

In the following example we estimate the rate of change of the national debt with respect to time. Here the function is defined not by a formula but by a table of values.

t	$D(t)$
1994	414.0
1996	469.5
1998	467.3
2000	456.4
2002	442.3

EXAMPLE 7 Let $D(t)$ be the Canadian gross public debt at time t . The table in the margin gives approximate values of this function by providing midyear estimates, in billions of dollars, from 1994 to 2002. Interpret and estimate the value of $D'(1998)$.

SOLUTION The derivative $D'(1998)$ means the rate of change of D with respect to t when $t = 1998$, that is, the rate of increase of the national debt in 1998.

According to Equation 5,

$$D'(1998) = \lim_{t \rightarrow 1998} \frac{D(t) - D(1998)}{t - 1998}$$

So we compute and tabulate values of the difference quotient (the average rates of change) as shown in the table at the left. From this table we see that $D'(1998)$ lies somewhere between -1.1 and -5.5 billion dollars per year. [Here we are making the reasonable assumption that the debt didn't fluctuate wildly between 1998 and 2002.] We estimate that the rate of change of the Canadian debt in 1998 was the average of these two numbers, namely

$$D'(1998) \approx -3.3 \text{ billion dollars per year}$$

t	$\frac{D(t) - D(1998)}{t - 1998}$
1994	13.3
1996	-1.1
2000	-5.5
2002	-6.3

■ A NOTE ON UNITS

The units for the average rate of change $\Delta D/\Delta t$ are the units for ΔD divided by the units for Δt , namely, billions of dollars per year. The instantaneous rate of change is the limit of the average rates of change, so it is measured in the same units: billions of dollars per year.

The minus sign means that the debt was *decreasing* at that time.


Another method would be to plot the debt function and estimate the slope of the tangent line when $t = 1998$. □

In Examples 3, 6, and 7 we saw three specific examples of rates of change: the velocity of an object is the rate of change of displacement with respect to time; marginal cost is the rate of change of production cost with respect to the number of items produced; the rate of change of the debt with respect to time is of interest in economics. Here is a small sample of other rates of change: In physics, the rate of change of work with respect to time is called *power*. Chemists who study a chemical reaction are interested in the rate of change in the concentration of a reactant with respect to time (called the *rate of reaction*). A biologist is interested in the rate of change of the population of a colony of bacteria with respect to time. In fact, the computation of rates of change is important in all of the natural sciences, in engineering, and even in the social sciences. Further examples will be given in Section 3.7.

All these rates of change are derivatives and can therefore be interpreted as slopes of tangents. This gives added significance to the solution of the tangent problem. Whenever we solve a problem involving tangent lines, we are not just solving a problem in geometry. We are also implicitly solving a great variety of problems involving rates of change in science and engineering.



3.1 EXERCISES

- A curve has equation $y = f(x)$.
 - Write an expression for the slope of the secant line through the points $P(3, f(3))$ and $Q(x, f(x))$.
 - Write an expression for the slope of the tangent line at P .

-  2. Graph the curve $y = \sin x$ in the viewing rectangles $[-2, 2]$ by $[-2, 2]$, $[-1, 1]$ by $[-1, 1]$, and $[-0.5, 0.5]$ by $[-0.5, 0.5]$.

What do you notice about the curve as you zoom in toward the origin?


- Find the slope of the tangent line to the parabola $y = 4x - x^2$ at the point $(1, 3)$
 - using Definition 1
 - using Equation 2
 - Find an equation of the tangent line in part (a).


-  (c) Graph the parabola and the tangent line. As a check on your work, zoom in toward the point $(1, 3)$ until the parabola and the tangent line are indistinguishable.
4. (a) Find the slope of the tangent line to the curve $y = x - x^3$ at the point $(1, 0)$
 (i) using Definition 1 (ii) using Equation 2
 (b) Find an equation of the tangent line in part (a).
 (c) Graph the curve and the tangent line in successively smaller viewing rectangles centered at $(1, 0)$ until the curve and the line appear to coincide.

5–8 Find an equation of the tangent line to the curve at the given point.

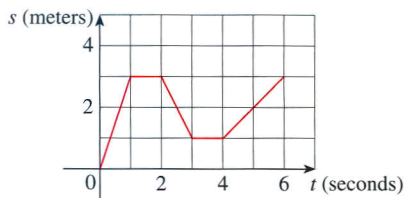
5. $y = \frac{x-1}{x-2}$, $(3, 2)$ 6. $y = 2x^3 - 5x$, $(-1, 3)$

7. $y = \sqrt{x}$, $(1, 1)$ 8. $y = \frac{2x}{(x+1)^2}$, $(0, 0)$

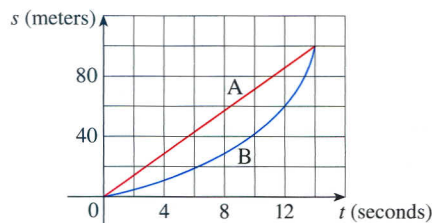
9. (a) Find the slope of the tangent to the curve $y = 3 + 4x^2 - 2x^3$ at the point where $x = a$.
 (b) Find equations of the tangent lines at the points $(1, 5)$ and $(2, 3)$.
 (c) Graph the curve and both tangents on a common screen.

10. (a) Find the slope of the tangent to the curve $y = 1/\sqrt{x}$ at the point where $x = a$.
 (b) Find equations of the tangent lines at the points $(1, 1)$ and $(4, \frac{1}{2})$.
 (c) Graph the curve and both tangents on a common screen.

11. (a) A particle starts by moving to the right along a horizontal line; the graph of its position function is shown. When is the particle moving to the right? Moving to the left? Standing still?
 (b) Draw a graph of the velocity function.



12. Shown are graphs of the position functions of two runners, A and B, who run a 100-m race and finish in a tie.



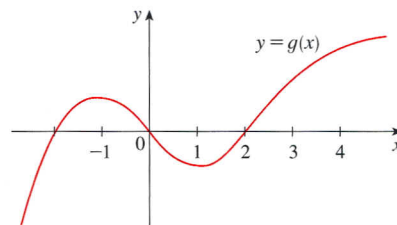
- (a) Describe and compare how the runners run the race.

- (b) At what time is the distance between the runners the greatest?
 (c) At what time do they have the same velocity?

13. If a ball is thrown into the air with a velocity of 10 m/s, its height (in meters) after t seconds is given by $y = 10t - 4.9t^2$. Find the velocity when $t = 2$.
14. If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height (in meters) after t seconds is given by $H = 10t - 1.86t^2$.
 (a) Find the velocity of the rock after one second.
 (b) Find the velocity of the rock when $t = a$.
 (c) When will the rock hit the surface?
 (d) With what velocity will the rock hit the surface?
15. The displacement (in meters) of a particle moving in a straight line is given by the equation of motion $s = 1/t^2$, where t is measured in seconds. Find the velocity of the particle at times $t = a$, $t = 1$, $t = 2$, and $t = 3$.
16. The displacement (in meters) of a particle moving in a straight line is given by $s = t^2 - 8t + 18$, where t is measured in seconds.
 (a) Find the average velocity over each time interval:
 (i) $[3, 4]$ (ii) $[3.5, 4]$
 (iii) $[4, 5]$ (iv) $[4, 4.5]$
 (b) Find the instantaneous velocity when $t = 4$.
 (c) Draw the graph of s as a function of t and draw the secant lines whose slopes are the average velocities in part (a) and the tangent line whose slope is the instantaneous velocity in part (b).

17. For the function g whose graph is given, arrange the following numbers in increasing order and explain your reasoning:

$$0 \quad g'(-2) \quad g'(0) \quad g'(2) \quad g'(4)$$



18. (a) Find an equation of the tangent line to the graph of $y = g(x)$ at $x = 5$ if $g(5) = -3$ and $g'(5) = 4$.
 (b) If the tangent line to $y = f(x)$ at $(4, 3)$ passes through the point $(0, 2)$, find $f(4)$ and $f'(4)$.
19. Sketch the graph of a function f for which $f(0) = 0$, $f'(0) = 3$, $f'(1) = 0$, and $f'(2) = -1$.
20. Sketch the graph of a function g for which $g(0) = g'(0) = 0$, $g'(-1) = -1$, $g'(1) = 3$, and $g'(2) = 1$.

21. If $f(x) = 3x^2 - 5x$, find $f'(2)$ and use it to find an equation of the tangent line to the parabola $y = 3x^2 - 5x$ at the point $(2, 2)$.
22. If $g(x) = 1 - x^3$, find $g'(0)$ and use it to find an equation of the tangent line to the curve $y = 1 - x^3$ at the point $(0, 1)$.
23. (a) If $F(x) = 5x/(1 + x^2)$, find $F'(2)$ and use it to find an equation of the tangent line to the curve $y = 5x/(1 + x^2)$ at the point $(2, 2)$.
- (b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.
24. (a) If $G(x) = 4x^2 - x^3$, find $G'(a)$ and use it to find equations of the tangent lines to the curve $y = 4x^2 - x^3$ at the points $(2, 8)$ and $(3, 9)$.
- (b) Illustrate part (a) by graphing the curve and the tangent lines on the same screen.

25–30 Find $f'(a)$.

25. $f(x) = 3 - 2x + 4x^2$ 26. $f(t) = t^4 - 5t$
27. $f(t) = \frac{2t + 1}{t + 3}$ 28. $f(x) = \frac{x^2 + 1}{x - 2}$
29. $f(x) = \frac{1}{\sqrt{x + 2}}$ 30. $f(x) = \sqrt{3x + 4}$

31–36 Each limit represents the derivative of some function f at some number a . State such an f and a in each case.

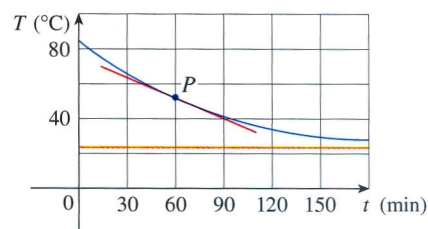
31. $\lim_{h \rightarrow 0} \frac{(1 + h)^{10} - 1}{h}$ 32. $\lim_{h \rightarrow 0} \frac{\sqrt[4]{16 + h} - 2}{h}$
33. $\lim_{x \rightarrow 5} \frac{2^x - 32}{x - 5}$ 34. $\lim_{x \rightarrow \pi/4} \frac{\tan x - 1}{x - \pi/4}$
35. $\lim_{h \rightarrow 0} \frac{\cos(\pi + h) + 1}{h}$ 36. $\lim_{t \rightarrow 1} \frac{t^4 + t - 2}{t - 1}$

37–38 A particle moves along a straight line with equation of motion $s = f(t)$, where s is measured in meters and t in seconds. Find the velocity and the speed when $t = 5$.

37. $f(t) = 100 + 50t - 4.9t^2$ 38. $f(t) = t^{-1} - t$

39. A warm can of soda pop is placed in a cold refrigerator. Sketch the graph of the temperature of the soda as a function of time. Is the initial rate of change of temperature greater or less than the rate of change after an hour?
40. A roast turkey is taken from an oven when its temperature has reached 85°C and is placed on a table in a room where the temperature is 24°C . The graph shows how the temperature of the turkey decreases and eventually approaches room

temperature. (In Section 7.5 we will be able to use Newton's Law of Cooling to find an equation for T as a function of time.) By measuring the slope of the tangent, estimate the rate of change of the temperature after an hour.



41. The table shows the estimated percentage P of the population of Europe that use cell phones. (Midyear estimates are given.)

Year	1998	1999	2000	2001	2002	2003
P	28	39	55	68	77	83

- (a) Find the average rate of cell phone growth
- (i) from 2000 to 2002 (ii) from 2000 to 2001
- (iii) from 1999 to 2000
- In each case, include the units.
- (b) Estimate the instantaneous rate of growth in 2000 by taking the average of two average rates of change. What are its units?
- (c) Estimate the instantaneous rate of growth in 2000 by measuring the slope of a tangent.

42. The number N of locations of a popular coffeehouse chain is given in the table. (The numbers of locations as of June 30 are given.)

Year	1998	1999	2000	2001	2002
N	1886	2135	3501	4709	5886

- (a) Find the average rate of growth
- (i) from 2000 to 2002 (ii) from 2000 to 2001
- (iii) from 1999 to 2000
- In each case, include the units.
- (b) Estimate the instantaneous rate of growth in 2000 by taking the average of two average rates of change. What are its units?
- (c) Estimate the instantaneous rate of growth in 2000 by measuring the slope of a tangent.

43. The cost (in dollars) of producing x units of a certain commodity is $C(x) = 5000 + 10x + 0.05x^2$.

- (a) Find the average rate of change of C with respect to x when the production level is changed
- (i) from $x = 100$ to $x = 105$
- (ii) from $x = 100$ to $x = 101$
- (b) Find the instantaneous rate of change of C with respect to x when $x = 100$. (This is called the *marginal cost*. Its significance will be explained in Section 3.7.)

44. If a cylindrical tank holds 100,000 liters of water, which can be drained from the bottom of the tank in an hour, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V(t) = 100,000 \left(1 - \frac{t}{60}\right)^2 \quad 0 \leq t \leq 60$$

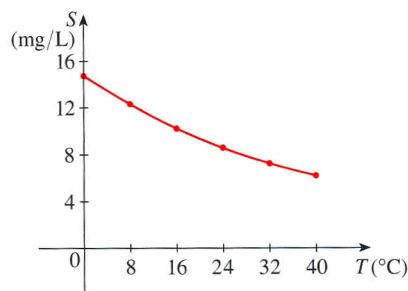
Find the rate at which the water is flowing out of the tank (the instantaneous rate of change of V with respect to t) as a function of t . What are its units? For times $t = 0, 10, 20, 30, 40, 50,$ and 60 min, find the flow rate and the amount of water remaining in the tank. Summarize your findings in a sentence or two. At what time is the flow rate the greatest? The least?

45. The cost of producing x kilograms of gold from a new gold mine is $C = f(x)$ dollars.
- What is the meaning of the derivative $f'(x)$? What are its units?
 - What does the statement $f'(50) = 36$ mean?
 - Do you think the values of $f'(x)$ will increase or decrease in the short term? What about the long term? Explain.
46. The number of bacteria after t hours in a controlled laboratory experiment is $n = f(t)$.
- What is the meaning of the derivative $f'(5)$? What are its units?
 - Suppose there is an unlimited amount of space and nutrients for the bacteria. Which do you think is larger, $f'(5)$ or $f'(10)$? If the supply of nutrients is limited, would that affect your conclusion? Explain.
47. Let $T(t)$ be the temperature (in $^{\circ}\text{C}$) in Seoul t hours after noon on August 21, 2004. The table shows values of this function recorded every two hours. What is the meaning of $T'(6)$? Estimate its value.

t	0	2	4	6	8	10
T	34.4	35.6	38.3	32.8	26.1	22.8

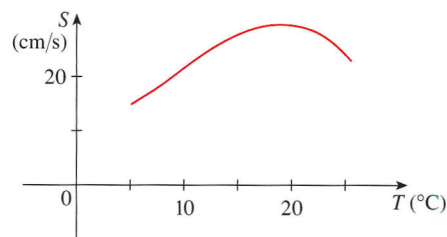
48. The quantity (in kilograms) of a gourmet ground coffee that is sold by a coffee company at a price of p dollars per kilogram is $Q = f(p)$.
- What is the meaning of the derivative $f'(8)$? What are its units?
 - Is $f'(8)$ positive or negative? Explain.
49. The quantity of oxygen that can dissolve in water depends on the temperature of the water. (So thermal pollution influences

- the oxygen content of water.) The graph shows how oxygen solubility S varies as a function of the water temperature T .
- What is the meaning of the derivative $S'(T)$? What are its units?
 - Estimate the value of $S'(16)$ and interpret it.



Adapted from *Environmental Science: Living Within the System of Nature*, 2d ed.; by Charles E. Kupchella, © 1989. Reprinted by permission of Prentice-Hall, Inc., Upper Saddle River, NJ.

50. The graph shows the influence of the temperature T on the maximum sustainable swimming speed S of Coho salmon.
- What is the meaning of the derivative $S'(T)$? What are its units?
 - Estimate the values of $S'(15)$ and $S'(25)$ and interpret them.



- 51–52 Determine whether $f'(0)$ exists.

$$51. f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$52. f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

slope of $f''(x)$, we have

$$f'''(x) = 6$$

for all values of x . So f''' is a constant function and its graph is a horizontal line. Therefore, for all values of x ,

$$f^{(4)}(x) = 0$$

□

We can interpret the third derivative physically in the case where the function is the position function $s = s(t)$ of an object that moves along a straight line. Because $s''' = (s'')' = a'$, the third derivative of the position function is the derivative of the acceleration function and is called the **jerk**:

$$j = \frac{da}{dt} = \frac{d^3s}{dt^3}$$

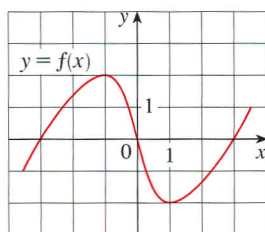
Thus the jerk j is the rate of change of acceleration. It is aptly named because a large jerk means a sudden change in acceleration, which causes an abrupt movement in a vehicle.

We have seen that one application of second and third derivatives occurs in analyzing the motion of objects using acceleration and jerk. We will investigate another application of second derivatives in Section 4.3, where we show how knowledge of f'' gives us information about the shape of the graph of f . In Chapter 12 we will see how second and higher derivatives enable us to represent functions as sums of infinite series.

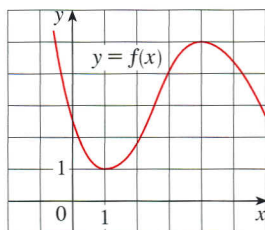
3.2 EXERCISES

1–2 Use the given graph to estimate the value of each derivative. Then sketch the graph of f' .

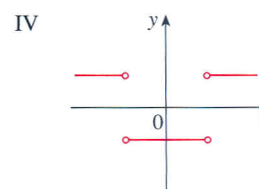
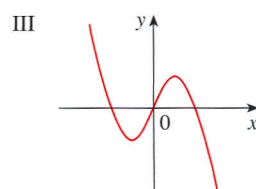
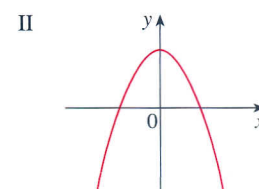
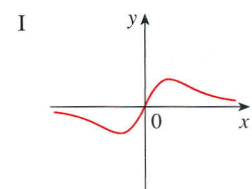
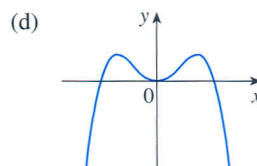
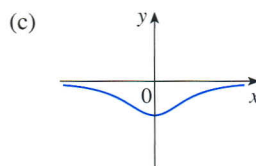
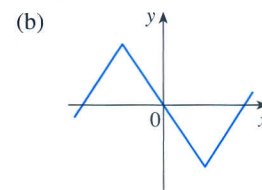
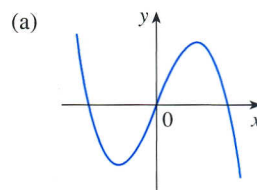
1. (a) $f'(-3)$ (b) $f'(-2)$
 (c) $f'(-1)$ (d) $f'(0)$
 (e) $f'(1)$ (f) $f'(2)$
 (g) $f'(3)$



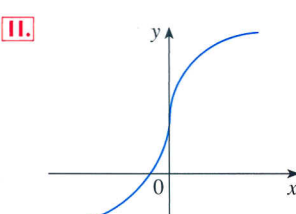
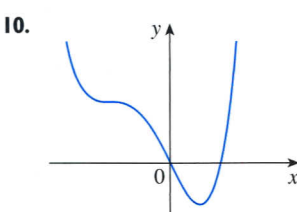
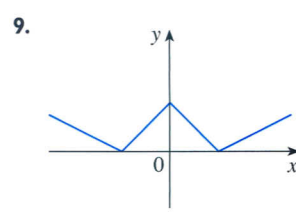
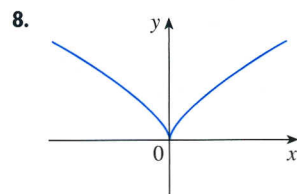
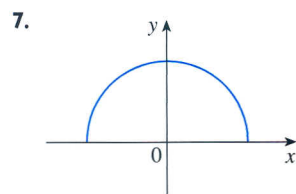
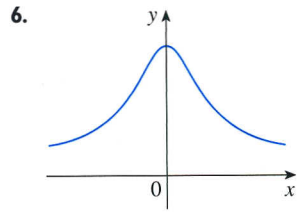
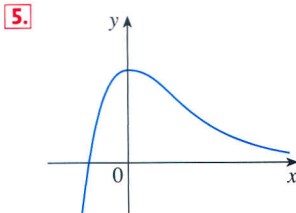
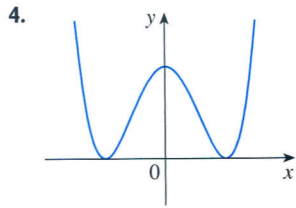
2. (a) $f'(0)$ (b) $f'(1)$
 (c) $f'(2)$ (d) $f'(3)$
 (e) $f'(4)$ (f) $f'(5)$



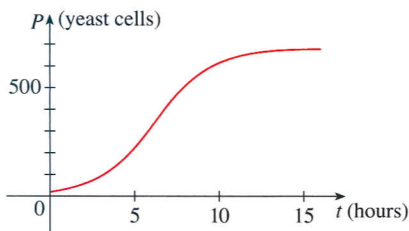
3. Match the graph of each function in (a)–(d) with the graph of its derivative in I–IV. Give reasons for your choices.



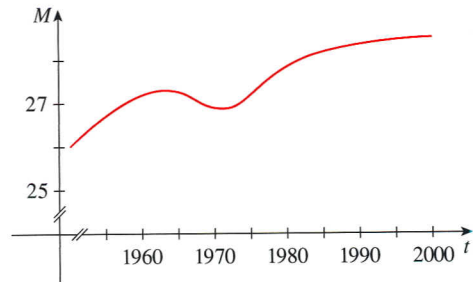
4–11 Trace or copy the graph of the given function f . (Assume that the axes have equal scales.) Then use the method of Example 1 to sketch the graph of f' below it.




12. Shown is the graph of the population function $P(t)$ for yeast cells in a laboratory culture. Use the method of Example 1 to graph the derivative $P'(t)$. What does the graph of P' tell us about the yeast population?



13. The graph shows how the average age of first marriage of Japanese men has varied in the last half of the 20th century. Sketch the graph of the derivative function $M'(t)$. During which years was the derivative negative?



14. Make a careful sketch of the graph of the sine function and below it sketch the graph of its derivative in the same manner as in Exercises 4–11. Can you guess what the derivative of the sine function is from its graph?

 **15.** Let $f(x) = x^2$.

- Estimate the values of $f'(0)$, $f'(\frac{1}{2})$, $f'(1)$, and $f'(2)$ by using a graphing device to zoom in on the graph of f .
- Use symmetry to deduce the values of $f'(-\frac{1}{2})$, $f'(-1)$, and $f'(-2)$.
- Use the results from parts (a) and (b) to guess a formula for $f'(x)$.
- Use the definition of a derivative to prove that your guess in part (c) is correct.

 **16.** Let $f(x) = x^3$.

- Estimate the values of $f'(0)$, $f'(\frac{1}{2})$, $f'(1)$, $f'(2)$, and $f'(3)$ by using a graphing device to zoom in on the graph of f .
- Use symmetry to deduce the values of $f'(-\frac{1}{2})$, $f'(-1)$, $f'(-2)$, and $f'(-3)$.
- Use the values from parts (a) and (b) to graph f' .
- Guess a formula for $f'(x)$.
- Use the definition of a derivative to prove that your guess in part (d) is correct.

17–27 Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

17. $f(x) = \frac{1}{2}x - \frac{1}{3}$


18. $f(x) = mx + b$

19. $f(t) = 5t - 9t^2$


20. $f(x) = 1.5x^2 - x + 3.7$

21. $f(x) = x^3 - 3x + 5$

22. $f(x) = x + \sqrt{x}$

 **23.** $g(x) = \sqrt{1 + 2x}$

24. $f(x) = \frac{3 + x}{1 - 3x}$

 **25.** $G(t) = \frac{4t}{t + 1}$

26. $g(t) = \frac{1}{\sqrt{t}}$

27. $f(x) = x^4$

28. (a) Sketch the graph of $f(x) = \sqrt{6-x}$ by starting with the graph of $y = \sqrt{x}$ and using the transformations of Section 1.3.
 (b) Use the graph from part (a) to sketch the graph of f' .
 (c) Use the definition of a derivative to find $f'(x)$. What are the domains of f and f' ?
 (d) Use a graphing device to graph f' and compare with your sketch in part (b).
29. (a) If $f(x) = x^4 + 2x$, find $f'(x)$.
 (b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f' .
30. (a) If $f(t) = t^2 - \sqrt{t}$, find $f'(t)$.
 (b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f' .

31. The unemployment rate $U(t)$ varies with time. The table gives the percentage of unemployed in the Australian labor force measured at midyear from 1995 to 2004.

t	$U(t)$	t	$U(t)$
1995	8.1	2000	6.2
1996	8.0	2001	6.9
1997	8.2	2002	6.5
1998	7.9	2003	6.2
1999	6.7	2004	5.6

- (a) What is the meaning of $U'(t)$? What are its units?
 (b) Construct a table of values for $U'(t)$.

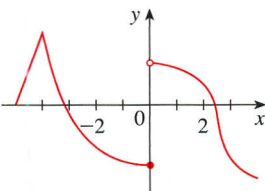
32. Let $P(t)$ be the percentage of the population of the Philippines over the age of 60 at time t . The table gives projections of values of this function from 1995 to 2020.

t	$P(t)$	t	$P(t)$
1995	5.2	2010	6.7
2000	5.5	2015	7.7
2005	6.1	2020	8.9

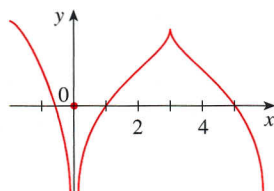
- (a) What is the meaning of $P'(t)$? What are its units?
 (b) Construct a table of estimated values for $P'(t)$.
 (c) Graph P and P' .

- 33–36 The graph of f is given. State, with reasons, the numbers at which f is not differentiable.

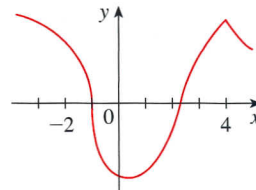
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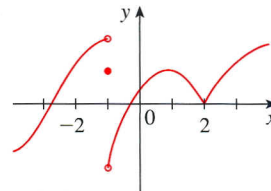
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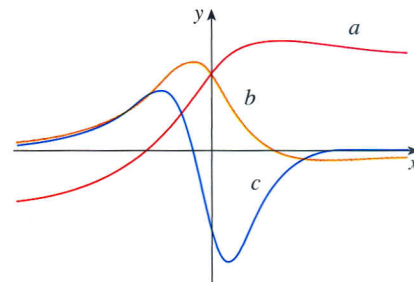
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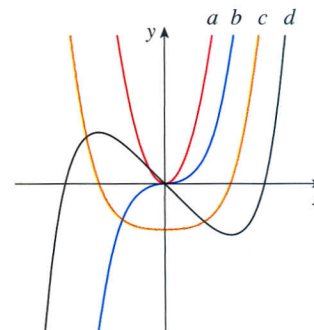
37. Graph the function $f(x) = x + \sqrt{|x|}$. Zoom in repeatedly, first toward the point $(-1, 0)$ and then toward the origin. What is different about the behavior of f in the vicinity of these two points? What do you conclude about the differentiability of f ?

38. Zoom in toward the points $(1, 0)$, $(0, 1)$, and $(-1, 0)$ on the graph of the function $g(x) = (x^2 - 1)^{2/3}$. What do you notice? Account for what you see in terms of the differentiability of g .

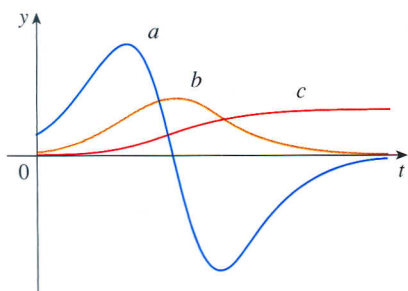
39. The figure shows the graphs of f , f' , and f'' . Identify each curve, and explain your choices.



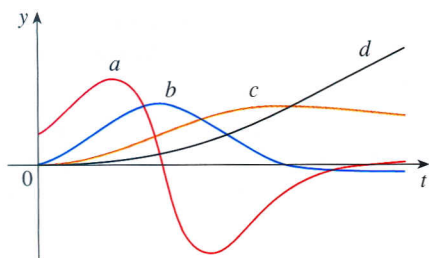
40. The figure shows graphs of f , f' , f'' , and f''' . Identify each curve, and explain your choices.



41. The figure shows the graphs of three functions. One is the position function of a car, one is the velocity of the car, and one is its acceleration. Identify each curve, and explain your choices.



42. The figure shows the graphs of four functions. One is the position function of a car, one is the velocity of the car, one is its acceleration, and one is its jerk. Identify each curve, and explain your choices.



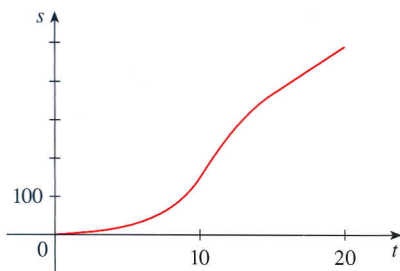
- 43–44 Use the definition of a derivative to find $f'(x)$ and $f''(x)$. Then graph f , f' , and f'' on a common screen and check to see if your answers are reasonable.

43. $f(x) = 1 + 4x - x^2$

44. $f(x) = 1/x$

45. If $f(x) = 2x^2 - x^3$, find $f'(x)$, $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$. Graph f , f' , f'' , and f''' on a common screen. Are the graphs consistent with the geometric interpretations of these derivatives?

46. (a) The graph of a position function of a car is shown, where s is measured in meters and t in seconds. Use it to graph the velocity and acceleration of the car. What is the acceleration at $t = 10$ seconds?



- (b) Use the acceleration curve from part (a) to estimate the jerk at $t = 10$ seconds. What are the units for jerk?
47. Let $f(x) = \sqrt[3]{x}$.
- (a) If $a \neq 0$, use Equation 3.1.5 to find $f'(a)$.

- (b) Show that $f'(0)$ does not exist.
- (c) Show that $y = \sqrt[3]{x}$ has a vertical tangent line at $(0, 0)$. (Recall the shape of the graph of f . See Figure 13 in Section 1.2.)

48. (a) If $g(x) = x^{2/3}$, show that $g'(0)$ does not exist.
- (b) If $a \neq 0$, find $g'(a)$.
- (c) Show that $y = x^{2/3}$ has a vertical tangent line at $(0, 0)$.
49. Illustrate part (c) by graphing $y = x^{2/3}$.

49. Show that the function $f(x) = |x - 6|$ is not differentiable at 6. Find a formula for f' and sketch its graph.

50. Where is the greatest integer function $f(x) = \llbracket x \rrbracket$ not differentiable? Find a formula for f' and sketch its graph.

51. (a) Sketch the graph of the function $f(x) = x|x|$.
- (b) For what values of x is f differentiable?
- (c) Find a formula for f' .

52. The left-hand and right-hand derivatives of f at a are defined by

$$f'_-(a) = \lim_{h \rightarrow 0^-} \frac{f(a+h) - f(a)}{h}$$

and
$$f'_+(a) = \lim_{h \rightarrow 0^+} \frac{f(a+h) - f(a)}{h}$$

if these limits exist. Then $f'(a)$ exists if and only if these one-sided derivatives exist and are equal.

- (a) Find $f'_-(4)$ and $f'_+(4)$ for the function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 5 - x & \text{if } 0 < x < 4 \\ \frac{1}{5 - x} & \text{if } x \geq 4 \end{cases}$$

- (b) Sketch the graph of f .
- (c) Where is f discontinuous?
- (d) Where is f not differentiable?

53. Recall that a function f is called *even* if $f(-x) = f(x)$ for all x in its domain and *odd* if $f(-x) = -f(x)$ for all such x . Prove each of the following.

- (a) The derivative of an even function is an odd function.
- (b) The derivative of an odd function is an even function.

54. When you turn on a hot-water faucet, the temperature T of the water depends on how long the water has been running.

- (a) Sketch a possible graph of T as a function of the time t that has elapsed since the faucet was turned on.
- (b) Describe how the rate of change of T with respect to t varies as t increases.
- (c) Sketch a graph of the derivative of T .

55. Let ℓ be the tangent line to the parabola $y = x^2$ at the point $(1, 1)$. The *angle of inclination* of ℓ is the angle ϕ that ℓ makes with the positive direction of the x -axis. Calculate ϕ correct to the nearest degree.

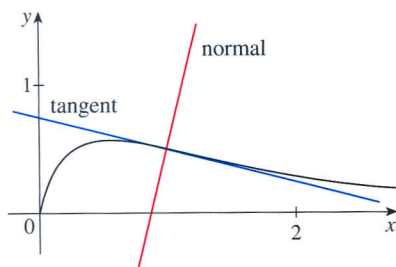


FIGURE 5

So the slope of the tangent line at $(1, \frac{1}{2})$ is

$$\left. \frac{dy}{dx} \right|_{x=1} = \frac{1 - 3 \cdot 1^2}{2\sqrt{1}(1 + 1^2)^2} = -\frac{1}{4}$$

We use the point-slope form to write an equation of the tangent line at $(1, \frac{1}{2})$:

$$y - \frac{1}{2} = -\frac{1}{4}(x - 1) \quad \text{or} \quad y = -\frac{1}{4}x + \frac{3}{4}$$

The slope of the normal line at $(1, \frac{1}{2})$ is the negative reciprocal of $-\frac{1}{4}$, namely 4, so an equation is

$$y - \frac{1}{2} = 4(x - 1) \quad \text{or} \quad y = 4x - \frac{7}{2}$$

The curve and its tangent and normal lines are graphed in Figure 5. □

EXAMPLE 13 At what points on the hyperbola $xy = 12$ is the tangent line parallel to the line $3x + y = 0$?

SOLUTION Since $xy = 12$ can be written as $y = 12/x$, we have

$$\frac{dy}{dx} = 12 \frac{d}{dx}(x^{-1}) = 12(-x^{-2}) = -\frac{12}{x^2}$$

Let the x -coordinate of one of the points in question be a . Then the slope of the tangent line at that point is $-12/a^2$. This tangent line will be parallel to the line $3x + y = 0$, or $y = -3x$, if it has the same slope, that is, -3 . Equating slopes, we get

$$-\frac{12}{a^2} = -3 \quad \text{or} \quad a^2 = 4 \quad \text{or} \quad a = \pm 2$$

Therefore the required points are $(2, 6)$ and $(-2, -6)$. The hyperbola and the tangents are shown in Figure 6. □

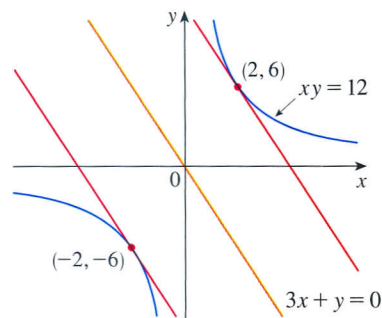


FIGURE 6

We summarize the differentiation formulas we have learned so far as follows.

TABLE OF DIFFERENTIATION FORMULAS

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$(cf)' = cf'$$

$$(f + g)' = f' + g'$$

$$(f - g)' = f' - g'$$

$$(fg)' = fg' + gf'$$

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

3.3 EXERCISES

1–20 Differentiate the function.

1. $f(x) = 186.5$

2. $f(x) = \sqrt{30}$

7. $f(t) = \frac{1}{4}(t^4 + 8)$

8. $f(t) = \frac{1}{2}t^6 - 3t^4 + t$

3. $f(x) = 5x - 1$

4. $F(x) = -4x^{10}$

9. $V(r) = \frac{4}{3}\pi r^3$

10. $R(t) = 5t^{-3/5}$

5. $f(x) = x^3 - 4x + 6$

6. $h(x) = (x - 2)(2x + 3)$

11. $Y(t) = 6t^{-9}$

12. $R(x) = \frac{\sqrt{10}}{x^7}$

13. $F(x) = \left(\frac{1}{2}x\right)^5$

14. $f(t) = \sqrt{t} - \frac{1}{\sqrt{t}}$

15. $Y(t) = 6t^{-9}$

16. $R(x) = \frac{\sqrt{10}}{x^7}$

17. $y = 4\pi^2$

18. $g(u) = \sqrt{2}u + \sqrt{3}u$

19. $u = \sqrt[3]{t} + 4\sqrt{t^5}$

20. $v = \left(\sqrt{x} + \frac{1}{\sqrt[3]{x}}\right)^2$

21. Find the derivative of $y = (x^2 + 1)(x^3 + 1)$ in two ways: by using the Product Rule and by performing the multiplication first. Do your answers agree?

22. Find the derivative of the function

$$F(x) = \frac{x - 3x\sqrt{x}}{\sqrt{x}}$$

in two ways: by using the Quotient Rule and by simplifying first. Show that your answers are equivalent. Which method do you prefer?

23–42 Differentiate.

23. $V(x) = (2x^3 + 3)(x^4 - 2x)$

24. $Y(u) = (u^{-2} + u^{-3})(u^5 - 2u^2)$

25. $F(y) = \left(\frac{1}{y^2} - \frac{3}{y^4}\right)(y + 5y^3)$

26. $y = \sqrt{x}(x - 1)$

27. $g(x) = \frac{3x - 1}{2x + 1}$

29. $y = \frac{x^3}{1 - x^2}$

31. $y = \frac{v^3 - 2v\sqrt{v}}{v}$

33. $y = \frac{t^2}{3t^2 - 2t + 1}$

35. $y = ax^2 + bx + c$

37. $y = \frac{r^2}{1 + \sqrt{r}}$

39. $y = \sqrt[3]{t}(t^2 + t + t^{-1})$

41. $f(x) = \frac{x}{x + \frac{c}{x}}$

28. $f(t) = \frac{2t}{4 + t^2}$

30. $y = \frac{x + 1}{x^3 + x - 2}$

32. $y = \frac{t^3 + t}{t^4 - 2}$

34. $g(t) = \frac{t - \sqrt{t}}{t^{1/3}}$

36. $y = A + \frac{B}{x} + \frac{C}{x^2}$

38. $y = \frac{cx}{1 + cx}$

40. $y = \frac{u^6 - 2u^3 + 5}{u^2}$

42. $f(x) = \frac{ax + b}{cx + d}$

43. The general polynomial of degree n has the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $a_n \neq 0$. Find the derivative of P .

44–46 Find $f'(x)$. Compare the graphs of f and f' and use them to explain why your answer is reasonable.

44. $f(x) = x/(x^2 - 1)$

45. $f(x) = 3x^{15} - 5x^3 + 3$

46. $f(x) = x + \frac{1}{x}$

47. (a) Use a graphing calculator or computer to graph the function $f(x) = x^4 - 3x^3 - 6x^2 + 7x + 30$ in the viewing rectangle $[-3, 5]$ by $[-10, 50]$.

(b) Using the graph in part (a) to estimate slopes, make a rough sketch, by hand, of the graph of f' . (See Example 1 in Section 3.2.)

(c) Calculate $f'(x)$ and use this expression, with a graphing device, to graph f' . Compare with your sketch in part (b).

48. (a) Use a graphing calculator or computer to graph the function $g(x) = x^2/(x^2 + 1)$ in the viewing rectangle $[-4, 4]$ by $[-1, 1.5]$.

(b) Using the graph in part (a) to estimate slopes, make a rough sketch, by hand, of the graph of g' . (See Example 1 in Section 3.2.)

(c) Calculate $g'(x)$ and use this expression, with a graphing device, to graph g' . Compare with your sketch in part (b).

49–50 Find an equation of the tangent line to the curve at the given point.

49. $y = \frac{2x}{x + 1}, (1, 1)$

50. $y = x^4 + 2x^2 - x, (1, 2)$

51. (a) The curve $y = 1/(1 + x^2)$ is called a **witch of Maria Agnesi**. Find an equation of the tangent line to this curve at the point $(-1, \frac{1}{2})$.

(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

52. (a) The curve $y = x/(1 + x^2)$ is called a **serpentine**. Find an equation of the tangent line to this curve at the point $(3, 0.3)$.

(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

53–56 Find equations of the tangent line and normal line to the curve at the given point.

53. $y = x + \sqrt{x}, (1, 2)$

54. $y = (1 + 2x)^2, (1, 9)$

55. $y = \frac{3x + 1}{x^2 + 1}, (1, 2)$

56. $y = \frac{\sqrt{x}}{x + 1}, (4, 0.4)$

57–60 Find the first and second derivatives of the function.

57. $f(x) = x^4 - 3x^3 + 16x$ **58.** $G(r) = \sqrt{r} + \sqrt[3]{r}$

59. $f(x) = \frac{x^2}{1 + 2x}$ **60.** $f(x) = \frac{1}{3 - x}$

- 61.** The equation of motion of a particle is $s = t^3 - 3t$, where s is in meters and t is in seconds. Find
- the velocity and acceleration as functions of t ,
 - the acceleration after 2 s, and
 - the acceleration when the velocity is 0.

- 62.** The equation of motion of a particle is $s = 2t^3 - 7t^2 + 4t + 1$, where s is in meters and t is in seconds.



- Find the velocity and acceleration as functions of t .
- Find the acceleration after 1 s.
- Graph the position, velocity, and acceleration functions on the same screen.

- 63.** Suppose that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$. Find the following values.

(a) $(fg)'(5)$ (b) $(f/g)'(5)$
 (c) $(g/f)'(5)$

- 64.** Find $h'(2)$, given that $f(2) = -3$, $g(2) = 4$, $f'(2) = -2$, and $g'(2) = 7$.

(a) $h(x) = 5f(x) - 4g(x)$ (b) $h(x) = f(x)g(x)$
 (c) $h(x) = \frac{f(x)}{g(x)}$ (d) $h(x) = \frac{g(x)}{1 + f(x)}$

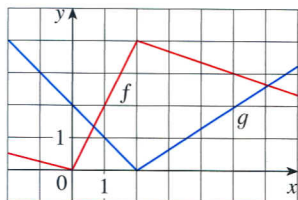
- 65.** If $f(x) = \sqrt{x}g(x)$, where $g(4) = 8$ and $g'(4) = 7$, find $f'(4)$.

- 66.** If $h(2) = 4$ and $h'(2) = -3$, find

$$\left. \frac{d}{dx} \left(\frac{h(x)}{x} \right) \right|_{x=2}$$

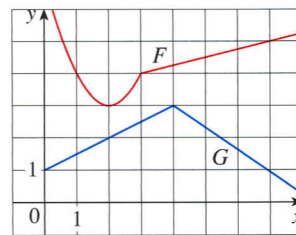
- 67.** If f and g are the functions whose graphs are shown, let $u(x) = f(x)g(x)$ and $v(x) = f(x)/g(x)$.

(a) Find $u'(1)$. (b) Find $v'(5)$.



- 68.** Let $P(x) = F(x)G(x)$ and $Q(x) = F(x)/G(x)$, where F and G are the functions whose graphs are shown.

(a) Find $P'(2)$. (b) Find $Q'(7)$.



- 69.** If g is a differentiable function, find an expression for the derivative of each of the following functions.

(a) $y = xg(x)$ (b) $y = \frac{x}{g(x)}$ (c) $y = \frac{g(x)}{x}$

- 70.** If f is a differentiable function, find an expression for the derivative of each of the following functions.

(a) $y = x^2f(x)$ (b) $y = \frac{f(x)}{x^2}$
 (c) $y = \frac{x^2}{f(x)}$ (d) $y = \frac{1 + xf(x)}{\sqrt{x}}$

- 71.** Find the points on the curve $y = 2x^3 + 3x^2 - 12x + 1$ where the tangent is horizontal.

- 72.** For what values of x does the graph of $f(x) = x^3 + 3x^2 + x + 3$ have a horizontal tangent?

- 73.** Show that the curve $y = 6x^3 + 5x - 3$ has no tangent line with slope 4.

- 74.** Find an equation of the tangent line to the curve $y = x\sqrt{x}$ that is parallel to the line $y = 1 + 3x$.

- 75.** Find equations of both lines that are tangent to the curve $y = 1 + x^3$ and are parallel to the line $12x - y = 1$.

- 76.** Find equations of the tangent lines to the curve

$$y = \frac{x - 1}{x + 1}$$

that are parallel to the line $x - 2y = 2$.

- 77.** Find an equation of the normal line to the parabola $y = x^2 - 5x + 4$ that is parallel to the line $x - 3y = 5$.

- 78.** Where does the normal line to the parabola $y = x - x^2$ at the point $(1, 0)$ intersect the parabola a second time? Illustrate with a sketch.

- 79.** Draw a diagram to show that there are two tangent lines to the parabola $y = x^2$ that pass through the point $(0, -4)$. Find the coordinates of the points where these tangent lines intersect the parabola.

80. (a) Find equations of both lines through the point $(2, -3)$ that are tangent to the parabola $y = x^2 + x$.
 (b) Show that there is no line through the point $(2, 7)$ that is tangent to the parabola. Then draw a diagram to see why.

81. (a) Use the Product Rule twice to prove that if f , g , and h are differentiable, then $(fgh)' = f'gh + fg'h + fgh'$.
 (b) Taking $f = g = h$ in part (a), show that

$$\frac{d}{dx} [f(x)]^3 = 3[f(x)]^2 f'(x)$$

- (c) Use part (b) to differentiate $y = (x^4 + 3x^3 + 17x + 82)^3$.
 82. Find the n th derivative of each function by calculating the first few derivatives and observing the pattern that occurs.
 (a) $f(x) = x^n$ (b) $f(x) = 1/x$

83. Find a second-degree polynomial P such that $P(2) = 5$, $P'(2) = 3$, and $P''(2) = 2$.

84. The equation $y'' + y' - 2y = x^2$ is called a **differential equation** because it involves an unknown function y and its derivatives y' and y'' . Find constants A , B , and C such that the function $y = Ax^2 + Bx + C$ satisfies this equation. (Differential equations will be studied in detail in Chapter 10.)

85. Find a cubic function $y = ax^3 + bx^2 + cx + d$ whose graph has horizontal tangents at the points $(-2, 6)$ and $(2, 0)$.

86. Find a parabola with equation $y = ax^2 + bx + c$ that has slope 4 at $x = 1$, slope -8 at $x = -1$, and passes through the point $(2, 15)$.

87. In this exercise we estimate the rate at which the total personal income is rising in the Richmond-Petersburg, Virginia, metropolitan area. In 1999, the population of this area was 961,400, and the population was increasing at roughly 9200 people per year. The average annual income was \$30,593 per capita, and this average was increasing at about \$1400 per year (a little above the national average of about \$1225 yearly). Use the Product Rule and these figures to estimate the rate at which total personal income was rising in the Richmond-Petersburg area in 1999. Explain the meaning of each term in the Product Rule.

88. A manufacturer produces bolts of a fabric with a fixed width. The quantity q of this fabric (measured in meters) that is sold is a function of the selling price p (in dollars per meter), so we can write $q = f(p)$. Then the total revenue earned with selling price p is $R(p) = pf(p)$.

- (a) What does it mean to say that $f(20) = 10,000$ and $f'(20) = -350$?
 (b) Assuming the values in part (a), find $R'(20)$ and interpret your answer.

89. Let

$$f(x) = \begin{cases} 2 - x & \text{if } x \leq 1 \\ x^2 - 2x + 2 & \text{if } x > 1 \end{cases}$$

Is f differentiable at 1? Sketch the graphs of f and f' .

90. At what numbers is the following function g differentiable?

$$g(x) = \begin{cases} -1 - 2x & \text{if } x < -1 \\ x^2 & \text{if } -1 \leq x \leq 1 \\ x & \text{if } x > 1 \end{cases}$$

Give a formula for g' and sketch the graphs of g and g' .

91. (a) For what values of x is the function $f(x) = |x^2 - 9|$ differentiable? Find a formula for f' .
 (b) Sketch the graphs of f and f' .

92. Where is the function $h(x) = |x - 1| + |x + 2|$ differentiable? Give a formula for h' and sketch the graphs of h and h' .

93. For what values of a and b is the line $2x + y = b$ tangent to the parabola $y = ax^2$ when $x = 2$?

94. (a) If $F(x) = f(x)g(x)$, where f and g have derivatives of all orders, show that $F'' = f''g + 2f'g' + fg''$.

(b) Find similar formulas for F''' and $F^{(4)}$.

(c) Guess a formula for $F^{(n)}$.

95. Find the value of c such that the line $y = \frac{3}{2}x + 6$ is tangent to the curve $y = c\sqrt{x}$.

96. Let

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ mx + b & \text{if } x > 2 \end{cases}$$

Find the values of m and b that make f differentiable everywhere.

97. An easy proof of the Quotient Rule can be given if we make the prior assumption that $F'(x)$ exists, where $F = f/g$. Write $f = Fg$; then differentiate using the Product Rule and solve the resulting equation for F' .

98. A tangent line is drawn to the hyperbola $xy = c$ at a point P .

(a) Show that the midpoint of the line segment cut from this tangent line by the coordinate axes is P .

(b) Show that the triangle formed by the tangent line and the coordinate axes always has the same area, no matter where P is located on the hyperbola.

99. Evaluate $\lim_{x \rightarrow 1} \frac{x^{1000} - 1}{x - 1}$.

100. Draw a diagram showing two perpendicular lines that intersect on the y -axis and are both tangent to the parabola $y = x^2$. Where do these lines intersect?

101. If $c > \frac{1}{2}$, how many lines through the point $(0, c)$ are normal lines to the parabola $y = x^2$? What if $c \leq \frac{1}{2}$?

102. Sketch the parabolas $y = x^2$ and $y = x^2 - 2x + 2$. Do you think there is a line that is tangent to both curves? If so, find its equation. If not, why not?

EXAMPLE 6 Calculate $\lim_{x \rightarrow 0} x \cot x$.

SOLUTION Here we divide numerator and denominator by x :

$$\begin{aligned} \lim_{x \rightarrow 0} x \cot x &= \lim_{x \rightarrow 0} \frac{x \cos x}{\sin x} = \lim_{x \rightarrow 0} \frac{\cos x}{\frac{\sin x}{x}} = \frac{\lim_{x \rightarrow 0} \cos x}{\lim_{x \rightarrow 0} \frac{\sin x}{x}} \\ &= \frac{\cos 0}{1} \quad (\text{by the continuity of cosine and Equation 2}) \\ &= 1 \end{aligned}$$

□

3.4 EXERCISES

1–16 Differentiate.

- | | |
|---|--|
| 1. $f(x) = x - 3 \sin x$ | 2. $f(x) = x \sin x$ |
| 3. $y = \sin x + 10 \tan x$ | 4. $y = 2 \csc x + 5 \cos x$ |
| 5. $g(t) = t^3 \cos t$ | 6. $g(t) = 4 \sec t + \tan t$ |
| 7. $h(\theta) = \theta \csc \theta - \cot \theta$ | 8. $y = u(a \cos u + b \cot u)$ |
| 9. $y = \frac{x}{2 - \tan x}$ | 10. $y = \frac{1 + \sin x}{x + \cos x}$ |
| 11. $f(\theta) = \frac{\sec \theta}{1 + \sec \theta}$ | 12. $y = \frac{1 - \sec x}{\tan x}$ |
| 13. $y = \frac{\sin x}{x^2}$ | 14. $y = \csc \theta (\theta + \cot \theta)$ |
| 15. $y = \sec \theta \tan \theta$ | 16. $y = x^2 \sin x \tan x$ |

17. Prove that $\frac{d}{dx} (\csc x) = -\csc x \cot x$.

18. Prove that $\frac{d}{dx} (\sec x) = \sec x \tan x$.

19. Prove that $\frac{d}{dx} (\cot x) = -\csc^2 x$.

20. Prove, using the definition of derivative, that if $f(x) = \cos x$, then $f'(x) = -\sin x$.

21–24 Find an equation of the tangent line to the curve at the given point.

21. $y = \sec x$, $(\pi/3, 2)$ 22. $y = (1 + x) \cos x$, $(0, 1)$

23. $y = x + \cos x$, $(0, 1)$ 24. $y = \frac{1}{\sin x + \cos x}$, $(0, 1)$

25. (a) Find an equation of the tangent line to the curve $y = 2x \sin x$ at the point $(\pi/2, \pi)$.

(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

26. (a) Find an equation of the tangent line to the curve $y = \sec x - 2 \cos x$ at the point $(\pi/3, 1)$.

(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

27. (a) If $f(x) = \sec x - x$, find $f'(x)$.

(b) Check to see that your answer to part (a) is reasonable by graphing both f and f' for $|x| < \pi/2$.

28. (a) If $f(x) = \sqrt{x} \sin x$, find $f'(x)$.

(b) Check to see that your answer to part (a) is reasonable by graphing both f and f' for $0 \leq x \leq 2\pi$.

29. If $H(\theta) = \theta \sin \theta$, find $H'(\theta)$ and $H''(\theta)$.

30. If $f(x) = \sec x$, find $f''(\pi/4)$.

31. (a) Use the Quotient Rule to differentiate the function

$$f(x) = \frac{\tan x - 1}{\sec x}$$

(b) Simplify the expression for $f(x)$ by writing it in terms of $\sin x$ and $\cos x$, and then find $f'(x)$.

(c) Show that your answers to parts (a) and (b) are equivalent.

32. Suppose $f(\pi/3) = 4$ and $f'(\pi/3) = -2$, and let

$$g(x) = f(x) \sin x$$

and

$$h(x) = \frac{\cos x}{f(x)}$$

Find (a) $g'(\pi/3)$ and (b) $h'(\pi/3)$.

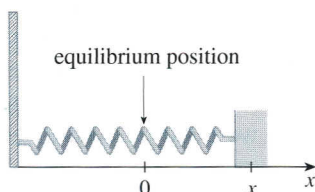
33. For what values of x does the graph of $f(x) = x + 2 \sin x$ have a horizontal tangent?

34. Find the points on the curve $y = (\cos x)/(2 + \sin x)$ at which the tangent is horizontal.

35. A mass on a spring vibrates horizontally on a smooth level surface (see the figure). Its equation of motion is $x(t) = 8 \sin t$, where t is in seconds and x in centimeters.

(a) Find the velocity and acceleration at time t .

- (b) Find the position, velocity, and acceleration of the mass at time $t = 2\pi/3$. In what direction is it moving at that time?



36. An elastic band is hung on a hook and a mass is hung on the lower end of the band. When the mass is pulled downward and then released, it vibrates vertically. The equation of motion is $s = 2 \cos t + 3 \sin t$, $t \geq 0$, where s is measured in centimeters and t in seconds. (Take the positive direction to be downward.)
- Find the velocity and acceleration at time t .
 - Graph the velocity and acceleration functions.
 - When does the mass pass through the equilibrium position for the first time?
 - How far from its equilibrium position does the mass travel?
 - When is the speed the greatest?

37. A ladder 6 m long rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall and let x be the distance from the bottom of the ladder to the wall. If the bottom of the ladder slides away from the wall, how fast does x change with respect to θ when $\theta = \pi/3$?

38. An object with mass m is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is

$$F = \frac{\mu mg}{\mu \sin \theta + \cos \theta}$$

where μ is a constant called the *coefficient of friction*.

- Find the rate of change of F with respect to θ .
 - When is this rate of change equal to 0?
- (c) If $m = 20$ kg, $g = 9.8$ m/s², and $\mu = 0.6$, draw the graph of F as a function of θ and use it to locate the value of θ for which $dF/d\theta = 0$. Is the value consistent with your answer to part (b)?

39–48 Find the limit.

39. $\lim_{x \rightarrow 0} \frac{\sin 3x}{x}$

40. $\lim_{x \rightarrow 0} \frac{\sin 4x}{\sin 6x}$

41. $\lim_{t \rightarrow 0} \frac{\tan 6t}{\sin 2t}$

42. $\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\sin \theta}$

43. $\lim_{\theta \rightarrow 0} \frac{\sin(\cos \theta)}{\sec \theta}$

44. $\lim_{t \rightarrow 0} \frac{\sin^2 3t}{t^2}$

45. $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta + \tan \theta}$

46. $\lim_{x \rightarrow 0} \frac{\sin(x^2)}{x}$

47. $\lim_{x \rightarrow \pi/4} \frac{1 - \tan x}{\sin x - \cos x}$

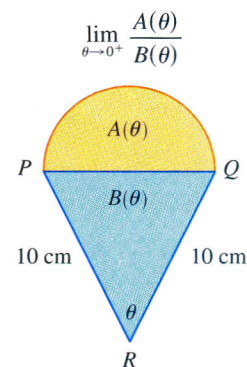
48. $\lim_{x \rightarrow 1} \frac{\sin(x-1)}{x^2 + x - 2}$

49. Differentiate each trigonometric identity to obtain a new (or familiar) identity.

(a) $\tan x = \frac{\sin x}{\cos x}$ (b) $\sec x = \frac{1}{\cos x}$

(c) $\sin x + \cos x = \frac{1 + \cot x}{\csc x}$

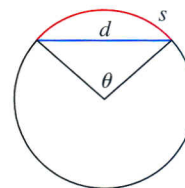
50. A semicircle with diameter PQ sits on an isosceles triangle PQR to form a region shaped like a two-dimensional ice-cream cone, as shown in the figure. If $A(\theta)$ is the area of the semicircle and $B(\theta)$ is the area of the triangle, find



$$\lim_{\theta \rightarrow 0^+} \frac{A(\theta)}{B(\theta)}$$

51. The figure shows a circular arc of length s and a chord of length d , both subtended by a central angle θ . Find

$$\lim_{\theta \rightarrow 0^+} \frac{s}{d}$$



3.5 THE CHAIN RULE

Suppose you are asked to differentiate the function

$$F(x) = \sqrt{x^2 + 1}$$

The differentiation formulas you learned in the previous sections of this chapter do not enable you to calculate $F'(x)$.

If we define ε to be 0 when $\Delta x = 0$, then ε becomes a continuous function of Δx . Thus, for a differentiable function f , we can write

$$\boxed{5} \quad \Delta y = f'(a) \Delta x + \varepsilon \Delta x \quad \text{where } \varepsilon \rightarrow 0 \text{ as } \Delta x \rightarrow 0$$

and ε is a continuous function of Δx . This property of differentiable functions is what enables us to prove the Chain Rule.

PROOF OF THE CHAIN RULE Suppose $u = g(x)$ is differentiable at a and $y = f(u)$ is differentiable at $b = g(a)$. If Δx is an increment in x and Δu and Δy are the corresponding increments in u and y , then we can use Equation 5 to write

$$\boxed{6} \quad \Delta u = g'(a) \Delta x + \varepsilon_1 \Delta x = [g'(a) + \varepsilon_1] \Delta x$$

where $\varepsilon_1 \rightarrow 0$ as $\Delta x \rightarrow 0$. Similarly

$$\boxed{7} \quad \Delta y = f'(b) \Delta u + \varepsilon_2 \Delta u = [f'(b) + \varepsilon_2] \Delta u$$

where $\varepsilon_2 \rightarrow 0$ as $\Delta u \rightarrow 0$. If we now substitute the expression for Δu from Equation 6 into Equation 7, we get

$$\Delta y = [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1] \Delta x$$

so

$$\frac{\Delta y}{\Delta x} = [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]$$

As $\Delta x \rightarrow 0$, Equation 6 shows that $\Delta u \rightarrow 0$. So both $\varepsilon_1 \rightarrow 0$ and $\varepsilon_2 \rightarrow 0$ as $\Delta x \rightarrow 0$. Therefore

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1] \\ &= f'(b)g'(a) = f'(g(a))g'(a) \end{aligned}$$

This proves the Chain Rule. □

3.5 EXERCISES

1–6 Write the composite function in the form $f(g(x))$. [Identify the inner function $u = g(x)$ and the outer function $y = f(u)$.] Then find the derivative dy/dx .

1. $y = \sin 4x$

2. $y = \sqrt{4 + 3x}$

3. $y = (1 - x^2)^{10}$

4. $y = \tan(\sin x)$

5. $y = \sqrt{\sin x}$

6. $y = \sin \sqrt{x}$

7–46 Find the derivative of the function.

7. $F(x) = (x^3 + 4x)^7$

8. $F(x) = (x^2 - x + 1)^3$

9. $F(x) = \sqrt[4]{1 + 2x + x^3}$

10. $f(x) = (1 + x^4)^{2/3}$

11. $g(t) = \frac{1}{(t^4 + 1)^3}$

12. $f(t) = \sqrt[3]{1 + \tan t}$

13. $y = \cos(a^3 + x^3)$

14. $y = a^3 + \cos^3 x$

15. $y = x \sec kx$

16. $y = 3 \cot(n\theta)$

17. $g(x) = (1 + 4x)^5(3 + x - x^2)^8$

18. $h(t) = (t^4 - 1)^3(t^3 + 1)^4$

19. $y = (2x - 5)^4(8x^2 - 5)^{-3}$

20. $y = (x^2 + 1)\sqrt[3]{x^2 + 2}$

21. $y = \left(\frac{x^2 + 1}{x^2 - 1}\right)^3$

22. $y = x \sin \sqrt{x}$

23. $y = \sin(x \cos x)$

24. $f(x) = \frac{x}{\sqrt{7 - 3x}}$

25. $F(z) = \sqrt{\frac{z - 1}{z + 1}}$

26. $G(y) = \frac{(y - 1)^4}{(y^2 + 2y)^5}$

27. $y = \frac{r}{\sqrt{r^2 + 1}}$

28. $y = \frac{\cos \pi x}{\sin \pi x + \cos \pi x}$

29. $y = \tan(\cos x)$ 30. $G(y) = \left(\frac{y^2}{y+1}\right)^5$
31. $y = \sin\sqrt{1+x^2}$ 32. $y = \tan^2(3\theta)$
33. $y = \sec^2 x + \tan^2 x$ 34. $y = x \sin \frac{1}{x}$
35. $y = \left(\frac{1 - \cos 2x}{1 + \cos 2x}\right)^4$ 36. $f(t) = \sqrt{\frac{t}{t^2 + 4}}$
37. $y = \cot^2(\sin \theta)$ 38. $y = (ax + \sqrt{x^2 + b^2})^{-2}$
39. $y = [x^2 + (1 - 3x)^5]^3$ 40. $y = \sin(\sin(\sin x))$
41. $y = \sqrt{x + \sqrt{x}}$ 42. $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$
43. $g(x) = (2r \sin rx + n)^p$ 44. $y = \cos^4(\sin^3 x)$
45. $y = \cos\sqrt{\sin(\tan \pi x)}$ 46. $y = [x + (x + \sin^2 x)^3]^4$

47–50 Find the first and second derivatives of the function.

47. $h(x) = \sqrt{x^2 + 1}$ 48. $y = \sin^2(\pi t)$
49. $H(t) = \tan 3t$ 50. $y = \frac{4x}{\sqrt{x+1}}$

51–54 Find an equation of the tangent line to the curve at the given point.

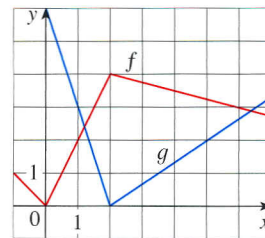
51. $y = \frac{8}{\sqrt{4+3x}}$, $(4, 2)$ 52. $y = \sin x + \cos 2x$, $(\pi/6, 1)$
53. $y = \sin(\sin x)$, $(\pi, 0)$ 54. $y = \sqrt{5+x^2}$, $(2, 3)$

55. (a) Find an equation of the tangent line to the curve $y = \tan(\pi x^2/4)$ at the point $(1, 1)$.
 (b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.
56. (a) The curve $y = |x|/\sqrt{2-x^2}$ is called a *bullet-nose curve*. Find an equation of the tangent line to this curve at the point $(1, 1)$.
 (b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.
57. (a) If $f(x) = x\sqrt{2-x^2}$, find $f'(x)$.
 (b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f' .
58. The function $f(x) = \sin(x + \sin 2x)$, $0 \leq x \leq \pi$, arises in applications to frequency modulation (FM) synthesis.
 (a) Use a graph of f produced by a graphing device to make a rough sketch of the graph of f' .
 (b) Calculate $f'(x)$ and use this expression, with a graphing device, to graph f' . Compare with your sketch in part (a).
59. Find all points on the graph of the function $f(x) = 2 \sin x + \sin^2 x$ at which the tangent line is horizontal.

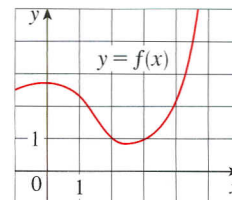
60. Find the x -coordinates of all points on the curve $y = \sin 2x - 2 \sin x$ at which the tangent line is horizontal.
61. If $F(x) = f(g(x))$, where $f(-2) = 8$, $f'(-2) = 4$, $f'(5) = 3$, $g(5) = -2$, and $g'(5) = 6$, find $F'(5)$.
62. If $h(x) = \sqrt{4 + 3f(x)}$, where $f(1) = 7$ and $f'(1) = 4$, find $h'(1)$.
63. A table of values for f , g , f' , and g' is given.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

- (a) If $h(x) = f(g(x))$, find $h'(1)$.
 (b) If $H(x) = g(f(x))$, find $H'(1)$.
64. Let f and g be the functions in Exercise 63.
 (a) If $F(x) = f(f(x))$, find $F'(2)$.
 (b) If $G(x) = g(g(x))$, find $G'(3)$.
65. If f and g are the functions whose graphs are shown, let $u(x) = f(g(x))$, $v(x) = g(f(x))$, and $w(x) = g(g(x))$. Find each derivative, if it exists. If it does not exist, explain why.
 (a) $u'(1)$ (b) $v'(1)$ (c) $w'(1)$



66. If f is the function whose graph is shown, let $h(x) = f(f(x))$ and $g(x) = f(x^2)$. Use the graph of f to estimate the value of each derivative.
 (a) $h'(2)$ (b) $g'(2)$



67. Suppose f is differentiable on \mathbb{R} . Let $F(x) = f(\cos x)$ and $G(x) = \cos(f(x))$. Find expressions for (a) $F'(x)$ and (b) $G'(x)$.
68. Suppose f is differentiable on \mathbb{R} and α is a real number. Let $F(x) = f(x^\alpha)$ and $G(x) = [f(x)]^\alpha$. Find expressions for (a) $F'(x)$ and (b) $G'(x)$.

69. Let $r(x) = f(g(h(x)))$, where $h(1) = 2$, $g(2) = 3$, $h'(1) = 4$, $g'(2) = 5$, and $f'(3) = 6$. Find $r'(1)$.
70. If g is a twice differentiable function and $f(x) = xg(x^2)$, find f'' in terms of g , g' , and g'' .
71. If $F(x) = f(3f(4f(x)))$, where $f(0) = 0$ and $f'(0) = 2$, find $F'(0)$.
72. If $F(x) = f(xf(xf(x)))$, where $f(1) = 2$, $f(2) = 3$, $f'(1) = 4$, $f'(2) = 5$, and $f'(3) = 6$, find $F'(1)$.

73–74 Find the given derivative by finding the first few derivatives and observing the pattern that occurs.

73. $D^{103} \cos 2x$

74. $D^{35} x \sin x$

- 75.** The displacement of a particle on a vibrating string is given by the equation

$$s(t) = 10 + \frac{1}{4} \sin(10\pi t)$$

where s is measured in centimeters and t in seconds. Find the velocity of the particle after t seconds.

- 76.** If the equation of motion of a particle is given by $s = A \cos(\omega t + \delta)$, the particle is said to undergo *simple harmonic motion*.
- (a) Find the velocity of the particle at time t .
- (b) When is the velocity 0?
- 77.** A Cepheid variable star is a star whose brightness alternately increases and decreases. The most easily visible such star is Delta Cephei, for which the interval between times of maximum brightness is 5.4 days. The average brightness of this star is 4.0 and its brightness changes by ± 0.35 . In view of these data, the brightness of Delta Cephei at time t , where t is measured in days, has been modeled by the function

$$B(t) = 4.0 + 0.35 \sin\left(\frac{2\pi t}{5.4}\right)$$

- (a) Find the rate of change of the brightness after t days.
- (b) Find, correct to two decimal places, the rate of increase after one day.
- 78.** In Example 4 in Section 1.3 we arrived at a model for the length of daylight (in hours) in Ankara, Turkey, on the t th day of the year:

$$L(t) = 12 + 2.8 \sin\left[\frac{2\pi}{365}(t - 80)\right]$$

Use this model to compare how the number of hours of daylight is increasing in Ankara on March 21 and May 21.

- 79.** A particle moves along a straight line with displacement $s(t)$, velocity $v(t)$, and acceleration $a(t)$. Show that

$$a(t) = v(t) \frac{dv}{ds}$$

Explain the difference between the meanings of the derivatives dv/dt and dv/ds .

- 80.** Air is being pumped into a spherical weather balloon. At any time t , the volume of the balloon is $V(t)$ and its radius is $r(t)$.
- (a) What do the derivatives dV/dr and dV/dt represent?
- (b) Express dV/dt in terms of dr/dt .

- CAS 81.** Computer algebra systems have commands that differentiate functions, but the form of the answer may not be convenient and so further commands may be necessary to simplify the answer.
- (a) Use a CAS to find the derivative in Example 5 and compare with the answer in that example. Then use the simplify command and compare again.
- (b) Use a CAS to find the derivative in Example 6. What happens if you use the simplify command? What happens if you use the factor command? Which form of the answer would be best for locating horizontal tangents?

- CAS 82.** (a) Use a CAS to differentiate the function

$$f(x) = \sqrt{\frac{x^4 - x + 1}{x^4 + x + 1}}$$

and to simplify the result.

- (b) Where does the graph of f have horizontal tangents?
- (c) Graph f and f' on the same screen. Are the graphs consistent with your answer to part (b)?
- 83.** Use the Chain Rule to prove the following.
- (a) The derivative of an even function is an odd function.
- (b) The derivative of an odd function is an even function.
- 84.** Use the Chain Rule and the Product Rule to give an alternative proof of the Quotient Rule.
[Hint: Write $f(x)/g(x) = f(x)[g(x)]^{-1}$.]
- 85.** (a) If n is a positive integer, prove that

$$\frac{d}{dx} (\sin^n x \cos nx) = n \sin^{n-1} x \cos(n+1)x$$

- (b) Find a formula for the derivative of $y = \cos^n x \cos nx$ that is similar to the one in part (a).

- 86.** Suppose $y = f(x)$ is a curve that always lies above the x -axis and never has a horizontal tangent, where f is differentiable everywhere. For what value of y is the rate of change of y^5 with respect to x eighty times the rate of change of y with respect to x ?

- 87.** Use the Chain Rule to show that if θ is measured in degrees, then

$$\frac{d}{d\theta} (\sin \theta) = \frac{\pi}{180} \cos \theta$$

(This gives one reason for the convention that radian measure is always used when dealing with trigonometric functions in calculus: The differentiation formulas would not be as simple if we used degree measure.)

88. (a) Write $|x| = \sqrt{x^2}$ and use the Chain Rule to show that

$$\frac{d}{dx} |x| = \frac{x}{|x|}$$

- (b) If $f(x) = |\sin x|$, find $f'(x)$ and sketch the graphs of f and f' . Where is f not differentiable?
 (c) If $g(x) = \sin |x|$, find $g'(x)$ and sketch the graphs of g and g' . Where is g not differentiable?

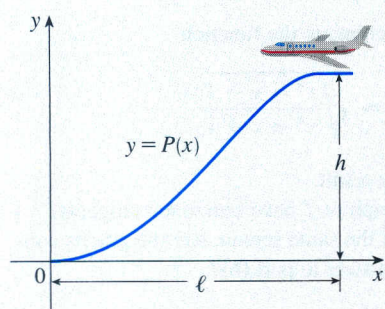
89. If $y = f(u)$ and $u = g(x)$, where f and g are twice differentiable functions, show that

$$\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} \left(\frac{du}{dx} \right)^2 + \frac{dy}{du} \frac{d^2u}{dx^2}$$

90. If $y = f(u)$ and $u = g(x)$, where f and g possess third derivatives, find a formula for d^3y/dx^3 similar to the one given in Exercise 89.

APPLIED PROJECT

WHERE SHOULD A PILOT START DESCENT?



An approach path for an aircraft landing is shown in the figure and satisfies the following conditions:

- The cruising altitude is h when descent starts at a horizontal distance ℓ from touchdown at the origin.
- The pilot must maintain a constant horizontal speed v throughout descent.
- The absolute value of the vertical acceleration should not exceed a constant k (which is much less than the acceleration due to gravity).

- Find a cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$ that satisfies condition (i) by imposing suitable conditions on $P(x)$ and $P'(x)$ at the start of descent and at touchdown.
- Use conditions (ii) and (iii) to show that

$$\frac{6hv^2}{\ell^2} \leq k$$

- Suppose that an airline decides not to allow vertical acceleration of a plane to exceed $k = 1385 \text{ km/h}^2$. If the cruising altitude of a plane is 11,000 m and the speed is 480 km/h, how far away from the airport should the pilot start descent?

-  4. Graph the approach path if the conditions stated in Problem 3 are satisfied.

3.6 IMPLICIT DIFFERENTIATION

The functions that we have met so far can be described by expressing one variable explicitly in terms of another variable—for example,

$$y = \sqrt{x^3 + 1} \quad \text{or} \quad y = x \sin x$$

or, in general, $y = f(x)$. Some functions, however, are defined implicitly by a relation between x and y such as

$$\text{1} \quad x^2 + y^2 = 25$$

or

$$\text{2} \quad x^3 + y^3 = 6xy$$

In some cases it is possible to solve such an equation for y as an explicit function (or several functions) of x . For instance, if we solve Equation 1 for y , we get $y = \pm\sqrt{25 - x^2}$, so two of the functions determined by the implicit Equation 1 are $f(x) = \sqrt{25 - x^2}$ and

3.6 EXERCISES

1-4

- (a) Find y' by implicit differentiation.
 (b) Solve the equation explicitly for y and differentiate to get y' in terms of x .
 (c) Check that your solutions to parts (a) and (b) are consistent by substituting the expression for y into your solution for part (a).

1. $xy + 2x + 3x^2 = 4$ 2. $4x^2 + 9y^2 = 36$

3. $\frac{1}{x} + \frac{1}{y} = 1$ 4. $\cos x + \sqrt{y} = 5$

5-20 Find dy/dx by implicit differentiation.

5. $x^3 + y^3 = 1$ 6. $2\sqrt{x} + \sqrt{y} = 3$

7. $x^3 + x^2y + 4y^2 = 6$ 8. $x^2 - 2xy + y^3 = c$

9. $x^4(x + y) = y^2(3x - y)$ 10. $y^5 + x^2y^3 = 1 + x^4y$

11. $x^2y^2 + x \sin y = 4$ 12. $1 + x = \sin(xy^2)$

13. $4 \cos x \sin y = 1$ 14. $y \sin(x^2) = x \sin(y^2)$

15. $\tan(x/y) = x + y$ 16. $\sqrt{x + y} = 1 + x^2y^2$

17. $\sqrt{xy} = 1 + x^2y$ 18. $\tan(x - y) = \frac{y}{1 + x^2}$

19. $xy = \cot(xy)$ 20. $\sin x + \cos y = \sin x \cos y$

21. If $f(x) + x^2[f(x)]^3 = 10$ and $f(1) = 2$, find $f'(1)$.

22. If $g(x) + x \sin g(x) = x^2$, find $g'(0)$.

23-24 Regard y as the independent variable and x as the dependent variable and use implicit differentiation to find dx/dy .

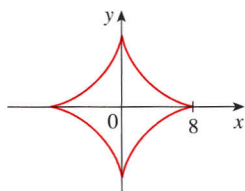
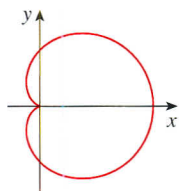
23. $x^4y^2 - x^3y + 2xy^3 = 0$ 24. $y \sec x = x \tan y$

25-30 Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

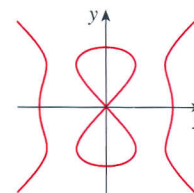
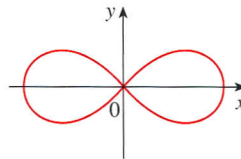
25. $x^2 + xy + y^2 = 3$, $(1, 1)$ (ellipse)

26. $x^2 + 2xy - y^2 + x = 2$, $(1, 2)$ (hyperbola)

27. $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$ 28. $x^{2/3} + y^{2/3} = 4$
 $(0, \frac{1}{2})$ (cardioid) $(-3\sqrt{3}, 1)$ (astroid)



29. $2(x^2 + y^2)^2 = 25(x^2 - y^2)$ 30. $y^2(y^2 - 4) = x^2(x^2 - 5)$
 $(3, 1)$ (lemniscate) $(0, -2)$ (devil's curve)



31. (a) The curve with equation $y^2 = 5x^4 - x^2$ is called a **kampyle of Eudoxus**. Find an equation of the tangent line to this curve at the point $(1, 2)$.
 (b) Illustrate part (a) by graphing the curve and the tangent line on a common screen. (If your graphing device will graph implicitly defined curves, then use that capability. If not, you can still graph this curve by graphing its upper and lower halves separately.)
32. (a) The curve with equation $y^2 = x^3 + 3x^2$ is called the **Tschirnhausen cubic**. Find an equation of the tangent line to this curve at the point $(1, -2)$.
 (b) At what points does this curve have horizontal tangents?
 (c) Illustrate parts (a) and (b) by graphing the curve and the tangent lines on a common screen.

33-36 Find y'' by implicit differentiation.

33. $9x^2 + y^2 = 9$

34. $\sqrt{x} + \sqrt{y} = 1$

35. $x^3 + y^3 = 1$

36. $x^4 + y^4 = a^4$

CAS 37. Fanciful shapes can be created by using the implicit plotting capabilities of computer algebra systems.

- (a) Graph the curve with equation

$$y(y^2 - 1)(y - 2) = x(x - 1)(x - 2)$$

- At how many points does this curve have horizontal tangents? Estimate the x -coordinates of these points.
 (b) Find equations of the tangent lines at the points $(0, 1)$ and $(0, 2)$.
 (c) Find the exact x -coordinates of the points in part (a).
 (d) Create even more fanciful curves by modifying the equation in part (a).

CAS 38. (a) The curve with equation

$$2y^3 + y^2 - y^5 = x^4 - 2x^3 + x^2$$

- has been likened to a bouncing wagon. Use a computer algebra system to graph this curve and discover why.
 (b) At how many points does this curve have horizontal tangent lines? Find the x -coordinates of these points.

39. Find the points on the lemniscate in Exercise 29 where the tangent is horizontal.
40. Show by implicit differentiation that the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) is

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$$

41. Find an equation of the tangent line to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) .

42. Show that the sum of the x - and y -intercepts of any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is equal to c .
43. Show, using implicit differentiation, that any tangent line at a point P to a circle with center O is perpendicular to the radius OP .
44. The Power Rule can be proved using implicit differentiation for the case where n is a rational number, $n = p/q$, and $y = f(x) = x^n$ is assumed beforehand to be a differentiable function. If $y = x^{p/q}$, then $y^q = x^p$. Use implicit differentiation to show that

$$y' = \frac{p}{q} x^{(p/q)-1}$$

45–48 Two curves are **orthogonal** if their tangent lines are perpendicular at each point of intersection. Show that the given families of curves are **orthogonal trajectories** of each other, that is, every curve in one family is orthogonal to every curve in the other family. Sketch both families of curves on the same axes.

45. $x^2 + y^2 = r^2$, $ax + by = 0$


46. $x^2 + y^2 = ax$, $x^2 + y^2 = by$

47. $y = cx^2$, $x^2 + 2y^2 = k$

48. $y = ax^3$, $x^2 + 3y^2 = b$

49. The equation $x^2 - xy + y^2 = 3$ represents a “rotated ellipse,” that is, an ellipse whose axes are not parallel to the coordinate axes. Find the points at which this ellipse crosses the x -axis and show that the tangent lines at these points are parallel.

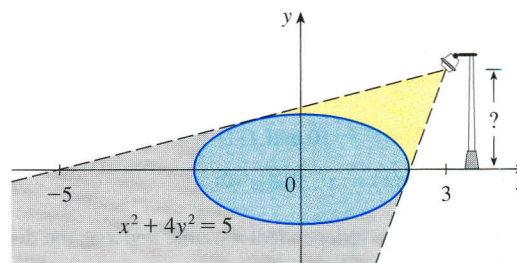
50. (a) Where does the normal line to the ellipse $x^2 - xy + y^2 = 3$ at the point $(-1, 1)$ intersect the ellipse a second time?

 (b) Illustrate part (a) by graphing the ellipse and the normal line.

51. Find all points on the curve $x^2 y^2 + xy = 2$ where the slope of the tangent line is -1 .

52. Find equations of both the tangent lines to the ellipse $x^2 + 4y^2 = 36$ that pass through the point $(12, 3)$.

53. The figure shows a lamp located three units to the right of the y -axis and a shadow created by the elliptical region $x^2 + 4y^2 \leq 5$. If the point $(-5, 0)$ is on the edge of the shadow, how far above the x -axis is the lamp located?



3.7 RATES OF CHANGE IN THE NATURAL AND SOCIAL SCIENCES

We know that if $y = f(x)$, then the derivative dy/dx can be interpreted as the rate of change of y with respect to x . In this section we examine some of the applications of this idea to physics, chemistry, biology, economics, and other sciences.

Let's recall from Section 3.1 the basic idea behind rates of change. If x changes from x_1 to x_2 , then the change in x is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

OTHER SCIENCES

Rates of change occur in all the sciences. A geologist is interested in knowing the rate at which an intruded body of molten rock cools by conduction of heat into surrounding rocks. An engineer wants to know the rate at which water flows into or out of a reservoir. An urban geographer is interested in the rate of change of the population density in a city as the distance from the city center increases. A meteorologist is concerned with the rate of change of atmospheric pressure with respect to height (see Exercise 17 in Section 7.5).

In psychology, those interested in learning theory study the so-called learning curve, which graphs the performance $P(t)$ of someone learning a skill as a function of the training time t . Of particular interest is the rate at which performance improves as time passes, that is, dP/dt .

In sociology, differential calculus is used in analyzing the spread of rumors (or innovations or fads or fashions). If $p(t)$ denotes the proportion of a population that knows a rumor by time t , then the derivative dp/dt represents the rate of spread of the rumor (see Exercise 59 in Section 7.2).

A SINGLE IDEA, MANY INTERPRETATIONS


Velocity, density, current, power, and temperature gradient in physics; rate of reaction and compressibility in chemistry; rate of growth and blood velocity gradient in biology; marginal cost and marginal profit in economics; rate of heat flow in geology; rate of improvement of performance in psychology; rate of spread of a rumor in sociology—these are all special cases of a single mathematical concept, the derivative.

This is an illustration of the fact that part of the power of mathematics lies in its abstractness. A single abstract mathematical concept (such as the derivative) can have different interpretations in each of the sciences. When we develop the properties of the mathematical concept once and for all, we can then turn around and apply these results to all of the sciences. This is much more efficient than developing properties of special concepts in each separate science. The French mathematician Joseph Fourier (1768–1830) put it succinctly: “Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them.”

3.7 EXERCISES

1–4 A particle moves according to a law of motion $s = f(t)$, $t \geq 0$, where t is measured in seconds and s in meters.

- Find the velocity at time t .
- What is the velocity after 3 s?
- When is the particle at rest?
- When is the particle moving in the positive direction?
- Find the total distance traveled during the first 8 s.
- Draw a diagram like Figure 2 to illustrate the motion of the particle.
- Find the acceleration at time t and after 3 s.

 (h) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 8$.

- When is the particle speeding up? When is it slowing down?

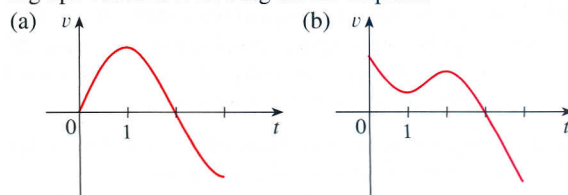
1. $f(t) = t^3 - 12t^2 + 36t$

2. $f(t) = 0.01t^4 - 0.04t^3$

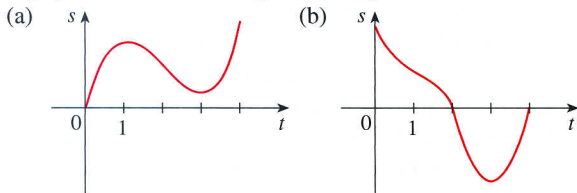
3. $f(t) = \cos(\pi t/4)$, $t \leq 10$

4. $f(t) = t/(1 + t^2)$

- 5.** Graphs of the *velocity* functions of two particles are shown, where t is measured in seconds. When is each particle speeding up? When is it slowing down? Explain.



6. Graphs of the *position* functions of two particles are shown, where t is measured in seconds. When is each particle speeding up? When is it slowing down? Explain.



7. The position function of a particle is given by $s = t^3 - 4.5t^2 - 7t$, $t \geq 0$.
- When does the particle reach a velocity of 5 m/s?
 - When is the acceleration 0? What is the significance of this value of t ?
8. If a ball is given a push so that it has an initial velocity of 5 m/s down a certain inclined plane, then the distance it has rolled after t seconds is $s = 5t + 3t^2$.
- Find the velocity after 2 s.
 - How long does it take for the velocity to reach 35 m/s?
9. If a stone is thrown vertically upward from the surface of the moon with a velocity of 10 m/s, its height (in meters) after t seconds is $h = 10t - 0.83t^2$.
- What is the velocity of the stone after 3 s?
 - What is the velocity of the stone after it has risen 25 m?
10. If a ball is thrown vertically upward with a velocity of 24.5 m/s, then its height after t seconds is $s = 24.5t - 4.9t^2$.
- What is the maximum height reached by the ball?
 - What is the velocity of the ball when it is 29.4 m above the ground on its way up? On its way down?
11. (a) A company makes computer chips from square wafers of silicon. It wants to keep the side length of a wafer very close to 15 mm and it wants to know how the area $A(x)$ of a wafer changes when the side length x changes. Find $A'(15)$ and explain its meaning in this situation.
- (b) Show that the rate of change of the area of a square with respect to its side length is half its perimeter. Try to explain geometrically why this is true by drawing a square whose side length x is increased by an amount Δx . How can you approximate the resulting change in area ΔA if Δx is small?
12. (a) Sodium chlorate crystals are easy to grow in the shape of cubes by allowing a solution of water and sodium chlorate to evaporate slowly. If V is the volume of such a cube with side length x , calculate dV/dx when $x = 3$ mm and explain its meaning.
- (b) Show that the rate of change of the volume of a cube with respect to its edge length is equal to half the surface area of the cube. Explain geometrically why this result is true by arguing by analogy with Exercise 11(b).
13. (a) Find the average rate of change of the area of a circle with respect to its radius r as r changes from
- 2 to 3
 - 2 to 2.5
 - 2 to 2.1

- Find the instantaneous rate of change when $r = 2$.
- Show that the rate of change of the area of a circle with respect to its radius (at any r) is equal to the circumference of the circle. Try to explain geometrically why this is true by drawing a circle whose radius is increased by an amount Δr . How can you approximate the resulting change in area ΔA if Δr is small?

14. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s. Find the rate at which the area within the circle is increasing after (a) 1 s, (b) 3 s, and (c) 5 s. What can you conclude?

15. A spherical balloon is being inflated. Find the rate of increase of the surface area ($S = 4\pi r^2$) with respect to the radius r when r is (a) 1 ft, (b) 2 ft, and (c) 3 ft. What conclusion can you make?

16. (a) The volume of a growing spherical cell is $V = \frac{4}{3}\pi r^3$, where the radius r is measured in micrometers ($1 \mu\text{m} = 10^{-6}$ m). Find the average rate of change of V with respect to r when r changes from
- 5 to 8 μm
 - 5 to 6 μm
 - 5 to 5.1 μm
- (b) Find the instantaneous rate of change of V with respect to r when $r = 5 \mu\text{m}$.
- (c) Show that the rate of change of the volume of a sphere with respect to its radius is equal to its surface area. Explain geometrically why this result is true. Argue by analogy with Exercise 13(c).

17. The mass of the part of a metal rod that lies between its left end and a point x meters to the right is $3x^2$ kg. Find the linear density (see Example 2) when x is (a) 1 m, (b) 2 m, and (c) 3 m. Where is the density the highest? The lowest?

18. If a tank holds 5000 liters of water, which drains from the bottom of the tank in 40 minutes, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V = 5000 \left(1 - \frac{t}{40} \right)^2 \quad 0 \leq t \leq 40$$

Find the rate at which water is draining from the tank after (a) 5 min, (b) 10 min, (c) 20 min, and (d) 40 min. At what time is the water flowing out the fastest? The slowest? Summarize your findings.

19. The quantity of charge Q in coulombs (C) that has passed through a point in a wire up to time t (measured in seconds) is given by $Q(t) = t^3 - 2t^2 + 6t + 2$. Find the current when (a) $t = 0.5$ s and (b) $t = 1$ s. [See Example 3. The unit of current is an ampere ($1 \text{ A} = 1 \text{ C/s}$.)] At what time is the current lowest?

20. Newton's Law of Gravitation says that the magnitude F of the force exerted by a body of mass m on a body of mass M is

$$F = \frac{GmM}{r^2}$$

where G is the gravitational constant and r is the distance between the bodies.

- Find dF/dr and explain its meaning. What does the minus sign indicate?
- Suppose it is known that the earth attracts an object with a force that decreases at the rate of 2 N/km when $r = 20,000$ km. How fast does this force change when $r = 10,000$ km?

21. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the product of the pressure and the volume remains constant: $PV = C$.

- Find the rate of change of volume with respect to pressure.
- A sample of gas is in a container at low pressure and is steadily compressed at constant temperature for 10 minutes. Is the volume decreasing more rapidly at the beginning or the end of the 10 minutes? Explain.
- Prove that the isothermal compressibility (see Example 5) is given by $\beta = 1/P$.

22. If, in Example 4, one molecule of the product C is formed from one molecule of the reactant A and one molecule of the reactant B, and the initial concentrations of A and B have a common value $[A] = [B] = a$ moles/L, then

$$[C] = a^2kt/(akt + 1)$$

where k is a constant.

- Find the rate of reaction at time t .
- Show that if $x = [C]$, then

$$\frac{dx}{dt} = k(a - x)^2$$

23. The table gives the population of the world in the 20th century.

Year	Population (in millions)	Year	Population (in millions)
1900	1650	1960	3040
1910	1750	1970	3710
1920	1860	1980	4450
1930	2070	1990	5280
1940	2300	2000	6080
1950	2560		

- Estimate the rate of population growth in 1920 and in 1980 by averaging the slopes of two secant lines.
- Use a graphing calculator or computer to find a cubic function (a third-degree polynomial) that models the data.
- Use your model in part (b) to find a model for the rate of population growth in the 20th century.
- Use part (c) to estimate the rates of growth in 1920 and 1980. Compare with your estimates in part (a).
- Estimate the rate of growth in 1985.

24. The table shows how the average age of first marriage of Japanese women varied in the last half of the 20th century.

t	$A(t)$	t	$A(t)$
1950	23.0	1980	25.2
1955	23.8	1985	25.5
1960	24.4	1990	25.9
1965	24.5	1995	26.3
1970	24.2	2000	27.0
1975	24.7		

- Use a graphing calculator or computer to model these data with a fourth-degree polynomial.
 - Use part (a) to find a model for $A'(t)$.
 - Estimate the rate of change of marriage age for women in 1990.
 - Graph the data points and the models for A and A' .
- 25.** Refer to the law of laminar flow given in Example 7. Consider a blood vessel with radius 0.01 cm, length 3 cm, pressure difference 3000 dynes/cm², and viscosity $\eta = 0.027$.
- Find the velocity of the blood along the centerline $r = 0$, at radius $r = 0.005$ cm, and at the wall $r = R = 0.01$ cm.
 - Find the velocity gradient at $r = 0$, $r = 0.005$, and $r = 0.01$.
 - Where is the velocity the greatest? Where is the velocity changing most?

26. The frequency of vibrations of a vibrating violin string is given by

$$f = \frac{1}{2L} \sqrt{\frac{T}{\rho}}$$

where L is the length of the string, T is its tension, and ρ is its linear density. [See Chapter 11 in D. E. Hall, *Musical Acoustics*, 3d ed. (Pacific Grove, CA: Brooks/Cole, 2002).]

- Find the rate of change of the frequency with respect to
 - the length (when T and ρ are constant),
 - the tension (when L and ρ are constant), and
 - the linear density (when L and T are constant).
 - The pitch of a note (how high or low the note sounds) is determined by the frequency f . (The higher the frequency, the higher the pitch.) Use the signs of the derivatives in part (a) to determine what happens to the pitch of a note
 - when the effective length of a string is decreased by placing a finger on the string so a shorter portion of the string vibrates,
 - when the tension is increased by turning a tuning peg,
 - when the linear density is increased by switching to another string.
- 27.** The cost, in dollars, of producing x meters of a certain fabric is
- $$C(x) = 1200 + 12x - 0.1x^2 + 0.0005x^3$$
- Find the marginal cost function.

- (b) Find $C'(200)$ and explain its meaning. What does it predict?
 (c) Compare $C'(200)$ with the cost of manufacturing the 201st meter of fabric.

28. The cost function for production of a commodity is

$$C(x) = 339 + 25x - 0.09x^2 + 0.0004x^3$$

- (a) Find and interpret $C'(100)$.
 (b) Compare $C'(100)$ with the cost of producing the 101st item.

29. If $p(x)$ is the total value of the production when there are x workers in a plant, then the *average productivity* of the workforce at the plant is

$$A(x) = \frac{p(x)}{x}$$

- (a) Find $A'(x)$. Why does the company want to hire more workers if $A'(x) > 0$?
 (b) Show that $A'(x) > 0$ if $p'(x)$ is greater than the average productivity.
30. If R denotes the reaction of the body to some stimulus of strength x , the *sensitivity* S is defined to be the rate of change of the reaction with respect to x . A particular example is that when the brightness x of a light source is increased, the eye reacts by decreasing the area R of the pupil. The experimental formula

$$R = \frac{40 + 24x^{0.4}}{1 + 4x^{0.4}}$$

has been used to model the dependence of R on x when R is measured in square millimeters and x is measured in appropriate units of brightness.

- (a) Find the sensitivity.
 (b) Illustrate part (a) by graphing both R and S as functions of x . Comment on the values of R and S at low levels of brightness. Is this what you would expect?
31. The gas law for an ideal gas at absolute temperature T (in kelvins), pressure P (in atmospheres), and volume V (in

liters) is $PV = nRT$, where n is the number of moles of the gas and $R = 0.0821$ is the gas constant. Suppose that, at a certain instant, $P = 8.0$ atm and is increasing at a rate of 0.10 atm/min and $V = 10$ L and is decreasing at a rate of 0.15 L/min. Find the rate of change of T with respect to time at that instant if $n = 10$ mol.

32. In a fish farm, a population of fish is introduced into a pond and harvested regularly. A model for the rate of change of the fish population is given by the equation

$$\frac{dP}{dt} = r_0 \left(1 - \frac{P(t)}{P_c} \right) P(t) - \beta P(t)$$

where r_0 is the birth rate of the fish, P_c is the maximum population that the pond can sustain (called the *carrying capacity*), and β is the percentage of the population that is harvested.

- (a) What value of dP/dt corresponds to a stable population?
 (b) If the pond can sustain 10,000 fish, the birth rate is 5%, and the harvesting rate is 4%, find the stable population level.
 (c) What happens if β is raised to 5%?
33. In the study of ecosystems, *predator-prey models* are often used to study the interaction between species. Consider populations of tundra wolves, given by $W(t)$, and caribou, given by $C(t)$, in northern Canada. The interaction has been modeled by the equations

$$\frac{dC}{dt} = aC - bCW \quad \frac{dW}{dt} = -cW + dCW$$

- (a) What values of dC/dt and dW/dt correspond to stable populations?
 (b) How would the statement “The caribou go extinct” be represented mathematically?
 (c) Suppose that $a = 0.05$, $b = 0.001$, $c = 0.05$, and $d = 0.0001$. Find all population pairs (C, W) that lead to stable populations. According to this model, is it possible for the two species to live in balance or will one or both species become extinct?

3.8 RELATED RATES

If we are pumping air into a balloon, both the volume and the radius of the balloon are increasing and their rates of increase are related to each other. But it is much easier to measure directly the rate of increase of the volume than the rate of increase of the radius.

In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time.

When $x = 0.06$ km and $y = 0.08$ km, the Pythagorean Theorem gives $z = 0.1$ km, so

$$\frac{dz}{dt} = \frac{1}{0.1} [0.06(-90) + 0.08(-100)] = -134$$

The cars are approaching each other at a rate of 134 km/h. □

EXAMPLE 5 A man walks along a straight path at a speed of 1.5 m/s. A searchlight is located on the ground 6 m from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 8 m from the point on the path closest to the searchlight?

SOLUTION We draw Figure 5 and let x be the distance from the man to the point on the path closest to the searchlight. We let θ be the angle between the beam of the searchlight and the perpendicular to the path.

We are given that $dx/dt = 1.5$ m/s and are asked to find $d\theta/dt$ when $x = 8$. The equation that relates x and θ can be written from Figure 5:

$$\frac{x}{6} = \tan \theta \quad x = 6 \tan \theta$$

Differentiating each side with respect to t , we get

$$\frac{dx}{dt} = 6 \sec^2 \theta \frac{d\theta}{dt}$$

so
$$\frac{d\theta}{dt} = \frac{1}{6} \cos^2 \theta \frac{dx}{dt} = \frac{1}{6} \cos^2 \theta (1.5) = \frac{1}{4} \cos^2 \theta$$

When $x = 8$, the length of the beam is 10, so $\cos \theta = \frac{4}{5}$ and

$$\frac{d\theta}{dt} = \frac{1}{4} \left(\frac{4}{5} \right)^2 = \frac{4}{25} = 0.16$$

The searchlight is rotating at a rate of 0.16 rad/s. □

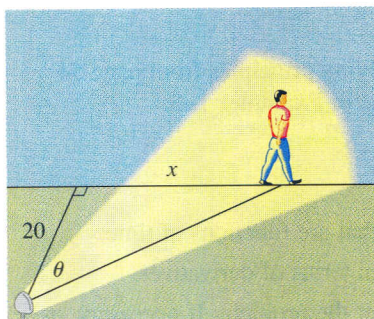


FIGURE 5

3.8 EXERCISES

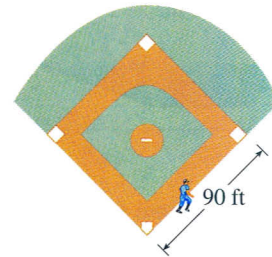
- If V is the volume of a cube with edge length x and the cube expands as time passes, find dV/dt in terms of dx/dt .
- (a) If A is the area of a circle with radius r and the circle expands as time passes, find dA/dt in terms of dr/dt .
(b) Suppose oil spills from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 1 m/s, how fast is the area of the spill increasing when the radius is 30 m?
- Each side of a square is increasing at a rate of 6 cm/s. At what rate is the area of the square increasing when the area of the square is 16 cm^2 ?
- The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?

5. A cylindrical tank with radius 5 m is being filled with water at a rate of $3 \text{ m}^3/\text{min}$. How fast is the height of the water increasing?
6. The radius of a sphere is increasing at a rate of 4 mm/s. How fast is the volume increasing when the diameter is 80 mm?
7. If $y = x^3 + 2x$ and $dx/dt = 5$, find dy/dt when $x = 2$.
8. If $x^2 + y^2 = 25$ and $dy/dt = 6$, find dx/dt when $y = 4$.
9. If $z^2 = x^2 + y^2$, $dx/dt = 2$, and $dy/dt = 3$, find dz/dt when $x = 5$ and $y = 12$.
10. A particle moves along the curve $y = \sqrt{1 + x^3}$. As it reaches the point $(2, 3)$, the y -coordinate is increasing at a rate of 4 cm/s. How fast is the x -coordinate of the point changing at that instant?

11–14

- (a) What quantities are given in the problem?
 - (b) What is the unknown?
 - (c) Draw a picture of the situation for any time t .
 - (d) Write an equation that relates the quantities.
 - (e) Finish solving the problem.
11. A plane flying horizontally at an altitude of 2 km and a speed of 800 km/h passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 3 km away from the station.
 12. If a snowball melts so that its surface area decreases at a rate of $1 \text{ cm}^2/\text{min}$, find the rate at which the diameter decreases when the diameter is 10 cm.
 13. A street light is mounted at the top of a 5-meter-tall pole. A man 2 m tall walks away from the pole with a speed of 1.5 m/s along a straight path. How fast is the tip of his shadow moving when he is 10 m from the pole?
 14. At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 PM?
-
15. Two cars start moving from the same point. One travels south at 30 km/h and the other travels west at 72 km/h. At what rate is the distance between the cars increasing two hours later?
 16. A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is the length of his shadow on the building decreasing when he is 4 m from the building?
 17. A man starts walking north at 1.2 m/s from a point P . Five minutes later a woman starts walking south at 1.6 m/s from a point 200 m due east of P . At what rate are the people moving apart 15 min after the woman starts walking?
 18. A baseball diamond is a square with side 90 ft. A batter hits the ball and runs toward first base with a speed of 24 ft/s.
 - (a) At what rate is his distance from second base decreasing when he is halfway to first base?

- (b) At what rate is his distance from third base increasing at the same moment?

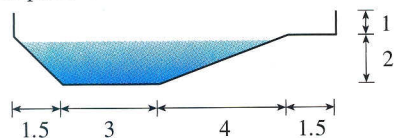


19. The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of $2 \text{ cm}^2/\text{min}$. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2 ?
20. A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8 m from the dock?

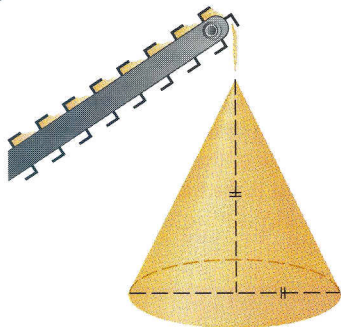


21. At noon, ship A is 100 km west of ship B. Ship A is sailing south at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 PM?
22. A particle is moving along the curve $y = \sqrt{x}$. As the particle passes through the point $(4, 2)$, its x -coordinate increases at a rate of 3 cm/s. How fast is the distance from the particle to the origin changing at this instant?
23. Water is leaking out of an inverted conical tank at a rate of $10,000 \text{ cm}^3/\text{min}$ at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank.
24. A trough is 6 m long and its ends have the shape of isosceles triangles that are 1 m across at the top and have a height of 50 cm. If the trough is being filled with water at a rate of $1.2 \text{ m}^3/\text{min}$, how fast is the water level rising when the water is 30 cm deep?
25. A water trough is 10 m long and a cross-section has the shape of an isosceles trapezoid that is 30 cm wide at the bottom, 80 cm wide at the top, and has height 50 cm. If the trough is being filled with water at the rate of $0.2 \text{ m}^3/\text{min}$, how fast is the water level rising when the water is 30 cm deep?
26. A swimming pool is 5 m wide, 10 m long, 1 m deep at the shallow end, and 3 m deep at its deepest point. A cross-section

is shown in the figure. If the pool is being filled at a rate of $0.1 \text{ m}^3/\text{min}$, how fast is the water level rising when the depth at the deepest point is 1 m ?



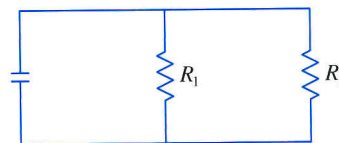
- 27.** Gravel is being dumped from a conveyor belt at a rate of $3 \text{ m}^3/\text{min}$, and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 3 m high?



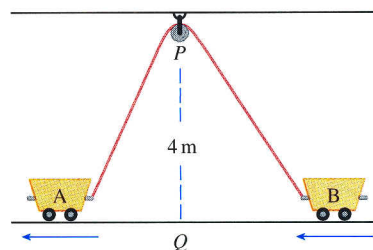
- 28.** A kite 50 m above the ground moves horizontally at a speed of 2 m/s . At what rate is the angle between the string and the horizontal decreasing when 100 m of string has been let out?
- 29.** Two sides of a triangle are 4 m and 5 m in length and the angle between them is increasing at a rate of 0.06 rad/s . Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is $\pi/3$.
- 30.** How fast is the angle between the ladder and the ground changing in Example 2 when the bottom of the ladder is 3 m from the wall?
- 31.** Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure P and volume V satisfy the equation $PV = C$, where C is a constant. Suppose that at a certain instant the volume is 600 cm^3 , the pressure is 150 kPa , and the pressure is increasing at a rate of 20 kPa/min . At what rate is the volume decreasing at this instant?
- 32.** When air expands adiabatically (without gaining or losing heat), its pressure P and volume V are related by the equation $PV^{1.4} = C$, where C is a constant. Suppose that at a certain instant the volume is 400 cm^3 and the pressure is 80 kPa and is decreasing at a rate of 10 kPa/min . At what rate is the volume increasing at this instant?
- 33.** If two resistors with resistances R_1 and R_2 are connected in parallel, as in the figure, then the total resistance R , measured in ohms (Ω), is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

If R_1 and R_2 are increasing at rates of $0.3 \text{ } \Omega/\text{s}$ and $0.2 \text{ } \Omega/\text{s}$, respectively, how fast is R changing when $R_1 = 80 \text{ } \Omega$ and $R_2 = 100 \text{ } \Omega$?



- 34.** Brain weight B as a function of body weight W in fish has been modeled by the power function $B = 0.007W^{2/3}$, where B and W are measured in grams. A model for body weight as a function of body length L (measured in centimeters) is $W = 0.12L^{2.53}$. If, over 10 million years, the average length of a certain species of fish evolved from 15 cm to 20 cm at a constant rate, how fast was this species' brain growing when the average length was 18 cm ?
- 35.** Two sides of a triangle have lengths 12 m and 15 m . The angle between them is increasing at a rate of $2^\circ/\text{min}$. How fast is the length of the third side increasing when the angle between the sides of fixed length is 60° ?
- 36.** Two carts, A and B, are connected by a rope 12 m long that passes over a pulley P (see the figure). The point Q is on the floor 4 m directly beneath P and between the carts. Cart A is being pulled away from Q at a speed of 0.5 m/s . How fast is cart B moving toward Q at the instant when cart A is 3 m from Q ?



- 37.** A television camera is positioned 1200 m from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. Also, the mechanism for focusing the camera has to take into account the increasing distance from the camera to the rising rocket. Let's assume the rocket rises vertically and its speed is 200 m/s when it has risen 900 m .
- (a) How fast is the distance from the television camera to the rocket changing at that moment?
- (b) If the television camera is always kept aimed at the rocket, how fast is the camera's angle of elevation changing at that same moment?
- 38.** A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P ?

39. A plane flies horizontally at an altitude of 5 km and passes directly over a tracking telescope on the ground. When the angle of elevation is $\pi/3$, this angle is decreasing at a rate of $\pi/6$ rad/min. How fast is the plane traveling at that time?
40. A Ferris wheel with a radius of 10 m is rotating at a rate of one revolution every 2 minutes. How fast is a rider rising when his seat is 16 m above ground level?
41. A plane flying with a constant speed of 300 km/h passes over a ground radar station at an altitude of 1 km and climbs at an angle of 30° . At what rate is the distance from the plane to the radar station increasing a minute later?
42. Two people start from the same point. One walks east at 4 km/h and the other walks northeast at 2 km/h. How fast is the distance between the people changing after 15 minutes?
43. A runner sprints around a circular track of radius 100 m at a constant speed of 7 m/s. The runner's friend is standing at a distance 200 m from the center of the track. How fast is the distance between the friends changing when the distance between them is 200 m?
44. The minute hand on a watch is 8 mm long and the hour hand is 4 mm long. How fast is the distance between the tips of the hands changing at one o'clock?

3.9 LINEAR APPROXIMATIONS AND DIFFERENTIALS

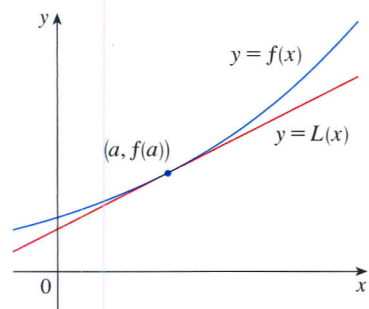


FIGURE 1

We have seen that a curve lies very close to its tangent line near the point of tangency. In fact, by zooming in toward a point on the graph of a differentiable function, we noticed that the graph looks more and more like its tangent line. (See Figure 2 in Section 3.1.) This observation is the basis for a method of finding approximate values of functions.

The idea is that it might be easy to calculate a value $f(a)$ of a function, but difficult (or even impossible) to compute nearby values of f . So we settle for the easily computed values of the linear function L whose graph is the tangent line of f at $(a, f(a))$. (See Figure 1.)

In other words, we use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a . An equation of this tangent line is

$$y = f(a) + f'(a)(x - a)$$

and the approximation

$$\boxed{1} \quad f(x) \approx f(a) + f'(a)(x - a)$$

is called the **linear approximation** or **tangent line approximation** of f at a . The linear function whose graph is this tangent line, that is,

$$\boxed{2} \quad L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at a .

EXAMPLE 1 Find the linearization of the function $f(x) = \sqrt{x + 3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

SOLUTION The derivative of $f(x) = (x + 3)^{1/2}$ is

$$f'(x) = \frac{1}{2}(x + 3)^{-1/2} = \frac{1}{2\sqrt{x + 3}}$$

and so we have $f(1) = 2$ and $f'(1) = \frac{1}{4}$. Putting these values into Equation 2, we see that the linearization is

$$L(x) = f(1) + f'(1)(x - 1) = 2 + \frac{1}{4}(x - 1) = \frac{7}{4} + \frac{x}{4}$$

V EXAMPLE 4 The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?

SOLUTION If the radius of the sphere is r , then its volume is $V = \frac{4}{3}\pi r^3$. If the error in the measured value of r is denoted by $dr = \Delta r$, then the corresponding error in the calculated value of V is ΔV , which can be approximated by the differential

$$dV = 4\pi r^2 dr$$

When $r = 21$ and $dr = 0.05$, this becomes

$$dV = 4\pi(21)^2 0.05 \approx 277$$

The maximum error in the calculated volume is about 277 cm^3 . □

NOTE Although the possible error in Example 4 may appear to be rather large, a better picture of the error is given by the **relative error**, which is computed by dividing the error by the total volume:

$$\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3} = 3 \frac{dr}{r}$$

Thus the relative error in the volume is about three times the relative error in the radius. In Example 4 the relative error in the radius is approximately $dr/r = 0.05/21 \approx 0.0024$ and it produces a relative error of about 0.007 in the volume. The errors could also be expressed as **percentage errors** of 0.24% in the radius and 0.7% in the volume.

3.9 EXERCISES

1–4 Find the linearization $L(x)$ of the function at a .

1. $f(x) = x^3$, $a = 1$ 2. $f(x) = 1/\sqrt{2+x}$, $a = 0$

3. $f(x) = \cos x$, $a = \pi/2$ 4. $f(x) = x^{3/4}$, $a = 16$

5. Find the linear approximation of the function $f(x) = \sqrt{1-x}$ at $a = 0$ and use it to approximate the numbers $\sqrt{0.9}$ and $\sqrt{0.99}$. Illustrate by graphing f and the tangent line.

6. Find the linear approximation of the function $g(x) = \sqrt[3]{1+x}$ at $a = 0$ and use it to approximate the numbers $\sqrt[3]{0.95}$ and $\sqrt[3]{1.1}$. Illustrate by graphing g and the tangent line.

7–10 Verify the given linear approximation at $a = 0$. Then determine the values of x for which the linear approximation is accurate to within 0.1.

7. $\sqrt{1+x} \approx 1 + \frac{1}{2}x$ 8. $\tan x \approx x$

9. $1/(1+2x)^4 \approx 1 - 8x$ 10. $1/\sqrt{4-x} \approx \frac{1}{2} + \frac{1}{16}x$

11–14 Find the differential of each function.

11. (a) $y = x^2 \sin 2x$ (b) $y = \sqrt{1+t^2}$

12. (a) $y = s/(1+2s)$ (b) $y = u \cos u$

13. (a) $y = \frac{u+1}{u-1}$ (b) $y = (1+r^3)^{-2}$

14. (a) $y = (t + \tan t)^5$ (b) $y = \sqrt{z+1/z}$

15–18 (a) Find the differential dy and (b) evaluate dy for the given values of x and dx .

15. $y = \sqrt{4+5x}$, $x = 0$, $dx = 0.04$

16. $y = 1/(x+1)$, $x = 1$, $dx = -0.01$

17. $y = \tan x$, $x = \pi/4$, $dx = -0.1$

18. $y = \cos x$, $x = \pi/3$, $dx = 0.05$

19–22 Compute Δy and dy for the given values of x and $dx = \Delta x$. Then sketch a diagram like Figure 5 showing the line segments with lengths dx , dy , and Δy .

19. $y = 2x - x^2$, $x = 2$, $\Delta x = -0.4$

20. $y = \sqrt{x}$, $x = 1$, $\Delta x = 1$

21. $y = 2/x$, $x = 4$, $\Delta x = 1$

22. $y = x^3$, $x = 1$, $\Delta x = 0.5$

23–28 Use a linear approximation (or differentials) to estimate the given number.

23. $(2.001)^5$ 24. $\sin 1^\circ$ 25. $(8.06)^{2/3}$
 26. $1/1002$ 27. $\tan 44^\circ$ 28. $\sqrt{99.8}$

29–30 Explain, in terms of linear approximations or differentials, why the approximation is reasonable.

29. $\sec 0.08 \approx 1$ 30. $(1.01)^6 \approx 1.06$

31. The edge of a cube was found to be 30 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum possible error, relative error, and percentage error in computing (a) the volume of the cube and (b) the surface area of the cube.

32. The radius of a circular disk is given as 24 cm with a maximum error in measurement of 0.2 cm.

- (a) Use differentials to estimate the maximum error in the calculated area of the disk.
 (b) What is the relative error? What is the percentage error?

33. The circumference of a sphere was measured to be 84 cm with a possible error of 0.5 cm.

- (a) Use differentials to estimate the maximum error in the calculated surface area. What is the relative error?
 (b) Use differentials to estimate the maximum error in the calculated volume. What is the relative error?

34. Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 m.

35. (a) Use differentials to find a formula for the approximate volume of a thin cylindrical shell with height h , inner radius r , and thickness Δr .

- (b) What is the error involved in using the formula from part (a)?

36. One side of a right triangle is known to be 20 cm long and the opposite angle is measured as 30° , with a possible error of $\pm 1^\circ$.

- (a) Use differentials to estimate the error in computing the length of the hypotenuse.
 (b) What is the percentage error?

37. If a current I passes through a resistor with resistance R , Ohm's Law states that the voltage drop is $V = RI$. If V is constant and R is measured with a certain error, use differentials to show that the relative error in calculating I is approximately the same (in magnitude) as the relative error in R .

38. When blood flows along a blood vessel, the flux F (the volume of blood per unit time that flows past a given point) is proportional to the fourth power of the radius R of the blood vessel:

$$F = kR^4$$

(This is known as Poiseuille's Law; we will show why it is true in Section 9.4.) A partially clogged artery can be expanded by an operation called angioplasty, in which a balloon-tipped catheter is inflated inside the artery in order to widen it and restore the normal blood flow.

Show that the relative change in F is about four times the relative change in R . How will a 5% increase in the radius affect the flow of blood?

39. Establish the following rules for working with differentials (where c denotes a constant and u and v are functions of x).

- (a) $dc = 0$ (b) $d(cu) = c du$
 (c) $d(u + v) = du + dv$ (d) $d(uv) = u dv + v du$
 (e) $d\left(\frac{u}{v}\right) = \frac{v du - u dv}{v^2}$ (f) $d(x^n) = nx^{n-1} dx$

40. On page 431 of *Physics: Calculus*, 2d ed., by Eugene Hecht (Pacific Grove, CA: Brooks/Cole, 2000), in the course of deriving the formula $T = 2\pi\sqrt{L/g}$ for the period of a pendulum of length L , the author obtains the equation $a_T = -g \sin \theta$ for the tangential acceleration of the bob of the pendulum. He then says, "for small angles, the value of θ in radians is very nearly the value of $\sin \theta$; they differ by less than 2% out to about 20° ."

(a) Verify the linear approximation at 0 for the sine function:

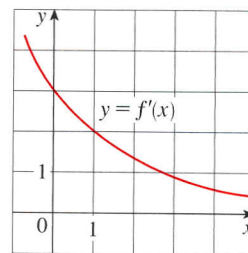
$$\sin x \approx x$$



(b) Use a graphing device to determine the values of x for which $\sin x$ and x differ by less than 2%. Then verify Hecht's statement by converting from radians to degrees.

41. Suppose that the only information we have about a function f is that $f(1) = 5$ and the graph of its derivative is as shown.

- (a) Use a linear approximation to estimate $f(0.9)$ and $f(1.1)$.
 (b) Are your estimates in part (a) too large or too small? Explain.



42. Suppose that we don't have a formula for $g(x)$ but we know that

$$g(2) = -4 \quad \text{and} \quad g'(x) = \sqrt{x^2 + 5}$$

for all x .

- (a) Use a linear approximation to estimate $g(1.95)$ and $g(2.05)$.
 (b) Are your estimates in part (a) too large or too small? Explain.

3 REVIEW

CONCEPT CHECK

- Write an expression for the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$.
- Suppose an object moves along a straight line with position $f(t)$ at time t . Write an expression for the instantaneous velocity of the object at time $t = a$. How can you interpret this velocity in terms of the graph of f ?
- If $y = f(x)$ and x changes from x_1 to x_2 , write expressions for the following.
 - The average rate of change of y with respect to x over the interval $[x_1, x_2]$.
 - The instantaneous rate of change of y with respect to x at $x = x_1$.
- Define the derivative $f'(a)$. Discuss two ways of interpreting this number.
- What does it mean for f to be differentiable at a ?
 - What is the relation between the differentiability and continuity of a function?
 - Sketch the graph of a function that is continuous but not differentiable at $a = 2$.
- Describe several ways in which a function can fail to be differentiable. Illustrate with sketches.
- What are the second and third derivatives of a function f ? If f is the position function of an object, how can you interpret f'' and f''' ?
- State each differentiation rule both in symbols and in words.

(a) The Power Rule	(b) The Constant Multiple Rule
(c) The Sum Rule	(d) The Difference Rule
(e) The Product Rule	(f) The Quotient Rule
(g) The Chain Rule	
- State the derivative of each function.

(a) $y = x^n$	(b) $y = \sin x$	(c) $y = \cos x$
(d) $y = \tan x$	(e) $y = \csc x$	(f) $y = \sec x$
(g) $y = \cot x$		
- Explain how implicit differentiation works.
- Write an expression for the linearization of f at a .
 - If $y = f(x)$, write an expression for the differential dy .
 - If $dx = \Delta x$, draw a picture showing the geometric meanings of Δy and dy .

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

- If f is continuous at a , then f is differentiable at a .
- If f and g are differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$
- If f and g are differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g'(x)$$
- If f and g are differentiable, then

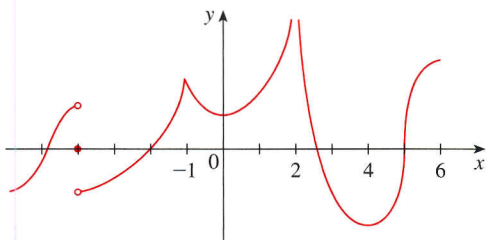
$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$
- If f is differentiable, then

$$\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}$$
- If f is differentiable, then

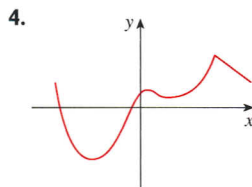
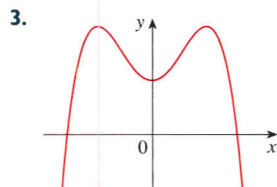
$$\frac{d}{dx}f(\sqrt{x}) = \frac{f'(x)}{2\sqrt{x}}$$
- $$\frac{d}{dx}|x^2 + x| = |2x + 1|$$
- If $f'(r)$ exists, then $\lim_{x \rightarrow r} f(x) = f(r)$.
- If $g(x) = x^5$, then $\lim_{x \rightarrow 2} \frac{g(x) - g(2)}{x - 2} = 80$.
- $$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$
- An equation of the tangent line to the parabola $y = x^2$ at $(-2, 4)$ is $y - 4 = 2x(x + 2)$.
- $$\frac{d}{dx}(\tan^2x) = \frac{d}{dx}(\sec^2x)$$

EXERCISES

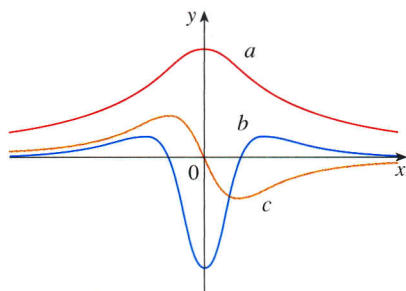
1. The displacement (in meters) of an object moving in a straight line is given by $s = 1 + 2t + \frac{1}{4}t^2$, where t is measured in seconds.
- (a) Find the average velocity over each time period.
- (i) [1, 3] (ii) [1, 2]
 (iii) [1, 1.5] (iv) [1, 1.1]
- (b) Find the instantaneous velocity when $t = 1$.
2. The graph of f is shown. State, with reasons, the numbers at which f is not differentiable.



3–4 Trace or copy the graph of the function. Then sketch a graph of its derivative directly beneath.



5. The figure shows the graphs of f , f' , and f'' . Identify each curve, and explain your choices.

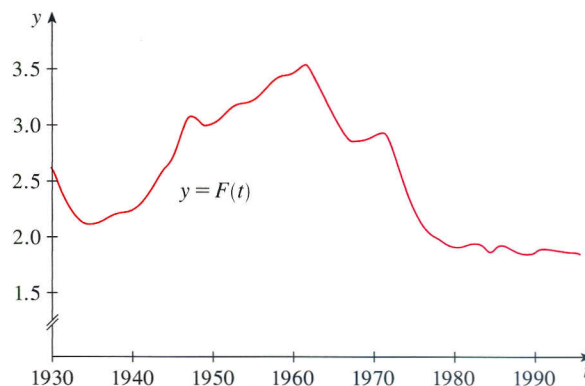


6. Find a function f and a number a such that

$$\lim_{h \rightarrow 0} \frac{(2+h)^6 - 64}{h} = f'(a)$$

7. The total cost of repaying a student loan at an interest rate of $r\%$ per year is $C = f(r)$.
- (a) What is the meaning of the derivative $f'(r)$? What are its units?
- (b) What does the statement $f'(10) = 1200$ mean?
- (c) Is $f'(r)$ always positive or does it change sign?

8. The total fertility rate at time t , denoted by $F(t)$, is an estimate of the average number of children born to each woman (assuming that current birth rates remain constant). The graph of the total fertility rate in Australia shows the fluctuations from 1930 to 2000.
- (a) Estimate the values of $F'(1946)$ and $F'(1973)$.
- (b) What are the meanings of these derivatives?
- (c) Can you suggest reasons for the values of these derivatives?



9. Let $E(t)$ be the value of the euro (the European currency) in terms of the US dollar at time t . The table gives values of this function, as of midyear, from 2000 to 2004. Interpret $E'(2002)$ and estimate its value.

t	2000	2001	2002	2003	2004
$E(t)$	0.955	0.847	0.986	1.149	1.218

- 10–11 Find $f'(x)$ from first principles, that is, directly from the definition of a derivative.

10. $f(x) = \frac{4-x}{3+x}$

11. $f(x) = x^3 + 5x + 4$

12. (a) If $f(x) = \sqrt{3-5x}$, use the definition of a derivative to find $f'(x)$.

(b) Find the domains of f and f' .

- (c) Graph f and f' on a common screen. Compare the graphs to see whether your answer to part (a) is reasonable.

13–40 Calculate y' .

13. $y = (x^4 - 3x^2 + 5)^3$

14. $y = \cos(\tan x)$

15. $y = \sqrt{x} + \frac{1}{\sqrt[3]{x^4}}$

16. $y = \frac{3x-2}{\sqrt{2x+1}}$

17. $y = 2x\sqrt{x^2+1}$

18. $y = \left(x + \frac{1}{x^2}\right)^{\sqrt{7}}$

19. $y = \frac{t}{1-t^2}$

20. $y = \sin(\cos x)$

21. $y = \tan \sqrt{1-x}$ 22. $y = \frac{1}{\sin(x - \sin x)}$
23. $xy^4 + x^2y = x + 3y$ 24. $y = \sec(1 + x^2)$
25. $y = \frac{\sec 2\theta}{1 + \tan 2\theta}$ 26. $x^2 \cos y + \sin 2y = xy$
27. $y = (1 - x^{-1})^{-1}$ 28. $y = 1/\sqrt[3]{x + \sqrt{x}}$
29. $\sin(xy) = x^2 - y$ 29. $y = \sqrt{\sin \sqrt{x}}$
31. $y = \cot(3x^2 + 5)$ 32. $y = \frac{(x + \lambda)^4}{x^4 + \lambda^4}$
33. $y = \sqrt{x} \cos \sqrt{x}$ 34. $y = \frac{\sin mx}{x}$
35. $y = \tan^2(\sin \theta)$ 36. $x \tan y = y - 1$
37. $y = \sqrt[3]{x \tan x}$ 38. $y = \frac{(x-1)(x-4)}{(x-2)(x-3)}$
39. $y = \sin(\tan \sqrt{1+x^3})$ 40. $y = \sin^2(\cos \sqrt{\sin \pi x})$

41. If $f(t) = \sqrt{4t+1}$, find $f''(2)$.
42. If $g(\theta) = \theta \sin \theta$, find $g''(\pi/6)$.
43. Find y'' if $x^6 + y^6 = 1$.
44. Find $f^{(n)}(x)$ if $f(x) = 1/(2-x)$.

45–46 Find the limit.

45. $\lim_{x \rightarrow 0} \frac{\sec x}{1 - \sin x}$ 46. $\lim_{t \rightarrow 0} \frac{t^3}{\tan^3 2t}$



47–48 Find an equation of the tangent to the curve at the given point.

47. $y = 4 \sin^2 x$, $(\pi/6, 1)$ 48. $y = \frac{x^2 - 1}{x^2 + 1}$, $(0, -1)$

49–50 Find equations of the tangent line and normal line to the curve at the given point.

49. $y = \sqrt{1 + 4 \sin x}$, $(0, 1)$

50. $x^2 + 4xy + y^2 = 13$, $(2, 1)$

51. (a) If $f(x) = x\sqrt{5-x}$, find $f'(x)$.
 (b) Find equations of the tangent lines to the curve $y = x\sqrt{5-x}$ at the points $(1, 2)$ and $(4, 4)$.
-  (c) Illustrate part (b) by graphing the curve and tangent lines on the same screen.
-  (d) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f' .
52. (a) If $f(x) = 4x - \tan x$, $-\pi/2 < x < \pi/2$, find f' and f'' .
 (b) Check to see that your answers to part (a) are reasonable by comparing the graphs of f , f' , and f'' .

53. At what points on the curve $y = \sin x + \cos x$, $0 \leq x \leq 2\pi$, is the tangent line horizontal?
54. Find the points on the ellipse $x^2 + 2y^2 = 1$ where the tangent line has slope 1.
55. Find a parabola $y = ax^2 + bx + c$ that passes through the point $(1, 4)$ and whose tangent lines at $x = -1$ and $x = 5$ have slopes 6 and -2 , respectively.
56. How many tangent lines to the curve $y = x/(x+1)$ pass through the point $(1, 2)$? At which points do these tangent lines touch the curve?
57. If $f(x) = (x-a)(x-b)(x-c)$, show that

$$\frac{f'(x)}{f(x)} = \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}$$

58. (a) By differentiating the double-angle formula

$$\cos 2x = \cos^2 x - \sin^2 x$$

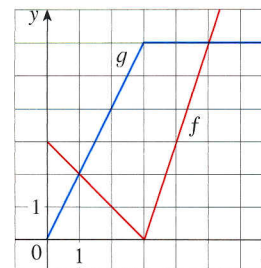
obtain the double-angle formula for the sine function.

- (b) By differentiating the addition formula

$$\sin(x+a) = \sin x \cos a + \cos x \sin a$$

obtain the addition formula for the cosine function.

59. Suppose that $h(x) = f(x)g(x)$ and $F(x) = f(g(x))$, where $f(2) = 3$, $g(2) = 5$, $g'(2) = 4$, $f'(2) = -2$, and $f'(5) = 11$. Find (a) $h'(2)$ and (b) $F'(2)$.
60. If f and g are the functions whose graphs are shown, let $P(x) = f(x)g(x)$, $Q(x) = f(x)/g(x)$, and $C(x) = f(g(x))$. Find (a) $P'(2)$, (b) $Q'(2)$, and (c) $C'(2)$.



61–68 Find f' in terms of g' .

61. $f(x) = x^2g(x)$ 62. $f(x) = g(x^2)$
63. $f(x) = [g(x)]^2$ 64. $f(x) = x^a g(x^b)$
65. $f(x) = g(g(x))$ 66. $f(x) = \sin(g(x))$
67. $f(x) = g(\sin x)$ 68. $f(x) = g(\tan \sqrt{x})$

69–71 Find h' in terms of f' and g' .

69. $h(x) = \frac{f(x)g(x)}{f(x) + g(x)}$ 70. $h(x) = \sqrt{\frac{f(x)}{g(x)}}$

71. $h(x) = f(g(\sin 4x))$

72. A particle moves along a horizontal line so that its coordinate at time t is $x = \sqrt{b^2 + c^2 t^2}$, $t \geq 0$, where b and c are positive constants.

- (a) Find the velocity and acceleration functions.
 (b) Show that the particle always moves in the positive direction.

73. A particle moves on a vertical line so that its coordinate at time t is $y = t^3 - 12t + 3$, $t \geq 0$.

- (a) Find the velocity and acceleration functions.
 (b) When is the particle moving upward and when is it moving downward?
 (c) Find the distance that the particle travels in the time interval $0 \leq t \leq 3$.
 (d) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 3$.
 (e) When is the particle speeding up? When is it slowing down?

74. The volume of a right circular cone is $V = \pi r^2 h/3$, where r is the radius of the base and h is the height.

- (a) Find the rate of change of the volume with respect to the height if the radius is constant.
 (b) Find the rate of change of the volume with respect to the radius if the height is constant.

75. The mass of part of a wire is $x(1 + \sqrt{x})$ kilograms, where x is measured in meters from one end of the wire. Find the linear density of the wire when $x = 4$ m.

76. The cost, in dollars, of producing x units of a certain commodity is

$$C(x) = 920 + 2x - 0.02x^2 + 0.00007x^3$$

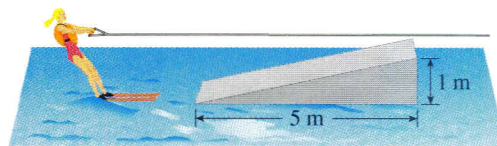
- (a) Find the marginal cost function.
 (b) Find $C'(100)$ and explain its meaning.
 (c) Compare $C'(100)$ with the cost of producing the 101st item.

77. The volume of a cube is increasing at a rate of $10 \text{ cm}^3/\text{min}$. How fast is the surface area increasing when the length of an edge is 30 cm ?

78. A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of $2 \text{ cm}^3/\text{s}$, how fast is the water level rising when the water is 5 cm deep?

79. A balloon is rising at a constant speed of 2 m/s . A boy is cycling along a straight road at a speed of 5 m/s . When he passes under the balloon, it is 15 m above him. How fast is the distance between the boy and the balloon increasing 3 s later?

80. A waterskier skis over the ramp shown in the figure at a speed of 10 m/s . How fast is she rising as she leaves the ramp?



81. The angle of elevation of the sun is decreasing at a rate of 0.25 rad/h . How fast is the shadow cast by a 400-m -tall building increasing when the angle of elevation of the sun is $\pi/6$?

82. (a) Find the linear approximation to $f(x) = \sqrt{25 - x^2}$ near 3 .
 (b) Illustrate part (a) by graphing f and the linear approximation.
 (c) For what values of x is the linear approximation accurate to within 0.1 ?

83. (a) Find the linearization of $f(x) = \sqrt[3]{1 + 3x}$ at $a = 0$. State the corresponding linear approximation and use it to give an approximate value for $\sqrt[3]{1.03}$.

- (b) Determine the values of x for which the linear approximation given in part (a) is accurate to within 0.1 .

84. Evaluate dy if $y = x^3 - 2x^2 + 1$, $x = 2$, and $dx = 0.2$.

85. A window has the shape of a square surmounted by a semi-circle. The base of the window is measured as having width 60 cm with a possible error in measurement of 0.1 cm . Use differentials to estimate the maximum error possible in computing the area of the window.

86–88 Express the limit as a derivative and evaluate.

86. $\lim_{x \rightarrow 1} \frac{x^{17} - 1}{x - 1}$

87. $\lim_{h \rightarrow 0} \frac{\sqrt[4]{16 + h} - 2}{h}$

88. $\lim_{\theta \rightarrow \pi/3} \frac{\cos \theta - 0.5}{\theta - \pi/3}$

89. Evaluate $\lim_{x \rightarrow 0} \frac{\sqrt{1 + \tan x} - \sqrt{1 + \sin x}}{x^3}$.

90. Suppose f is a differentiable function such that $f(g(x)) = x$ and $f'(x) = 1 + [f(x)]^2$. Show that $g'(x) = 1/(1 + x^2)$.

91. Find $f'(x)$ if it is known that

$$\frac{d}{dx} [f(2x)] = x^2$$

92. Show that the length of the portion of any tangent line to the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ cut off by the coordinate axes is constant.

Before you look at the example, cover up the solution and try it yourself first.

EXAMPLE 1 How many lines are tangent to both of the parabolas $y = -1 - x^2$ and $y = 1 + x^2$? Find the coordinates of the points at which these tangents touch the parabolas.

SOLUTION To gain insight into this problem, it is essential to draw a diagram. So we sketch the parabolas $y = 1 + x^2$ (which is the standard parabola $y = x^2$ shifted 1 unit upward) and $y = -1 - x^2$ (which is obtained by reflecting the first parabola about the x -axis). If we try to draw a line tangent to both parabolas, we soon discover that there are only two possibilities, as illustrated in Figure 1.

Let P be a point at which one of these tangents touches the upper parabola and let a be its x -coordinate. (The choice of notation for the unknown is important. Of course we could have used b or c or x_0 or x_1 instead of a . However, it's not advisable to use x in place of a because that x could be confused with the variable x in the equation of the parabola.) Then, since P lies on the parabola $y = 1 + x^2$, its y -coordinate must be $1 + a^2$. Because of the symmetry shown in Figure 1, the coordinates of the point Q where the tangent touches the lower parabola must be $(-a, -(1 + a^2))$.

To use the given information that the line is a tangent, we equate the slope of the line PQ to the slope of the tangent line at P . We have

$$m_{PQ} = \frac{1 + a^2 - (-1 - a^2)}{a - (-a)} = \frac{1 + a^2}{a}$$

If $f(x) = 1 + x^2$, then the slope of the tangent line at P is $f'(a) = 2a$. Thus the condition that we need to use is that

$$\frac{1 + a^2}{a} = 2a$$

Solving this equation, we get $1 + a^2 = 2a^2$, so $a^2 = 1$ and $a = \pm 1$. Therefore the points are $(1, 2)$ and $(-1, -2)$. By symmetry, the two remaining points are $(-1, 2)$ and $(1, -2)$. □

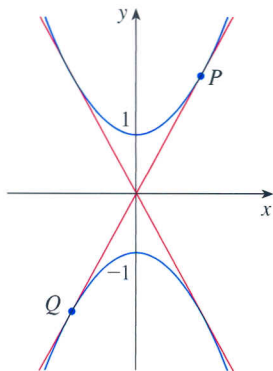
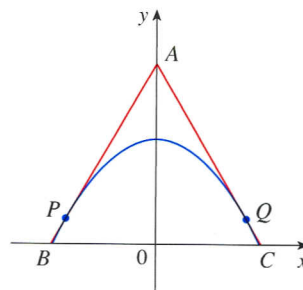


FIGURE 1

PROBLEMS

- Find points P and Q on the parabola $y = 1 - x^2$ so that the triangle ABC formed by the x -axis and the tangent lines at P and Q is an equilateral triangle.



2. Find the point where the curves $y = x^3 - 3x + 4$ and $y = 3(x^2 - x)$ are tangent to each other, that is, have a common tangent line. Illustrate by sketching both curves and the common tangent.
3. Show that the tangent lines to the parabola $y = ax^2 + bx + c$ at any two points with x -coordinates p and q must intersect at a point whose x -coordinate is halfway between p and q .
4. Show that

$$\frac{d}{dx} \left(\frac{\sin^2 x}{1 + \cot x} + \frac{\cos^2 x}{1 + \tan x} \right) = -\cos 2x$$

5. Suppose f is a function that satisfies the equation

$$f(x + y) = f(x) + f(y) + x^2 y + xy^2$$

for all real numbers x and y . Suppose also that

$$\lim_{x \rightarrow 0} \frac{f(x)}{x} = 1$$

- (a) Find $f(0)$. (b) Find $f'(0)$. (c) Find $f'(x)$.

6. A car is traveling at night along a highway shaped like a parabola with its vertex at the origin (see the figure). The car starts at a point 100 m west and 100 m north of the origin and travels in an easterly direction. There is a statue located 100 m east and 50 m north of the origin. At what point on the highway will the car's headlights illuminate the statue?

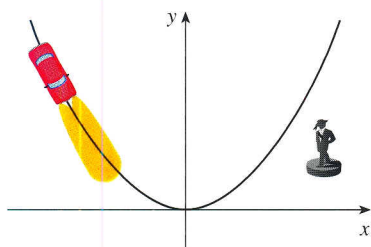
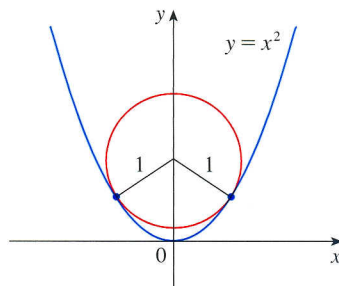


FIGURE FOR PROBLEM 6

7. Prove that $\frac{d^n}{dx^n} (\sin^4 x + \cos^4 x) = 4^{n-1} \cos(4x + n\pi/2)$.
8. Find the n th derivative of the function $f(x) = x^n/(1 - x)$.
9. The figure shows a circle with radius 1 inscribed in the parabola $y = x^2$. Find the center of the circle.



10. If f is differentiable at a , where $a > 0$, evaluate the following limit in terms of $f'(a)$:

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}}$$

PROBLEMS PLUS

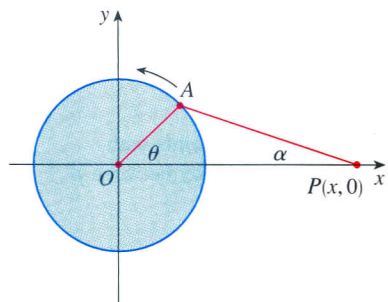


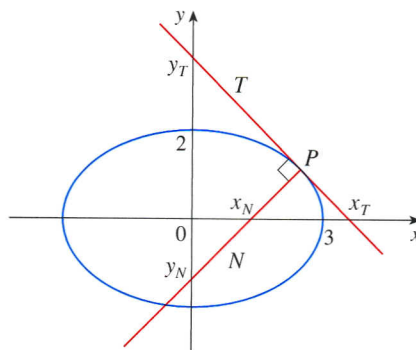
FIGURE FOR PROBLEM 11

11. The figure shows a rotating wheel with radius 40 cm and a connecting rod AP with length 1.2 m. The pin P slides back and forth along the x -axis as the wheel rotates counterclockwise at a rate of 360 revolutions per minute.
- Find the angular velocity of the connecting rod, $d\alpha/dt$, in radians per second, when $\theta = \pi/3$.
 - Express the distance $x = |OP|$ in terms of θ .
 - Find an expression for the velocity of the pin P in terms of θ .

12. Tangent lines T_1 and T_2 are drawn at two points P_1 and P_2 on the parabola $y = x^2$ and they intersect at a point P . Another tangent line T is drawn at a point between P_1 and P_2 ; it intersects T_1 at Q_1 and T_2 at Q_2 . Show that

$$\frac{|PQ_1|}{|PP_1|} + \frac{|PQ_2|}{|PP_2|} = 1$$

13. Let T and N be the tangent and normal lines to the ellipse $x^2/9 + y^2/4 = 1$ at any point P on the ellipse in the first quadrant. Let x_T and y_T be the x - and y -intercepts of T and x_N and y_N be the intercepts of N . As P moves along the ellipse in the first quadrant (but not on the axes), what values can x_T , y_T , x_N , and y_N take on? First try to guess the answers just by looking at the figure. Then use calculus to solve the problem and see how good your intuition is.



14. Evaluate $\lim_{x \rightarrow 0} \frac{\sin(3+x)^2 - \sin 9}{x}$.
15. (a) Use the identity for $\tan(x - y)$ (see Equation 14b in Appendix D) to show that if two lines L_1 and L_2 intersect at an angle α , then

$$\tan \alpha = \frac{m_2 - m_1}{1 + m_1 m_2}$$

where m_1 and m_2 are the slopes of L_1 and L_2 , respectively.

- (b) The **angle between the curves** C_1 and C_2 at a point of intersection P is defined to be the angle between the tangent lines to C_1 and C_2 at P (if these tangent lines exist). Use part (a) to find, correct to the nearest degree, the angle between each pair of curves at each point of intersection.
- $y = x^2$ and $y = (x - 2)^2$
 - $x^2 - y^2 = 3$ and $x^2 - 4x + y^2 + 3 = 0$
16. Let $P(x_1, y_1)$ be a point on the parabola $y^2 = 4px$ with focus $F(p, 0)$. Let α be the angle between the parabola and the line segment FP , and let β be the angle between the horizontal line $y = y_1$ and the parabola as in the figure. Prove that $\alpha = \beta$. (Thus, by a principle of geometrical optics, light from a source placed at F will be reflected along a line parallel to the

x -axis. This explains why *paraboloids*, the surfaces obtained by rotating parabolas about their axes, are used as the shape of some automobile headlights and mirrors for telescopes.)

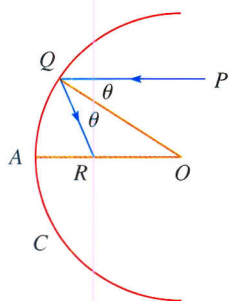
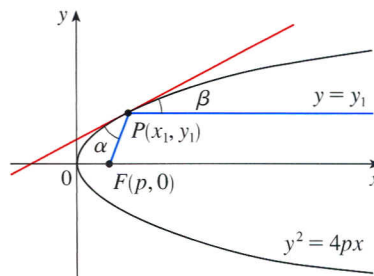


FIGURE FOR PROBLEM 17

17. Suppose that we replace the parabolic mirror of Problem 16 by a spherical mirror. Although the mirror has no focus, we can show the existence of an *approximate* focus. In the figure, C is a semicircle with center O . A ray of light coming in toward the mirror parallel to the axis along the line PQ will be reflected to the point R on the axis so that $\angle PQO = \angle OQR$ (the angle of incidence is equal to the angle of reflection). What happens to the point R as P is taken closer and closer to the axis?

18. If f and g are differentiable functions with $f(0) = g(0) = 0$ and $g'(0) \neq 0$, show that

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)}$$

19. Evaluate $\lim_{x \rightarrow 0} \frac{\sin(a + 2x) - 2 \sin(a + x) + \sin a}{x^2}$.
20. Given an ellipse $x^2/a^2 + y^2/b^2 = 1$, where $a \neq b$, find the equation of the set of all points from which there are two tangents to the curve whose slopes are (a) reciprocals and (b) negative reciprocals.
21. Find the two points on the curve $y = x^4 - 2x^2 - x$ that have a common tangent line.
22. Suppose that three points on the parabola $y = x^2$ have the property that their normal lines intersect at a common point. Show that the sum of their x -coordinates is 0.
23. A *lattice point* in the plane is a point with integer coordinates. Suppose that circles with radius r are drawn using all lattice points as centers. Find the smallest value of r such that any line with slope $\frac{2}{3}$ intersects some of these circles.
24. A cone of radius r centimeters and height h centimeters is lowered point first at a rate of 1 cm/s into a tall cylinder of radius R centimeters that is partially filled with water. How fast is the water level rising at the instant the cone is completely submerged?
25. A container in the shape of an inverted cone has height 16 cm and radius 5 cm at the top. It is partially filled with a liquid that oozes through the sides at a rate proportional to the area of the container that is in contact with the liquid. (The surface area of a cone is πrl , where r is the radius and l is the slant height.) If we pour the liquid into the container at a rate of $2 \text{ cm}^3/\text{min}$, then the height of the liquid decreases at a rate of 0.3 cm/min when the height is 10 cm. If our goal is to keep the liquid at a constant height of 10 cm, at what rate should we pour the liquid into the container?
- CAS** 26. (a) The cubic function $f(x) = x(x - 2)(x - 6)$ has three distinct zeros: 0, 2, and 6. Graph f and its tangent lines at the *average* of each pair of zeros. What do you notice?
 (b) Suppose the cubic function $f(x) = (x - a)(x - b)(x - c)$ has three distinct zeros: a , b , and c . Prove, with the help of a computer algebra system, that a tangent line drawn at the average of the zeros a and b intersects the graph of f at the third zero.