.

51. Let a and b be positive numbers. Find the length of the shortest line segment that is cut off by the first quadrant and passes through the point (a, b).

52. At which points on the curve $y = 1 + 40x^3 - 3x^5$ does the tangent line have the largest slope?

53. (a) If $C(x)$ is the cost of producing x units of a commodity, then the **average cost** per unit is $c(x) = C(x)/x$. Show that if the average cost is a minimum, then the marginal cost equals the average cost.
(b) If $C(x) = 16{,}000 + 200x + 4x^{3/2}$, in dollars, find (i) the cost, average cost, and marginal cost at a production level of 1000 units; (ii) the production level that will minimize the average cost; and (iii) the minimum average cost.

54. (a) Show that if the profit $P(x)$ is a maximum, then the marginal revenue equals the marginal cost.
(b) If $C(x) = 16{,}000 + 500x - 1.6x^2 + 0.004x^3$ is the cost function and $p(x) = 1700 - 7x$ is the demand function, find the production level that will maximize profit.

55. A baseball team plays in a stadium that holds 55,000 spectators. With ticket prices at \$10, the average attendance had been 27,000. When ticket prices were lowered to \$8, the average attendance rose to 33,000.
(a) Find the demand function, assuming that it is linear.
(b) How should ticket prices be set to maximize revenue?

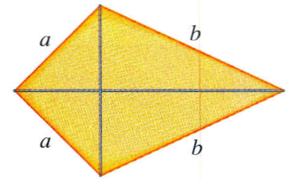
56. During the summer months Terry makes and sells necklaces on the beach. Last summer he sold the necklaces for \$10 each and his sales averaged 20 per day. When he increased the price by \$1, he found that the average decreased by two sales per day.
(a) Find the demand function, assuming that it is linear.
(b) If the material for each necklace costs Terry \$6, what should the selling price be to maximize his profit?

57. A manufacturer has been selling 1000 television sets a week at \$450 each. A market survey indicates that for each \$10 rebate offered to the buyer, the number of sets sold will increase by 100 per week.
(a) Find the demand function.
(b) How large a rebate should the company offer the buyer in order to maximize its revenue?
(c) If its weekly cost function is $C(x) = 68{,}000 + 150x$, how should the manufacturer set the size of the rebate in order to maximize its profit?

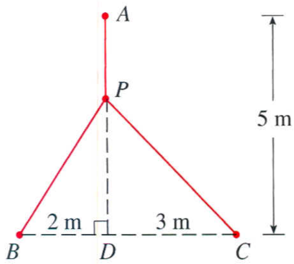
58. The manager of a 100-unit apartment complex knows from experience that all units will be occupied if the rent is \$800 per month. A market survey suggests that, on average, one additional unit will remain vacant for each \$10 increase in rent. What rent should the manager charge to maximize revenue?

59. Show that of all the isosceles triangles with a given perimeter, the one with the greatest area is equilateral.

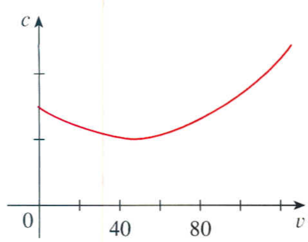
CAS **60.** The frame for a kite is to be made from six pieces of wood. The four exterior pieces have been cut with the lengths indicated in the figure. To maximize the area of the kite, how long should the diagonal pieces be?



61. A point P needs to be located somewhere on the line AD so that the total length L of cables linking P to the points A, B, and C is minimized (see the figure). Express L as a function of $x = |AP|$ and use the graphs of L and dL/dx to estimate the minimum value.



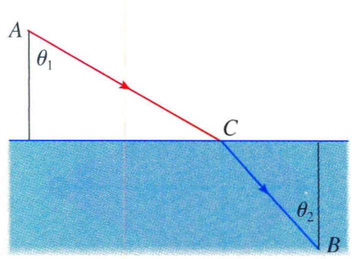
62. The graph shows the fuel consumption c of a car (measured in liters per hour) as a function of the speed v of the car. At very low speeds the engine runs inefficiently, so initially c decreases as the speed increases. But at high speeds the fuel consumption increases. You can see that $c(v)$ is minimized for this car when $v \approx 48$ km/h. However, for fuel efficiency, what must be minimized is not the consumption in gallons per hour but rather the fuel consumption in liters *per kilometer.* Let's call this consumption G. Using the graph, estimate the speed at which G has its minimum value.



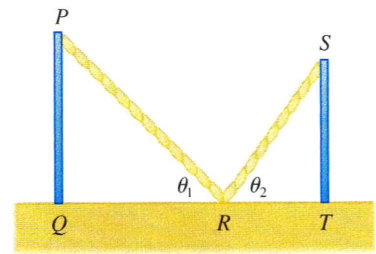
63. Let v_1 be the velocity of light in air and v_2 the velocity of light in water. According to Fermat's Principle, a ray of light will travel from a point A in the air to a point B in the water by a path ACB that minimizes the time taken. Show that

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2}$$

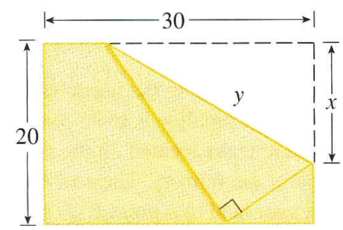
where θ_1 (the angle of incidence) and θ_2 (the angle of refraction) are as shown. This equation is known as Snell's Law.



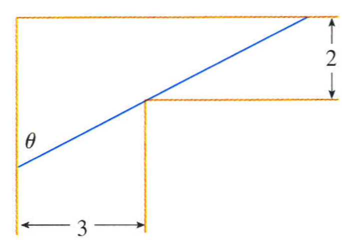
64. Two vertical poles PQ and ST are secured by a rope PRS going from the top of the first pole to a point R on the ground between the poles and then to the top of the second pole as in the figure. Show that the shortest length of such a rope occurs when $\theta_1 = \theta_2$.



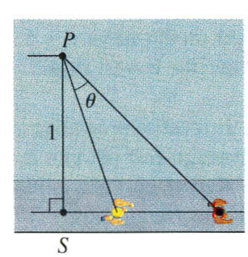
65. The upper right-hand corner of a piece of paper, 30 cm by 20 cm, as in the figure, is folded over to the bottom edge. How would you fold it so as to minimize the length of the fold? In other words, how would you choose x to minimize y?



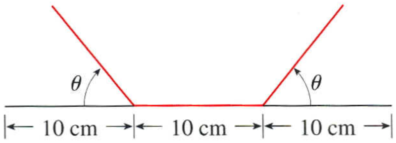
66. A steel pipe is being carried down a hallway 3 m wide. At the end of the hall there is a right-angled turn into a narrower hallway 2 m wide. What is the length of the longest pipe that can be carried horizontally around the corner?



67. An observer stands at a point P, one unit away from a track. Two runners start at the point S in the figure and run along the track. One runner runs three times as fast as the other. Find the maximum value of the observer's angle of sight θ between the runners. [*Hint:* Maximize $\tan \theta$.]



68. A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one-third of the sheet on each side through an angle θ. How should θ be chosen so that the gutter will carry the maximum amount of water?

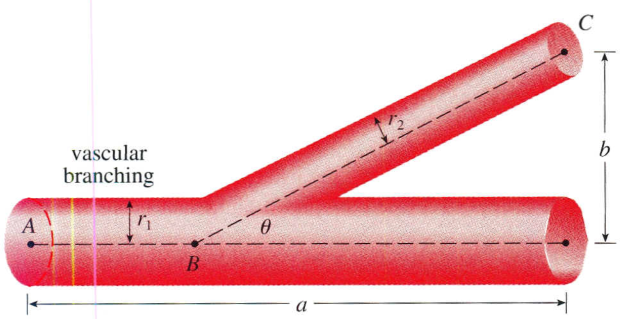


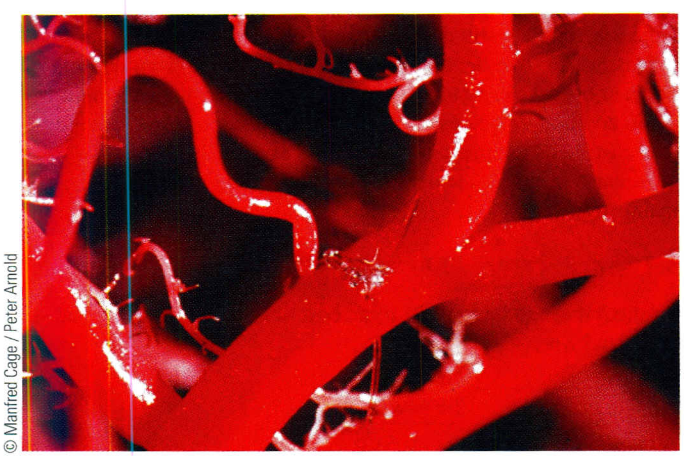
69. Find the maximum area of a rectangle that can be circumscribed about a given rectangle with length L and width W. [*Hint:* Express the area as a function of an angle θ.]

70. The blood vascular system consists of blood vessels (arteries, arterioles, capillaries, and veins) that convey blood from the heart to the organs and back to the heart. This system should work so as to minimize the energy expended by the heart in pumping the blood. In particular, this energy is reduced when the resistance of the blood is lowered. One of Poiseuille's Laws gives the resistance R of the blood as

$$R = C\frac{L}{r^4}$$

where L is the length of the blood vessel, r is the radius, and C is a positive constant determined by the viscosity of the blood. (Poiseuille established this law experimentally, but it also follows from Equation 9.4.2.) The figure shows a main blood vessel with radius r_1 branching at an angle θ into a smaller vessel with radius r_2





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(a) Use Poiseuille's Law to show that the total resistance of the blood along the path ABC is

$$R = C\left(\frac{a - b\cot\theta}{r_1^4} + \frac{b\csc\theta}{r_2^4}\right)$$

where a and b are the distances shown in the figure.

(b) Prove that this resistance is minimized when

$$\cos\theta = \frac{r_2^4}{r_1^4}$$

(c) Find the optimal branching angle (correct to the nearest degree) when the radius of the smaller blood vessel is two-thirds the radius of the larger vessel.

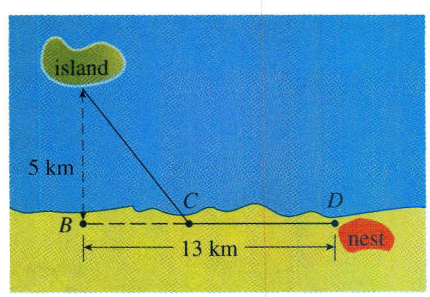
71. Ornithologists have determined that some species of birds tend to avoid flights over large bodies of water during daylight hours. It is believed that more energy is required to fly over water than land because air generally rises over land and falls over water during the day. A bird with these tendencies is released from an island that is 5 km from the nearest point B on a straight shoreline, flies to a point C on the shoreline, and then flies along the shoreline to its nesting area D. Assume that the bird instinctively chooses a path that will minimize its energy expenditure. Points B and D are 13 km apart.

(a) In general, if it takes 1.4 times as much energy to fly over water as land, to what point C should the bird fly in order to minimize the total energy expended in returning to its nesting area?

(b) Let W and L denote the energy (in joules) per kilometer flown over water and land, respectively. What would a large value of the ratio W/L mean in terms of the bird's flight? What would a small value mean? Determine the ratio W/L corresponding to the minimum expenditure of energy.

(c) What should the value of W/L be in order for the bird to fly directly to its nesting area D? What should the value of W/L be for the bird to fly to B and then along the shore to D?

(d) If the ornithologists observe that birds of a certain species reach the shore at a point 4 km from B, how many times more energy does it take a bird to fly over water than land?



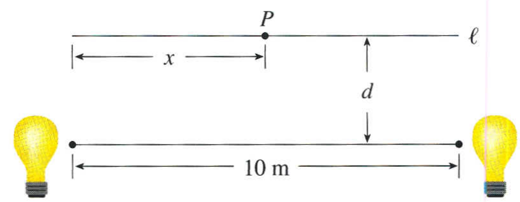
72. Two light sources of identical strength are placed 10 m apart. An object is to be placed at a point P on a line ℓ parallel to the line joining the light sources and at a distance d meters from it (see the figure). We want to locate P on ℓ so that the intensity of illumination is minimized. We need to use the fact that the intensity of illumination for a single source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source.

(a) Find an expression for the intensity $I(x)$ at the point P.

(b) If $d = 5$ m, use graphs of $I(x)$ and $I'(x)$ to show that the intensity is minimized when $x = 5$ m, that is, when P is at the midpoint of ℓ.

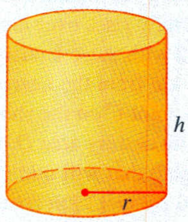
(c) If $d = 10$ m, show that the intensity (perhaps surprisingly) is *not* minimized at the midpoint.

(d) Somewhere between $d = 5$ m and $d = 10$ m there is a transitional value of d at which the point of minimal illumination abruptly changes. Estimate this value of d by graphical methods. Then find the exact value of d.



APPLIED PROJECT

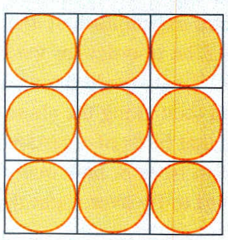
THE SHAPE OF A CAN



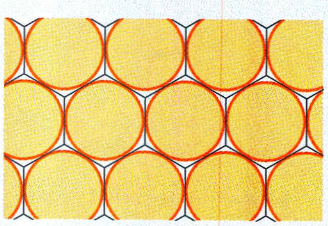
In this project we investigate the most economical shape for a can. We first interpret this to mean that the volume V of a cylindrical can is given and we need to find the height h and radius r that minimize the cost of the metal to make the can (see the figure). If we disregard any waste metal in the manufacturing process, then the problem is to minimize the surface area of the cylinder. We solved this problem in Example 2 in Section 4.7 and we found that $h = 2r$; that is, the height should be the same as the diameter. But if you go to your cupboard or your supermarket with a ruler, you will discover that the height is usually greater than the diameter and the ratio h/r varies from 2 up to about 3.8. Let's see if we can explain this phenomenon.

1. The material for the cans is cut from sheets of metal. The cylindrical sides are formed by bending rectangles; these rectangles are cut from the sheet with little or no waste. But if the top and bottom discs are cut from squares of side $2r$ (as in the figure), this leaves considerable waste metal, which may be recycled but has little or no value to the can makers. If this is the case, show that the amount of metal used is minimized when

$$\frac{h}{r} = \frac{8}{\pi} \approx 2.55$$



Discs cut from squares



Discs cut from hexagons

2. A more efficient packing of the discs is obtained by dividing the metal sheet into hexagons and cutting the circular lids and bases from the hexagons (see the figure). Show that if this strategy is adopted, then

$$\frac{h}{r} = \frac{4\sqrt{3}}{\pi} \approx 2.21$$

3. The values of h/r that we found in Problems 1 and 2 are a little closer to the ones that actually occur on supermarket shelves, but they still don't account for everything. If we look more closely at some real cans, we see that the lid and the base are formed from discs with radius larger than r that are bent over the ends of the can. If we allow for this we would increase h/r. More significantly, in addition to the cost of the metal we need to incorporate the manufacturing of the can into the cost. Let's assume that most of the expense is incurred in joining the sides to the rims of the cans. If we cut the discs from hexagons as in Problem 2, then the total cost is proportional to

$$4\sqrt{3}\, r^2 + 2\pi r h + k(4\pi r + h)$$

where k is the reciprocal of the length that can be joined for the cost of one unit area of metal. Show that this expression is minimized when

$$\frac{\sqrt[3]{V}}{k} = \sqrt[3]{\frac{\pi h}{r}} \cdot \frac{2\pi - h/r}{\pi h/r - 4\sqrt{3}}$$

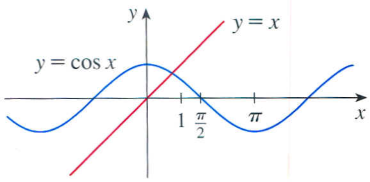


FIGURE 6

In order to guess a suitable value for x_1 we sketch the graphs of $y = \cos x$ and $y = x$ in Figure 6. It appears that they intersect at a point whose x-coordinate is somewhat less than 1, so let's take $x_1 = 1$ as a convenient first approximation. Then, remembering to put our calculator in radian mode, we get

$$x_2 \approx 0.75036387$$
$$x_3 \approx 0.73911289$$
$$x_4 \approx 0.73908513$$
$$x_5 \approx 0.73908513$$

Since x_4 and x_5 agree to six decimal places (eight, in fact), we conclude that the root of the equation, correct to six decimal places, is 0.739085. □

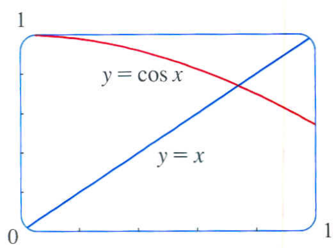


FIGURE 7

Instead of using the rough sketch in Figure 6 to get a starting approximation for Newton's method in Example 3, we could have used the more accurate graph that a calculator or computer provides. Figure 7 suggests that we use $x_1 = 0.75$ as the initial approximation. Then Newton's method gives

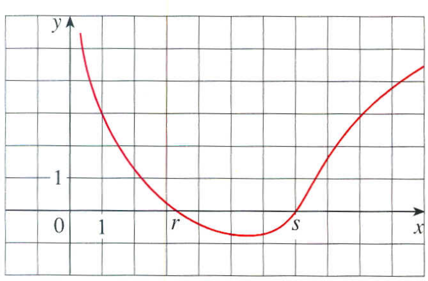
$$x_2 \approx 0.73911114 \qquad x_3 \approx 0.73908513 \qquad x_4 \approx 0.73908513$$

and so we obtain the same answer as before, but with one fewer step.

You might wonder why we bother at all with Newton's method if a graphing device is available. Isn't it easier to zoom in repeatedly and find the roots as we did in Section 1.4? If only one or two decimal places of accuracy are required, then indeed Newton's method is inappropriate and a graphing device suffices. But if six or eight decimal places are required, then repeated zooming becomes tiresome. It is usually faster and more efficient to use a computer and Newton's method in tandem—the graphing device to get started and Newton's method to finish.

4.8 EXERCISES

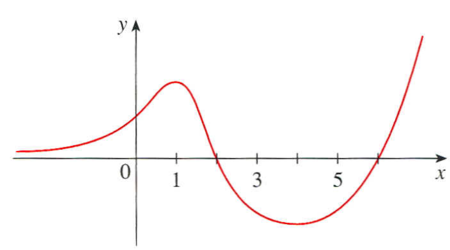
1. The figure shows the graph of a function f. Suppose that Newton's method is used to approximate the root r of the equation $f(x) = 0$ with initial approximation $x_1 = 1$.
   (a) Draw the tangent lines that are used to find x_2 and x_3, and estimate the numerical values of x_2 and x_3.
   (b) Would $x_1 = 5$ be a better first approximation? Explain.



2. Follow the instructions for Exercise 1(a) but use $x_1 = 9$ as the starting approximation for finding the root s.

3. Suppose the line $y = 5x - 4$ is tangent to the curve $y = f(x)$ when $x = 3$. If Newton's method is used to locate a root of the equation $f(x) = 0$ and the initial approximation is $x_1 = 3$, find the second approximation x_2.

4. For each initial approximation, determine graphically what happens if Newton's method is used for the function whose graph is shown.
   (a) $x_1 = 0$ (b) $x_1 = 1$ (c) $x_1 = 3$
   (d) $x_1 = 4$ (e) $x_1 = 5$



5–8 Use Newton's method with the specified initial approximation x_1 to find x_3, the third approximation to the root of the given equation. (Give your answer to four decimal places.)

5. $x^3 + 2x - 4 = 0, \quad x_1 = 1$

6. $\frac{1}{3}x^3 + \frac{1}{2}x^2 + 3 = 0, \quad x_1 = -3$

7. $x^5 - x - 1 = 0, \quad x_1 = 1$

8. $x^5 + 2 = 0, \quad x_1 = -1$

9. Use Newton's method with initial approximation $x_1 = -1$ to find x_2, the second approximation to the root of the equation $x^3 + x + 3 = 0$. Explain how the method works by first graphing the function and its tangent line at $(-1, 1)$.

10. Use Newton's method with initial approximation $x_1 = 1$ to find x_2, the second approximation to the root of the equation $x^4 - x - 1 = 0$. Explain how the method works by first graphing the function and its tangent line at $(1, -1)$.

11–12 Use Newton's method to approximate the given number correct to eight decimal places.

11. $\sqrt[5]{20}$ **12.** $\sqrt[100]{100}$

13–16 Use Newton's method to approximate the indicated root of the equation correct to six decimal places.

13. The root of $2x^3 - 6x^2 + 3x + 1 = 0$ in the interval $[2, 3]$

14. The root of $x^4 + x - 4 = 0$ in the interval $[1, 2]$

15. The positive root of $\sin x = x^2$

16. The positive root of $2 \cos x = x^4$

17–22 Use Newton's method to find all roots of the equation correct to six decimal places.

17. $x^4 = 1 + x$ **18.** $x^5 = 5x - 2$

19. $\sqrt[3]{x} = x^2 - 1$ **20.** $\dfrac{1}{x} = 1 + x^3$

21. $\cos x = \sqrt{x}$ **22.** $\sqrt{x + 3} = x^2$

23–26 Use Newton's method to find all the roots of the equation correct to eight decimal places. Start by drawing a graph to find initial approximations.

23. $x^6 - x^5 - 6x^4 - x^2 + x + 10 = 0$

24. $x^2(4 - x^2) = \dfrac{4}{x^2 + 1}$

25. $x^2\sqrt{2 - x - x^2} = 1$ **26.** $3 \sin(x^2) = 2x$

27. (a) Apply Newton's method to the equation $x^2 - a = 0$ to derive the following square-root algorithm (used by the ancient Babylonians to compute $\sqrt{a}$):

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

(b) Use part (a) to compute $\sqrt{1000}$ correct to six decimal places.

28. (a) Apply Newton's method to the equation $1/x - a = 0$ to derive the following reciprocal algorithm:

$$x_{n+1} = 2x_n - ax_n^2$$

(This algorithm enables a computer to find reciprocals without actually dividing.)

(b) Use part (a) to compute $1/1.6984$ correct to six decimal places.

29. Explain why Newton's method doesn't work for finding the root of the equation $x^3 - 3x + 6 = 0$ if the initial approximation is chosen to be $x_1 = 1$.

30. (a) Use Newton's method with $x_1 = 1$ to find the root of the equation $x^3 - x = 1$ correct to six decimal places.

(b) Solve the equation in part (a) using $x_1 = 0.6$ as the initial approximation.

(c) Solve the equation in part (a) using $x_1 = 0.57$. (You definitely need a programmable calculator for this part.)

(d) Graph $f(x) = x^3 - x - 1$ and its tangent lines at $x_1 = 1$, 0.6, and 0.57 to explain why Newton's method is so sensitive to the value of the initial approximation.

31. Explain why Newton's method fails when applied to the equation $\sqrt[3]{x} = 0$ with any initial approximation $x_1 \neq 0$. Illustrate your explanation with a sketch.

32. If

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ -\sqrt{-x} & \text{if } x < 0 \end{cases}$$

then the root of the equation $f(x) = 0$ is $x = 0$. Explain why Newton's method fails to find the root no matter which initial approximation $x_1 \neq 0$ is used. Illustrate your explanation with a sketch.

33. (a) Use Newton's method to find the critical numbers of the function $f(x) = x^6 - x^4 + 3x^3 - 2x$ correct to six decimal places.

(b) Find the absolute minimum value of f correct to four decimal places.

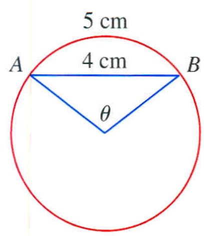
34. Use Newton's method to find the absolute maximum value of the function $f(x) = x \cos x$, $0 \leq x \leq \pi$, correct to six decimal places.

35. Use Newton's method to find the coordinates of the inflection point of the curve $y = x^3 + \cos x$ correct to six decimal places.

36. Of the infinitely many lines that are tangent to the curve $y = -\sin x$ and pass through the origin, there is one that has the largest slope. Use Newton's method to find the slope of that line correct to six decimal places.

37. Use Newton's method to find the coordinates, correct to six decimal places, of the point on the parabola $y = (x - 1)^2$ that is closest to the origin.

38. In the figure, the length of the chord AB is 4 cm and the length of the arc AB is 5 cm. Find the central angle θ, in radians, correct to four decimal places. Then give the answer to the nearest degree.



39. A car dealer sells a new car for \$18,000. He also offers to sell the same car for payments of \$375 per month for five years. What monthly interest rate is this dealer charging?

To solve this problem you will need to use the formula for the present value A of an annuity consisting of n equal payments of size R with interest rate i per time period:

$$A = \frac{R}{i}[1 - (1 + i)^{-n}]$$

Replacing i by x, show that

$$48x(1 + x)^{60} - (1 + x)^{60} + 1 = 0$$

Use Newton's method to solve this equation.

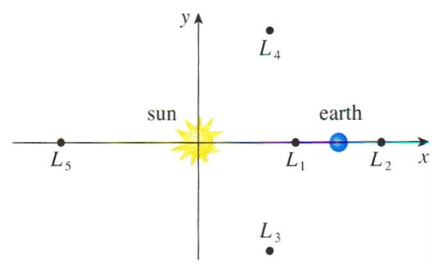
40. The figure shows the sun located at the origin and the earth at the point $(1, 0)$. (The unit here is the distance between the centers of the earth and the sun, called an *astronomical unit:* 1 AU $\approx 1.496 \times 10^8$ km.) There are five locations L_1, L_2, L_3, L_4, and L_5 in this plane of rotation of the earth about the sun where a satellite remains motionless with respect to the earth because the forces acting on the satellite (including the gravitational attractions of the earth and the sun) balance each other. These locations are called *libration points.* (A solar research satellite has been placed at one of these libration points.) If m_1 is the mass of the sun, m_2 is the mass of the earth, and $r = m_2/(m_1 + m_2)$, it turns out that the x-coordinate of L_1 is the unique root of the fifth-degree equation

$$p(x) = x^5 - (2 + r)x^4 + (1 + 2r)x^3 - (1 - r)x^2$$
$$+ 2(1 - r)x + r - 1 = 0$$

and the x-coordinate of L_2 is the root of the equation

$$p(x) - 2rx^2 = 0$$

Using the value $r \approx 3.04042 \times 10^{-6}$, find the locations of the libration points (a) L_1 and (b) L_2.



4.9 ANTIDERIVATIVES

A physicist who knows the velocity of a particle might wish to know its position at a given time. An engineer who can measure the variable rate at which water is leaking from a tank wants to know the amount leaked over a certain time period. A biologist who knows the rate at which a bacteria population is increasing might want to deduce what the size of the population will be at some future time. In each case, the problem is to find a function F whose derivative is a known function f. If such a function F exists, it is called an *antiderivative* of f.

DEFINITION A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

For instance, let $f(x) = x^2$. It isn't difficult to discover an antiderivative of f if we keep the Power Rule in mind. In fact, if $F(x) = \frac{1}{3}x^3$, then $F'(x) = x^2 = f(x)$. But the function $G(x) = \frac{1}{3}x^3 + 100$ also satisfies $G'(x) = x^2$. Therefore both F and G are antiderivatives of f. Indeed, any function of the form $H(x) = \frac{1}{3}x^3 + C$, where C is a constant, is an antiderivative of f. The question arises: Are there any others?

■ Figure 4 shows the position function of the ball in Example 7. The graph corroborates the conclusions we reached: The ball reaches its maximum height after 1.5 s and hits the ground after 7.1 s.

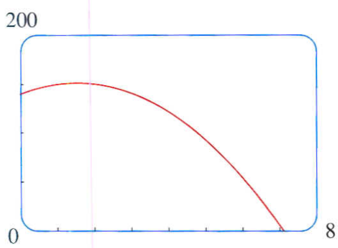


FIGURE 4

The expression for $s(t)$ is valid until the ball hits the ground. This happens when $s(t) = 0$, that is, when

$$-4.9t^2 - 15t - 140 = 0$$

Using the quadratic formula to solve this equation, we get

$$t = \frac{15 \pm \sqrt{2969}}{9.8}$$

We reject the solution with the minus sign since it gives a negative value for t. Therefore the ball hits the ground after

$$\frac{15 + \sqrt{2969}}{9.8} \approx 7.1 \text{ s}$$ □

4.9 EXERCISES

1–18 Find the most general antiderivative of the function. (Check your answer by differentiation.)

1. $f(x) = x - 3$

2. $f(x) = \frac{1}{2}x^2 - 2x + 6$

3. $f(x) = \frac{1}{2} + \frac{3}{4}x^2 - \frac{4}{5}x^3$

4. $f(x) = 8x^9 - 3x^6 + 12x^3$

5. $f(x) = (x + 1)(2x - 1)$

6. $f(x) = x(2 - x)^2$

7. $f(x) = 5x^{1/4} - 7x^{3/4}$

8. $f(x) = 2x + 3x^{1.7}$

9. $f(x) = 6\sqrt{x} - \sqrt[6]{x}$

10. $f(x) = \sqrt[4]{x^3} + \sqrt[3]{x^4}$

11. $f(x) = \dfrac{10}{x^9}$

12. $g(x) = \dfrac{5 - 4x^3 + 2x^6}{x^6}$

13. $f(u) = \dfrac{u^4 + 3\sqrt{u}}{u^2}$

14. $f(t) = 3\cos t - 4\sin t$

15. $g(\theta) = \cos\theta - 5\sin\theta$

16. $f(\theta) = 6\theta^2 - 7\sec^2\theta$

17. $f(t) = 2\sec t\,\tan t + \frac{1}{2}t^{-1/2}$

18. $f(x) = 2\sqrt{x} + 6\cos x$

19–20 Find the antiderivative F of f that satisfies the given condition. Check your answer by comparing the graphs of f and F.

19. $f(x) = 5x^4 - 2x^5$, $F(0) = 4$

20. $f(x) = x + 2\sin x$, $F(0) = -6$

21–40 Find f.

21. $f''(x) = 6x + 12x^2$

22. $f''(x) = 2 + x^3 + x^6$

23. $f''(x) = \frac{2}{3}x^{2/3}$

24. $f''(x) = 6x + \sin x$

25. $f'''(t) = 60t^2$

26. $f'''(t) = t - \sqrt{t}$

27. $f'(x) = 1 - 6x$, $f(0) = 8$

28. $f'(x) = 8x^3 + 12x + 3$, $f(1) = 6$

29. $f'(x) = \sqrt{x}(6 + 5x)$, $f(1) = 10$

30. $f'(x) = 2x - 3/x^4$, $x > 0$, $f(1) = 3$

31. $f'(t) = 2\cos t + \sec^2 t$, $-\pi/2 < t < \pi/2$, $f(\pi/3) = 4$

32. $f'(x) = x^{-1/3}$, $f(1) = 1$, $f(-1) = -1$

33. $f''(x) = 24x^2 + 2x + 10$, $f(1) = 5$, $f'(1) = -3$

34. $f''(x) = 4 - 6x - 40x^3$, $f(0) = 2$, $f'(0) = 1$

35. $f''(\theta) = \sin\theta + \cos\theta$, $f(0) = 3$, $f'(0) = 4$

36. $f''(t) = 3/\sqrt{t}$, $f(4) = 20$, $f'(4) = 7$

37. $f''(x) = 2 - 12x$, $f(0) = 9$, $f(2) = 15$

38. $f''(x) = 20x^3 + 12x^2 + 4$, $f(0) = 8$, $f(1) = 5$

39. $f''(x) = 2 + \cos x$, $f(0) = -1$, $f(\pi/2) = 0$

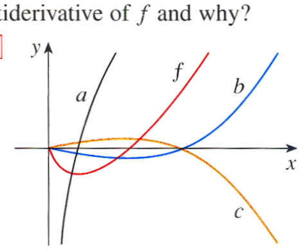
40. $f'''(x) = \cos x$, $f(0) = 1$, $f'(0) = 2$, $f''(0) = 3$

41. Given that the graph of f passes through the point $(1, 6)$ and that the slope of its tangent line at $(x, f(x))$ is $2x + 1$, find $f(2)$.

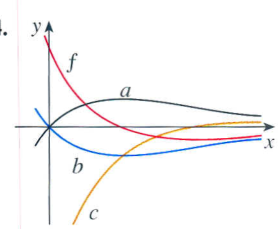
42. Find a function f such that $f'(x) = x^3$ and the line $x + y = 0$ is tangent to the graph of f.

43–44 The graph of a function f is shown. Which graph is an antiderivative of f and why?

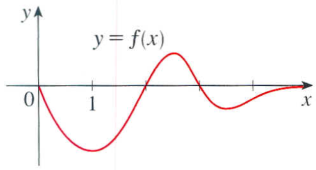
43.



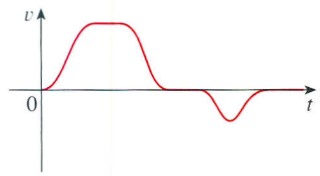
44.



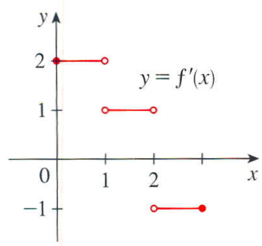
45. The graph of a function is shown in the figure. Make a rough sketch of an antiderivative F, given that $F(0) = 1$.



46. The graph of the velocity function of a particle is shown in the figure. Sketch the graph of the position function.



47. The graph of f' is shown in the figure. Sketch the graph of f if f is continuous and $f(0) = -1$.



48. (a) Use a graphing device to graph $f(x) = 2x - 3\sqrt{x}$.
(b) Starting with the graph in part (a), sketch a rough graph of the antiderivative F that satisfies $F(0) = 1$.
(c) Use the rules of this section to find an expression for $F(x)$.
(d) Graph F using the expression in part (c). Compare with your sketch in part (b).

49–50 Draw a graph of f and use it to make a rough sketch of the antiderivative that passes through the origin.

49. $f(x) = \dfrac{\sin x}{1 + x^2}, \quad -2\pi \leqslant x \leqslant 2\pi$

50. $f(x) = \sqrt{x^4 - 2x^2 + 2} - 1, \quad -1.5 \leqslant x \leqslant 1.5$

51–56 A particle is moving with the given data. Find the position of the particle.

51. $v(t) = \sin t - \cos t, \quad s(0) = 0$

52. $v(t) = 1.5\sqrt{t}, \quad s(4) = 10$

53. $a(t) = t - 2, \quad s(0) = 1, \quad v(0) = 3$

54. $a(t) = \cos t + \sin t, \quad s(0) = 0, \quad v(0) = 5$

55. $a(t) = 10 \sin t + 3 \cos t, \quad s(0) = 0, \quad s(2\pi) = 12$

56. $a(t) = t^2 - 4t + 6, \quad s(0) = 0, \quad s(1) = 20$

57. A stone is dropped from the upper observation deck (the Space Deck) of the CN Tower, 450 m above the ground.
(a) Find the distance of the stone above ground level at time t.
(b) How long does it take the stone to reach the ground?
(c) With what velocity does it strike the ground?
(d) If the stone is thrown downward with a speed of 5 m/s, how long does it take to reach the ground?

58. Show that for motion in a straight line with constant acceleration a, initial velocity v_0, and initial displacement s_0, the displacement after time t is

$$s = \tfrac{1}{2}at^2 + v_0 t + s_0$$

59. An object is projected upward with initial velocity v_0 meters per second from a point s_0 meters above the ground. Show that

$$[v(t)]^2 = v_0^2 - 19.6[s(t) - s_0]$$

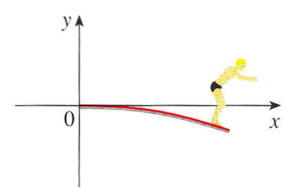
60. Two balls are thrown upward from the edge of the cliff in Example 7. The first is thrown with a speed of 15 m/s and the other is thrown a second later with a speed of 8 m/s. Do the balls ever pass each other?

61. A stone was dropped off a cliff and hit the ground with a speed of 40 m/s. What is the height of the cliff?

62. If a diver of mass m stands at the end of a diving board with length L and linear density ρ, then the board takes on the shape of a curve $y = f(x)$, where

$$EIy'' = mg(L - x) + \tfrac{1}{2}\rho g(L - x)^2$$

E and I are positive constants that depend on the material of the board and g (< 0) is the acceleration due to gravity.
(a) Find an expression for the shape of the curve.
(b) Use $f(L)$ to estimate the distance below the horizontal at the end of the board.



63. A company estimates that the marginal cost (in dollars per item) of producing x items is $1.92 - 0.002x$. If the cost of producing one item is \$562, find the cost of producing 100 items.

64. The linear density of a rod of length 1 m is given by $\rho(x) = 1/\sqrt{x}$, in grams per centimeter, where x is measured in centimeters from one end of the rod. Find the mass of the rod.

65. Since raindrops grow as they fall, their surface area increases and therefore the resistance to their falling increases. A raindrop has an initial downward velocity of 10 m/s and its downward acceleration is

$$a = \begin{cases} 9 - 0.9t & \text{if } 0 \leqslant t \leqslant 10 \\ 0 & \text{if } t > 10 \end{cases}$$

If the raindrop is initially 500 m above the ground, how long does it take to fall?

66. A car is traveling at 80 km/h when the brakes are fully applied, producing a constant deceleration of 7 m/s^2. What is the distance traveled before the car comes to a stop?

67. What constant acceleration is required to increase the speed of a car from 50 km/h to 80 km/h in 5 s?

68. A car braked with a constant deceleration of 5 m/s^2, producing skid marks measuring 60 m before coming to a stop. How fast was the car traveling when the brakes were first applied?

69. A car is traveling at 100 km/h when the driver sees an accident 80 m ahead and slams on the brakes. What constant deceleration is required to stop the car in time to avoid a pileup?

70. A model rocket is fired vertically upward from rest. Its acceleration for the first three seconds is $a(t) = 18t$, at which time the fuel is exhausted and it becomes a freely "falling" body. Fourteen seconds later, the rocket's parachute opens, and the (downward) velocity slows linearly to -5.5 m/s in five seconds. The rocket then "floats" to the ground at that rate.

(a) Determine the position function s and the velocity function v (for all times t). Sketch the graphs of s and v.

(b) At what time does the rocket reach its maximum height, and what is that height?

(c) At what time does the rocket land?

71. A high-speed bullet train accelerates and decelerates at the rate of 1.2 m/s^2. Its maximum cruising speed is 145 km/h.

(a) What is the maximum distance the train can travel if it accelerates from rest until it reaches its cruising speed and then runs at that speed for 15 minutes?

(b) Suppose that the train starts from rest and must come to a complete stop in 15 minutes. What is the maximum distance it can travel under these conditions?

(c) Find the minimum time that the train takes to travel between two consecutive stations that are 72 km apart.

(d) The trip from one station to the next takes 37.5 minutes. How far apart are the stations?

4 REVIEW

CONCEPT CHECK

1. Explain the difference between an absolute maximum and a local maximum. Illustrate with a sketch.

2. (a) What does the Extreme Value Theorem say?
(b) Explain how the Closed Interval Method works.

3. (a) State Fermat's Theorem.
(b) Define a critical number of f.

4. (a) State Rolle's Theorem.
(b) State the Mean Value Theorem and give a geometric interpretation.

5. (a) State the Increasing/Decreasing Test.
(b) What does it mean to say that f is concave upward on an interval I?
(c) State the Concavity Test.
(d) What are inflection points? How do you find them?

6. (a) State the First Derivative Test.
(b) State the Second Derivative Test.
(c) What are the relative advantages and disadvantages of these tests?

7. Explain the meaning of each of the following statements.
(a) $\lim_{x\to\infty} f(x) = L$ (b) $\lim_{x\to-\infty} f(x) = L$ (c) $\lim_{x\to\infty} f(x) = \infty$
(d) The curve $y = f(x)$ has the horizontal asymptote $y = L$.

8. If you have a graphing calculator or computer, why do you need calculus to graph a function?

9. (a) Given an initial approximation x_1 to a root of the equation $f(x) = 0$, explain geometrically, with a diagram, how the second approximation x_2 in Newton's method is obtained.
(b) Write an expression for x_2 in terms of x_1, $f(x_1)$, and $f'(x_1)$.
(c) Write an expression for x_{n+1} in terms of x_n, $f(x_n)$, and $f'(x_n)$.
(d) Under what circumstances is Newton's method likely to fail or to work very slowly?

10. (a) What is an antiderivative of a function f?
(b) Suppose F_1 and F_2 are both antiderivatives of f on an interval I. How are F_1 and F_2 related?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If $f'(c) = 0$, then f has a local maximum or minimum at c.
2. If f has an absolute minimum value at c, then $f'(c) = 0$.
3. If f is continuous on (a, b), then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in (a, b).
4. If f is differentiable and $f(-1) = f(1)$, then there is a number c such that $|c| < 1$ and $f'(c) = 0$.
5. If $f'(x) < 0$ for $1 < x < 6$, then f is decreasing on $(1, 6)$.
6. If $f''(2) = 0$, then $(2, f(2))$ is an inflection point of the curve $y = f(x)$.
7. If $f'(x) = g'(x)$ for $0 < x < 1$, then $f(x) = g(x)$ for $0 < x < 1$.
8. There exists a function f such that $f(1) = -2$, $f(3) = 0$, and $f'(x) > 1$ for all x.
9. There exists a function f such that $f(x) > 0$, $f'(x) < 0$, and $f''(x) > 0$ for all x.
10. There exists a function f such that $f(x) < 0$, $f'(x) < 0$, and $f''(x) > 0$ for all x.
11. If f and g are increasing on an interval I, then $f + g$ is increasing on I.
12. If f and g are increasing on an interval I, then $f - g$ is increasing on I.
13. If f and g are increasing on an interval I, then fg is increasing on I.
14. If f and g are positive increasing functions on an interval I, then fg is increasing on I.
15. If f is increasing and $f(x) > 0$ on I, then $g(x) = 1/f(x)$ is decreasing on I.
16. If f is even, then f' is even.
17. If f is periodic, then f' is periodic.
18. The most general antiderivative of $f(x) = x^{-2}$ is
$$F(x) = -\frac{1}{x} + C$$
19. If $f'(x)$ exists and is nonzero for all x, then $f(1) \neq f(0)$.

EXERCISES

1–6 Find the local and absolute extreme values of the function on the given interval.

1. $f(x) = x^3 - 6x^2 + 9x + 1$, $[2, 4]$
2. $f(x) = x\sqrt{1 - x}$, $[-1, 1]$
3. $f(x) = \dfrac{3x - 4}{x^2 + 1}$, $[-2, 2]$
4. $f(x) = (x^2 + 2x)^3$, $[-2, 1]$
5. $f(x) = x + \sin 2x$, $[0, \pi]$
6. $f(x) = \sin x + \cos^2 x$, $[0, \pi]$

7–12 Find the limit.

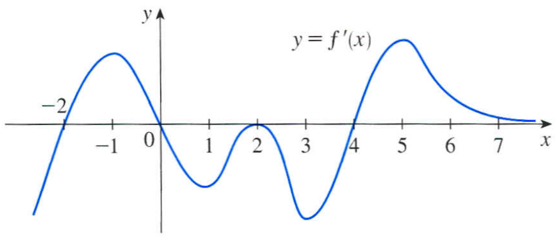
7. $\displaystyle\lim_{x\to\infty} \frac{3x^4 + x - 5}{6x^4 - 2x^2 + 1}$
8. $\displaystyle\lim_{t\to\infty} \frac{t^3 - t + 2}{(2t - 1)(t^2 + t + 1)}$
9. $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{4x^2 + 1}}{3x - 1}$
10. $\displaystyle\lim_{x\to-\infty} (x^2 + x^3)$
11. $\displaystyle\lim_{x\to\infty} \left(\sqrt{4x^2 + 3x} - 2x\right)$
12. $\displaystyle\lim_{x\to\infty} \frac{\sin^4 x}{\sqrt{x}}$

13–15 Sketch the graph of a function that satisfies the given conditions:

13. $f(0) = 0$, $f'(-2) = f'(1) = f'(9) = 0$,
$\lim_{x\to\infty} f(x) = 0$, $\lim_{x\to 6} f(x) = -\infty$,
$f'(x) < 0$ on $(-\infty, -2)$, $(1, 6)$, and $(9, \infty)$,
$f'(x) > 0$ on $(-2, 1)$ and $(6, 9)$,
$f''(x) > 0$ on $(-\infty, 0)$ and $(12, \infty)$,
$f''(x) < 0$ on $(0, 6)$ and $(6, 12)$
14. $f(0) = 0$, f is continuous and even,
$f'(x) = 2x$ if $0 < x < 1$, $f'(x) = -1$ if $1 < x < 3$,
$f'(x) = 1$ if $x > 3$
15. f is odd, $f'(x) < 0$ for $0 < x < 2$,
$f'(x) > 0$ for $x > 2$, $f''(x) > 0$ for $0 < x < 3$,
$f''(x) < 0$ for $x > 3$, $\lim_{x\to\infty} f(x) = -2$

16. The figure shows the graph of the *derivative* f' of a function f.
(a) On what intervals is f increasing or decreasing?
(b) For what values of x does f have a local maximum or minimum?

(c) Sketch the graph of f''.
(d) Sketch a possible graph of f.



17–28 Use the guidelines of Section 4.5 to sketch the curve.

17. $y = 2 - 2x - x^3$

18. $y = x^3 - 6x^2 - 15x + 4$

19. $y = x^4 - 3x^3 + 3x^2 - x$

20. $y = \dfrac{1}{1 - x^2}$

21. $y = \dfrac{1}{x(x - 3)^2}$

22. $y = \dfrac{1}{x^2} - \dfrac{1}{(x - 2)^2}$

23. $y = x^2/(x + 8)$

24. $y = \sqrt{1 - x} + \sqrt{1 + x}$

25. $y = x\sqrt{2 + x}$

26. $y = \sqrt[3]{x^2 + 1}$

27. $y = \sin^2 x - 2 \cos x$

28. $y = 4x - \tan x, \quad -\pi/2 < x < \pi/2$

29–32 Produce graphs of f that reveal all the important aspects of the curve. Use graphs of f' and f'' to estimate the intervals of increase and decrease, extreme values, intervals of concavity, and inflection points. In Exercise 29 use calculus to find these quantities exactly.

29. $f(x) = \dfrac{x^2 - 1}{x^3}$

30. $f(x) = \dfrac{x^3 - x}{x^2 + x + 3}$

31. $f(x) = 3x^6 - 5x^5 + x^4 - 5x^3 - 2x^2 + 2$

32. $f(x) = x^2 + 6.5 \sin x, \quad -5 \leqslant x \leqslant 5$

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33. Show that the equation $3x + 2 \cos x + 5 = 0$ has exactly one real root.

34. Suppose that f is continuous on $[0, 4]$, $f(0) = 1$, and $2 \leqslant f'(x) \leqslant 5$ for all x in $(0, 4)$. Show that $9 \leqslant f(4) \leqslant 21$.

35. By applying the Mean Value Theorem to the function $f(x) = x^{1/5}$ on the interval $[32, 33]$, show that

$$2 < \sqrt[5]{33} < 2.0125$$

36. For what values of the constants a and b is $(1, 6)$ a point of inflection of the curve $y = x^3 + ax^2 + bx + 1$?

37. Let $g(x) = f(x^2)$, where f is twice differentiable for all x, $f'(x) > 0$ for all $x \neq 0$, and f is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$.
(a) At what numbers does g have an extreme value?
(b) Discuss the concavity of g.

38. Find two positive integers such that the sum of the first number and four times the second number is 1000 and the product of the numbers is as large as possible.

39. Show that the shortest distance from the point (x_1, y_1) to the straight line $Ax + By + C = 0$ is

$$\frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

40. Find the point on the hyperbola $xy = 8$ that is closest to the point $(3, 0)$.

41. Find the smallest possible area of an isosceles triangle that is circumscribed about a circle of radius r.

42. Find the volume of the largest circular cone that can be inscribed in a sphere of radius r.

43. In ΔABC, D lies on AB, $CD \perp AB$, $|AD| = |BD| = 4$ cm, and $|CD| = 5$ cm. Where should a point P be chosen on CD so that the sum $|PA| + |PB| + |PC|$ is a minimum?

44. Solve Exercise 43 when $|CD| = 2$ cm.

45. The velocity of a wave of length L in deep water is

$$v = K\sqrt{\frac{L}{C} + \frac{C}{L}}$$

where K and C are known positive constants. What is the length of the wave that gives the minimum velocity?

46. A metal storage tank with volume V is to be constructed in the shape of a right circular cylinder surmounted by a hemisphere. What dimensions will require the least amount of metal?

47. A hockey team plays in an arena with a seating capacity of 15,000 spectators. With the ticket price set at \$12, average attendance at a game has been 11,000. A market survey indicates that for each dollar the ticket price is lowered, average attendance will increase by 1000. How should the owners of the team set the ticket price to maximize their revenue from ticket sales?

48. A manufacturer determines that the cost of making x units of a commodity is $C(x) = 1800 + 25x - 0.2x^2 + 0.001x^3$ and the demand function is $p(x) = 48.2 - 0.03x$.
(a) Graph the cost and revenue functions and use the graphs to estimate the production level for maximum profit.
(b) Use calculus to find the production level for maximum profit.
(c) Estimate the production level that minimizes the average cost.

49. Use Newton's method to find the root of the equation $x^5 - x^4 + 3x^2 - 3x - 2 = 0$ in the interval $[1, 2]$ correct to six decimal places.

50. Use Newton's method to find all roots of the equation $\sin x = x^2 - 3x + 1$ correct to six decimal places.

51. Use Newton's method to find the absolute maximum value of the function $f(t) = \cos t + t - t^2$ correct to eight decimal places.

52. Use the guidelines in Section 4.5 to sketch the curve $y = x \sin x$, $0 \leqslant x \leqslant 2\pi$. Use Newton's method when necessary.

53–58 Find f.

53. $f'(x) = \sqrt{x^3} + \sqrt[3]{x^2}$

54. $f'(x) = 8x - 3 \sec^2 x$

55. $f'(t) = 2t - 3 \sin t, \quad f(0) = 5$

56. $f'(u) = \dfrac{u^2 + \sqrt{u}}{u}, \quad f(1) = 3$

57. $f''(x) = 1 - 6x + 48x^2, \quad f(0) = 1, \quad f'(0) = 2$

58. $f''(x) = 2x^3 + 3x^2 - 4x + 5, \quad f(0) = 2, \quad f(1) = 0$

59–60 A particle is moving with the given data. Find the position of the particle.

59. $v(t) = 2t - \sin t, \quad s(0) = 3$

60. $a(t) = \sin t + 3 \cos t, \quad s(0) = 0, \quad v(0) = 2$

61. Use a graphing device to draw a graph of the function $f(x) = x^2 \sin(x^2)$, $0 \leqslant x \leqslant \pi$, and use that graph to sketch the antiderivative F of f that satisfies the initial condition $F(0) = 0$.

62. Investigate the family of curves given by

$$f(x) = x^4 + x^3 + cx^2$$

In particular you should determine the transitional value of c at which the number of critical numbers changes and the transitional value at which the number of inflection points changes. Illustrate the various possible shapes with graphs.

63. A canister is dropped from a helicopter 500 m above the ground. Its parachute does not open, but the canister has been designed to withstand an impact velocity of 100 m/s. Will it burst?

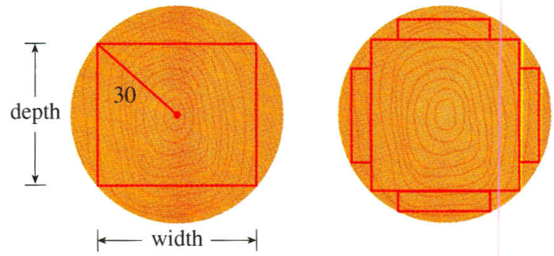
64. In an automobile race along a straight road, car A passed car B twice. Prove that at some time during the race their accelerations were equal. State the assumptions that you make.

65. A rectangular beam will be cut from a cylindrical log of radius 30 cm.

(a) Show that the beam of maximal cross-sectional area is a square.

(b) Four rectangular planks will be cut from the four sections of the log that remain after cutting the square beam. Determine the dimensions of the planks that will have maximal cross-sectional area.

(c) Suppose that the strength of a rectangular beam is proportional to the product of its width and the square of its depth. Find the dimensions of the strongest beam that can be cut from the cylindrical log.



66. If a projectile is fired with an initial velocity v at an angle of inclination θ from the horizontal, then its trajectory, neglecting air resistance, is the parabola

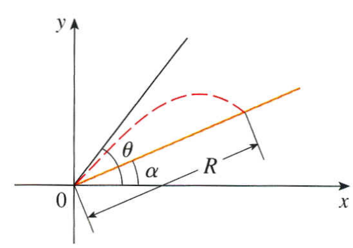
$$y = (\tan \theta)x - \frac{g}{2v^2 \cos^2\theta} x^2 \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

(a) Suppose the projectile is fired from the base of a plane that is inclined at an angle α, $\alpha > 0$, from the horizontal, as shown in the figure. Show that the range of the projectile, measured up the slope, is given by

$$R(\theta) = \frac{2v^2 \cos \theta \, \sin(\theta - \alpha)}{g \cos^2 \alpha}$$

(b) Determine θ so that R is a maximum.

(c) Suppose the plane is at an angle α *below* the horizontal. Determine the range R in this case, and determine the angle at which the projectile should be fired to maximize R.



PROBLEMS PLUS

PROBLEMS

1. Show that $|\sin x - \cos x| \leqslant \sqrt{2}$ for all x.

2. Show that $x^2y^2(4 - x^2)(4 - y^2) \leqslant 16$ for all numbers x and y such that $|x| \leqslant 2$ and $|y| \leqslant 2$.

3. Let a and b be positive numbers. Show that not both of the numbers $a(1 - b)$ and $b(1 - a)$ can be greater than $\frac{1}{4}$.

4. Find the point on the parabola $y = 1 - x^2$ at which the tangent line cuts from the first quadrant the triangle with the smallest area.

5. Find the highest and lowest points on the curve $x^2 + xy + y^2 = 12$.

6. Water is flowing at a constant rate into a spherical tank. Let $V(t)$ be the volume of water in the tank and $H(t)$ be the height of the water in the tank at time t.
   (a) What are the meanings of $V'(t)$ and $H'(t)$? Are these derivatives positive, negative, or zero?
   (b) Is $V''(t)$ positive, negative, or zero? Explain.
   (c) Let t_1, t_2, and t_3 be the times when the tank is one-quarter full, half full, and three-quarters full, respectively. Are the values $H''(t_1)$, $H''(t_2)$, and $H''(t_3)$ positive, negative, or zero? Why?

7. Find the absolute maximum value of the function

$$f(x) = \frac{1}{1 + |x|} + \frac{1}{1 + |x - 2|}$$

8. Find a function f such that $f'(-1) = \frac{1}{2}$, $f'(0) = 0$, and $f''(x) > 0$ for all x, or prove that such a function cannot exist.

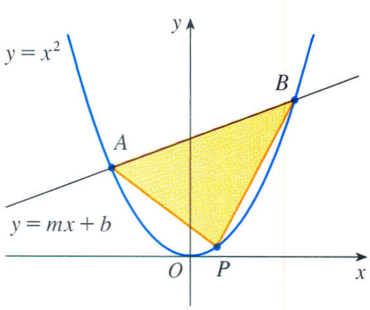


FIGURE FOR PROBLEM 9

9. The line $y = mx + b$ intersects the parabola $y = x^2$ in points A and B. (See the figure.) Find the point P on the arc AOB of the parabola that maximizes the area of the triangle PAB.

10. Sketch the graph of a function f such that $f'(x) < 0$ for all x, $f''(x) > 0$ for $|x| > 1$, $f''(x) < 0$ for $|x| < 1$, and $\lim_{x \to \pm\infty} [f(x) + x] = 0$.

11. Determine the values of the number a for which the function f has no critical number:

$$f(x) = (a^2 + a - 6)\cos 2x + (a - 2)x + \cos 1$$

12. Sketch the region in the plane consisting of all points (x, y) such that

$$2xy \leqslant |x - y| \leqslant x^2 + y^2$$

13. Let ABC be a triangle with $\angle BAC = 120°$ and $|AB| \cdot |AC| = 1$.
   (a) Express the length of the angle bisector AD in terms of $x = |AB|$.
   (b) Find the largest possible value of $|AD|$.

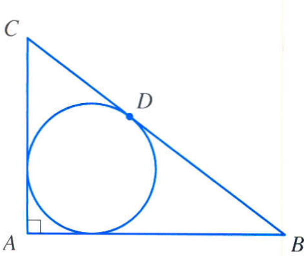


FIGURE FOR PROBLEM 14

14. (a) Let ABC be a triangle with right angle A and hypotenuse $a = |BC|$. (See the figure.) If the inscribed circle touches the hypotenuse at D, show that

$$|CD| = \tfrac{1}{2}(|BC| + |AC| - |AB|)$$

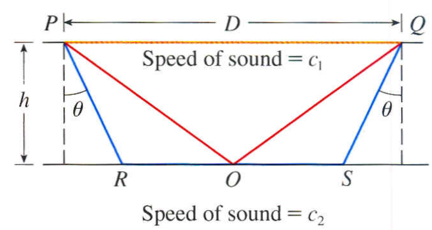
   (b) If $\theta = \frac{1}{2}\angle C$, express the radius r of the inscribed circle in terms of a and θ.
   (c) If a is fixed and θ varies, find the maximum value of r.

15. A triangle with sides a, b, and c varies with time t, but its area never changes. Let θ be the angle opposite the side of length a and suppose θ always remains acute.
   (a) Express $d\theta/dt$ in terms of b, c, θ, db/dt, and dc/dt.
   (b) Express da/dt in terms of the quantities in part (a).

16. $ABCD$ is a square piece of paper with sides of length 1 m. A quarter-circle is drawn from B to D with center A. The piece of paper is folded along EF, with E on AB and F on AD, so that A falls on the quarter-circle. Determine the maximum and minimum areas that the triangle AEF can have.

17. The speeds of sound c_1 in an upper layer and c_2 in a lower layer of rock and the thickness h of the upper layer can be determined by seismic exploration if the speed of sound in the lower layer is greater than the speed in the upper layer. A dynamite charge is detonated at a point P and the transmitted signals are recorded at a point Q, which is a distance D from P. The first signal to arrive at Q travels along the surface and takes T_1 seconds. The next signal travels from P to a point R, from R to S in the lower layer, and then to Q, taking T_2 seconds. The third signal is reflected off the lower layer at the midpoint O of RS and takes T_3 seconds to reach Q.

(a) Express T_1, T_2, and T_3 in terms of D, h, c_1, c_2, and θ.
(b) Show that T_2 is a minimum when $\sin\theta = c_1/c_2$.
(c) Suppose that $D = 1$ km, $T_1 = 0.26$ s, $T_2 = 0.32$ s, and $T_3 = 0.34$ s. Find c_1, c_2, and h.



Note: Geophysicists use this technique when studying the structure of the earth's crust, whether searching for oil or examining fault lines.

18. For what values of c is there a straight line that intersects the curve
$y = x^4 + cx^3 + 12x^2 - 5x + 2$ in four distinct points?

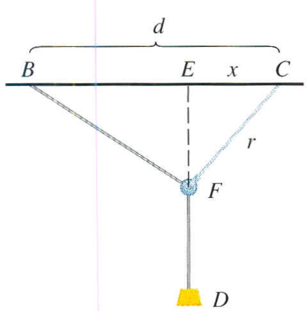


FIGURE FOR PROBLEM 19

19. One of the problems posed by the Marquis de l'Hospital in his calculus textbook *Analyse des Infiniment Petits* concerns a pulley that is attached to the ceiling of a room at a point C by a rope of length r. At another point B on the ceiling, at a distance d from C (where $d > r$), a rope of length ℓ is attached and passed through the pulley at F and connected to a weight W. The weight is released and comes to rest at its equilibrium position D. As l'Hospital argued, this happens when the distance $|ED|$ is maximized. Show that when the system reaches equilibrium, the value of x is

$$\frac{r}{4d}\left(r + \sqrt{r^2 + 8d^2}\right)$$

Notice that this expression is independent of both W and ℓ.

20. Given a sphere with radius r, find the height of a pyramid of minimum volume whose base is a square and whose base and triangular faces are all tangent to the sphere. What if the base of the pyramid is a regular n-gon? (A regular n-gon is a polygon with n equal sides and angles.) (Use the fact that the volume of a pyramid is $\frac{1}{3}Ah$, where A is the area of the base.)

21. Assume that a snowball melts so that its volume decreases at a rate proportional to its surface area. If it takes three hours for the snowball to decrease to half its original volume, how much longer will it take for the snowball to melt completely?

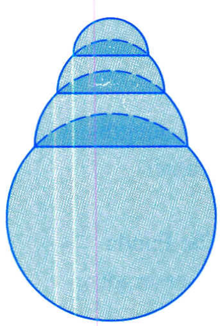


FIGURE FOR PROBLEM 22

22. A hemispherical bubble is placed on a spherical bubble of radius 1. A smaller hemispherical bubble is then placed on the first one. This process is continued until n chambers, including the sphere, are formed. (The figure shows the case $n = 4$.) Use mathematical induction to prove that the maximum height of any bubble tower with n chambers is $1 + \sqrt{n}$.