2.1 EXERCISES

1. A tank holds 1000 liters of water, which drains from the bottom of the tank in half an hour. The values in the table show the volume V of water remaining in the tank (in liters) after t minutes.

t (min)	5	10	15	20	25	30
V (L)	694	444	250	111	28	0

(a) If P is the point (15, 250) on the graph of V, find the slopes of the secant lines PQ when Q is the point on the graph with $t = 5, 10, 20, 25$, and 30.

(b) Estimate the slope of the tangent line at P by averaging the slopes of two secant lines.

(c) Use a graph of the function to estimate the slope of the tangent line at P. (This slope represents the rate at which the water is flowing from the tank after 15 minutes.)

2. A cardiac monitor is used to measure the heart rate of a patient after surgery. It compiles the number of heartbeats after t minutes. When the data in the table are graphed, the slope of the tangent line represents the heart rate in beats per minute.

t (min)	36	38	40	42	44
Heartbeats	2530	2661	2806	2948	3080

The monitor estimates this value by calculating the slope of a secant line. Use the data to estimate the patient's heart rate after 42 minutes using the secant line between the points with the given values of t.

(a) $t = 36$ and $t = 42$ (b) $t = 38$ and $t = 42$
(c) $t = 40$ and $t = 42$ (d) $t = 42$ and $t = 44$

What are your conclusions?

3. The point $P(1, \frac{1}{2})$ lies on the curve $y = x/(1 + x)$.

(a) If Q is the point $(x, x/(1 + x))$, use your calculator to find the slope of the secant line PQ (correct to six decimal places) for the following values of x:

(i) 0.5 (ii) 0.9 (iii) 0.99 (iv) 0.999
(v) 1.5 (vi) 1.1 (vii) 1.01 (viii) 1.001

(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(1, \frac{1}{2})$.

(c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(1, \frac{1}{2})$.

4. The point $P(3, 1)$ lies on the curve $y = \sqrt{x - 2}$.

(a) If Q is the point $(x, \sqrt{x - 2})$, use your calculator to find the slope of the secant line PQ (correct to six decimal places) for the following values of x:

(i) 2.5 (ii) 2.9 (iii) 2.99 (iv) 2.999
(v) 3.5 (vi) 3.1 (vii) 3.01 (viii) 3.001

(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(3, 1)$.

(c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(3, 1)$.

(d) Sketch the curve, two of the secant lines, and the tangent line.

5. If a ball is thrown into the air with a velocity of 10 m/s, its height in meters t seconds later is given by $y = 10t - 4.9t^2$.

(a) Find the average velocity for the time period beginning when $t = 1.5$ and lasting

(i) 0.5 second (ii) 0.1 second
(iii) 0.05 second (iv) 0.01 second

(b) Estimate the instantaneous velocity when $t = 1.5$.

6. If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height in meters t seconds later is given by $y = 10t - 1.86t^2$.

(a) Find the average velocity over the given time intervals:

(i) [1, 2] (ii) [1, 1.5] (iii) [1, 1.1]
(iv) [1, 1.01] (v) [1, 1.001]

(b) Estimate the instantaneous velocity when $t = 1$.

7. The table shows the position of a cyclist.

t (seconds)	0	1	2	3	4	5
s (meters)	0	1.4	5.1	10.7	17.7	25.8

(a) Find the average velocity for each time period:

(i) [1, 3] (ii) [2, 3] (iii) [3, 5] (iv) [3, 4]

(b) Use the graph of s as a function of t to estimate the instantaneous velocity when $t = 3$.

8. The displacement (in centimeters) of a particle moving back and forth along a straight line is given by the equation of motion $s = 2 \sin \pi t + 3 \cos \pi t$, where t is measured in seconds.

(a) Find the average velocity during each time period:

(i) [1, 2] (ii) [1, 1.1]
(iii) [1, 1.01] (iv) [1, 1.001]

(b) Estimate the instantaneous velocity of the particle when $t = 1$.

9. The point $P(1, 0)$ lies on the curve $y = \sin(10\pi/x)$.

(a) If Q is the point $(x, \sin(10\pi/x))$, find the slope of the secant line PQ (correct to four decimal places) for $x = 2, 1.5, 1.4, 1.3, 1.2, 1.1, 0.5, 0.6, 0.7, 0.8$, and 0.9. Do the slopes appear to be approaching a limit?

(b) Use a graph of the curve to explain why the slopes of the secant lines in part (a) are not close to the slope of the tangent line at P.

(c) By choosing appropriate secant lines, estimate the slope of the tangent line at P.

EXAMPLE 9 Find $\lim_{x \to 3^+} \frac{2x}{x-3}$ and $\lim_{x \to 3^-} \frac{2x}{x-3}$.

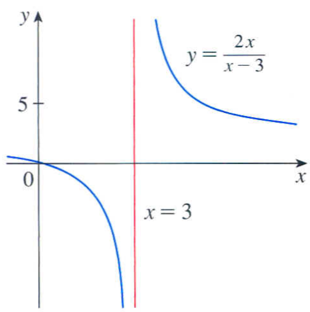


FIGURE 15

SOLUTION If x is close to 3 but larger than 3, then the denominator $x - 3$ is a small positive number and $2x$ is close to 6. So the quotient $2x/(x - 3)$ is a large *positive* number. Thus, intuitively, we see that

$$\lim_{x \to 3^+} \frac{2x}{x-3} = \infty$$

Likewise, if x is close to 3 but smaller than 3, then $x - 3$ is a small negative number but $2x$ is still a positive number (close to 6). So $2x/(x - 3)$ is a numerically large *negative* number. Thus

$$\lim_{x \to 3^-} \frac{2x}{x-3} = -\infty$$

The graph of the curve $y = 2x/(x - 3)$ is given in Figure 15. The line $x = 3$ is a vertical asymptote. □

EXAMPLE 10 Find the vertical asymptotes of $f(x) = \tan x$.

SOLUTION Because

$$\tan x = \frac{\sin x}{\cos x}$$

there are potential vertical asymptotes where $\cos x = 0$. In fact, since $\cos x \to 0^+$ as $x \to (\pi/2)^-$ and $\cos x \to 0^-$ as $x \to (\pi/2)^+$, whereas $\sin x$ is positive when x is near $\pi/2$, we have

$$\lim_{x \to (\pi/2)^-} \tan x = \infty \qquad \text{and} \qquad \lim_{x \to (\pi/2)^+} \tan x = -\infty$$

This shows that the line $x = \pi/2$ is a vertical asymptote. Similar reasoning shows that the lines $x = (2n + 1)\pi/2$, where n is an integer, are all vertical asymptotes of $f(x) = \tan x$. The graph in Figure 16 confirms this.

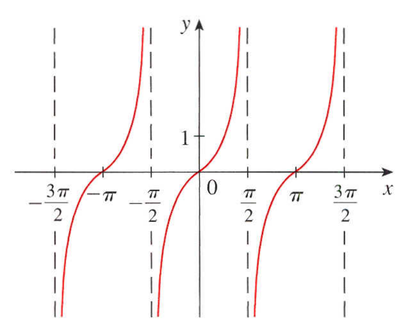


FIGURE 16
$y = \tan x$

□

2.2 EXERCISES

1. Explain in your own words what is meant by the equation

$$\lim_{x \to 2} f(x) = 5$$

Is it possible for this statement to be true and yet $f(2) = 3$? Explain.

2. Explain what it means to say that

$$\lim_{x \to 1^-} f(x) = 3 \qquad \text{and} \qquad \lim_{x \to 1^+} f(x) = 7$$

In this situation is it possible that $\lim_{x \to 1} f(x)$ exists? Explain.

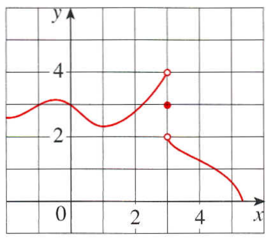
3. Explain the meaning of each of the following.

(a) $\lim_{x \to -3} f(x) = \infty$ (b) $\lim_{x \to 4^+} f(x) = -\infty$

4. For the function f whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to 0} f(x)$ (b) $\lim_{x \to 3^-} f(x)$ (c) $\lim_{x \to 3^+} f(x)$

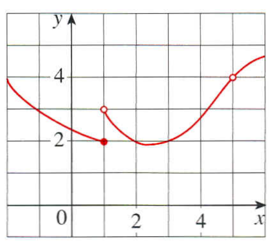
(d) $\lim_{x \to 3} f(x)$ (e) $f(3)$



5. Use the given graph of f to state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to 1^-} f(x)$ (b) $\lim_{x \to 1^+} f(x)$ (c) $\lim_{x \to 1} f(x)$

(d) $\lim_{x \to 5} f(x)$ (e) $f(5)$



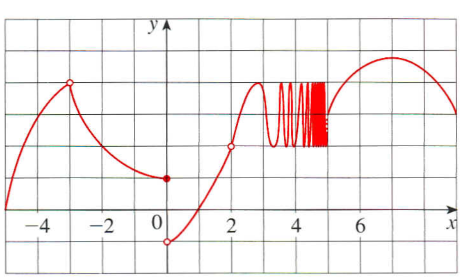
6. For the function h whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to -3^-} h(x)$ (b) $\lim_{x \to -3^+} h(x)$ (c) $\lim_{x \to -3} h(x)$

(d) $h(-3)$ (e) $\lim_{x \to 0^-} h(x)$ (f) $\lim_{x \to 0^+} h(x)$

(g) $\lim_{x \to 0} h(x)$ (h) $h(0)$ (i) $\lim_{x \to 2} h(x)$

(j) $h(2)$ (k) $\lim_{x \to 5^+} h(x)$ (l) $\lim_{x \to 5^-} h(x)$

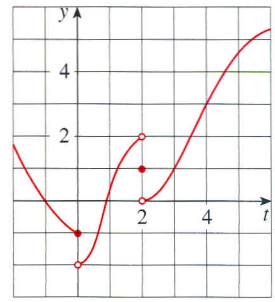


7. For the function g whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{t \to 0^-} g(t)$ (b) $\lim_{t \to 0^+} g(t)$ (c) $\lim_{t \to 0} g(t)$

(d) $\lim_{t \to 2^-} g(t)$ (e) $\lim_{t \to 2^+} g(t)$ (f) $\lim_{t \to 2} g(t)$

(g) $g(2)$ (h) $\lim_{t \to 4} g(t)$

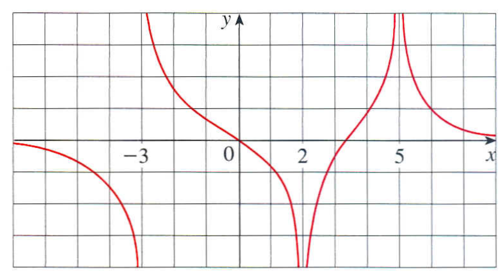


8. For the function R whose graph is shown, state the following.

(a) $\lim_{x \to 2} R(x)$ (b) $\lim_{x \to 5} R(x)$

(c) $\lim_{x \to -3^-} R(x)$ (d) $\lim_{x \to -3^+} R(x)$

(e) The equations of the vertical asymptotes.

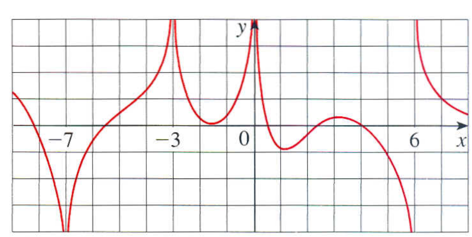


9. For the function f whose graph is shown, state the following.

(a) $\lim_{x \to -7} f(x)$ (b) $\lim_{x \to -3} f(x)$ (c) $\lim_{x \to 0} f(x)$

(d) $\lim_{x \to 6^-} f(x)$ (e) $\lim_{x \to 6^+} f(x)$

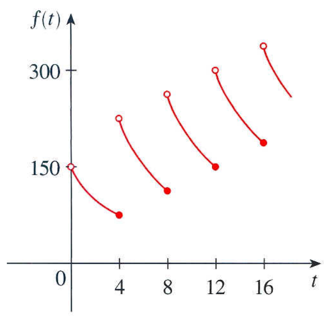
(f) The equations of the vertical asymptotes.



10. A patient receives a 150-mg injection of a drug every 4 hours. The graph shows the amount $f(t)$ of the drug in the bloodstream after t hours. Find

$$\lim_{t \to 12^-} f(t) \qquad \text{and} \qquad \lim_{t \to 12^+} f(t)$$

and explain the significance of these one-sided limits.



11. Use the graph of the function $f(x) = 1/(1 + 2^{1/x})$ to state the value of each limit, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to 0^-} f(x)$ (b) $\lim_{x \to 0^+} f(x)$ (c) $\lim_{x \to 0} f(x)$

12. Sketch the graph of the following function and use it to determine the values of a for which $\lim_{x \to a} f(x)$ exists:

$$f(x) = \begin{cases} 2 - x & \text{if } x < -1 \\ x & \text{if } -1 \leqslant x < 1 \\ (x - 1)^2 & \text{if } x \geqslant 1 \end{cases}$$

13–16 Sketch the graph of an example of a function f that satisfies all of the given conditions.

13. $\lim_{x \to 1^-} f(x) = 2$, $\lim_{x \to 1^+} f(x) = -2$, $f(1) = 2$

14. $\lim_{x \to 0^-} f(x) = 1$, $\lim_{x \to 0^+} f(x) = -1$, $\lim_{x \to 2^-} f(x) = 0$,
$\lim_{x \to 2^+} f(x) = 1$, $f(2) = 1$, $f(0)$ is undefined

15. $\lim_{x \to 3^+} f(x) = 4$, $\lim_{x \to 3^-} f(x) = 2$, $\lim_{x \to -2} f(x) = 2$,
$f(3) = 3$, $f(-2) = 1$

16. $\lim_{x \to 1} f(x) = 3$, $\lim_{x \to 4^-} f(x) = 3$, $\lim_{x \to 4^+} f(x) = -3$,
$f(1) = 1$, $f(4) = -1$

17–20 Guess the value of the limit (if it exists) by evaluating the function at the given numbers (correct to six decimal places).

17. $\lim_{x \to 2} \dfrac{x^2 - 2x}{x^2 - x - 2}$, $x = 2.5, 2.1, 2.05, 2.01, 2.005, 2.001,$
$1.9, 1.95, 1.99, 1.995, 1.999$

18. $\lim_{x \to -1} \dfrac{x^2 - 2x}{x^2 - x - 2}$,
$x = 0, -0.5, -0.9, -0.95, -0.99, -0.999,$
$-2, -1.5, -1.1, -1.01, -1.001$

19. $\lim_{x \to 0} \dfrac{\sin x}{x + \tan x}$, $x = \pm 1, \pm 0.5, \pm 0.2, \pm 0.1, \pm 0.05, \pm 0.01$

20. $\lim_{x \to 16} \dfrac{\sqrt{x} - 4}{x - 16}$, $x = 17, 16.5, 16.1, 16.05, 16.01,$
$15, 15.5, 15.9, 15.95, 15.99$

21–24 Use a table of values to estimate the value of the limit. If you have a graphing device, use it to confirm your result graphically.

21. $\lim_{x \to 0} \dfrac{\sqrt{x + 4} - 2}{x}$

22. $\lim_{x \to 0} \dfrac{\tan 3x}{\tan 5x}$

23. $\lim_{x \to 1} \dfrac{x^6 - 1}{x^{10} - 1}$

24. $\lim_{x \to 0} \dfrac{9^x - 5^x}{x}$

25–32 Determine the infinite limit.

25. $\lim_{x \to 5^+} \dfrac{6}{x - 5}$

26. $\lim_{x \to 5^-} \dfrac{6}{x - 5}$

27. $\lim_{x \to 1} \dfrac{2 - x}{(x - 1)^2}$

28. $\lim_{x \to 0} \dfrac{x - 1}{x^2(x + 2)}$

29. $\lim_{x \to -2^+} \dfrac{x - 1}{x^2(x + 2)}$

30. $\lim_{x \to \pi^-} \csc x$

31. $\lim_{x \to (-\pi/2)^-} \sec x$

32. $\lim_{x \to 1^+} \dfrac{x + 1}{x \sin \pi x}$

33. Determine $\lim_{x \to 1^-} \dfrac{1}{x^3 - 1}$ and $\lim_{x \to 1^+} \dfrac{1}{x^3 - 1}$

(a) by evaluating $f(x) = 1/(x^3 - 1)$ for values of x that approach 1 from the left and from the right,

(b) by reasoning as in Example 9, and

(c) from a graph of f.

34. (a) Find the vertical asymptotes of the function

$$y = \frac{x^2 + 1}{3x - 2x^2}$$

(b) Confirm your answer to part (a) by graphing the function.

35. (a) Estimate the value of the limit $\lim_{x \to 0} (1 + x)^{1/x}$ to five decimal places.

(b) Illustrate part (a) by graphing the function $y = (1 + x)^{1/x}$.

36. (a) By graphing the function $f(x) = (\tan 4x)/x$ and zooming in toward the point where the graph crosses the y-axis, estimate the value of $\lim_{x \to 0} f(x)$.

(b) Check your answer in part (a) by evaluating $f(x)$ for values of x that approach 0.

37. (a) Evaluate the function $f(x) = x^2 - (2^x/1000)$ for $x = 1$, 0.8, 0.6, 0.4, 0.2, 0.1, and 0.05, and guess the value of

$$\lim_{x \to 0} \left(x^2 - \frac{2^x}{1000} \right)$$

(b) Evaluate $f(x)$ for $x = 0.04$, 0.02, 0.01, 0.005, 0.003, and 0.001. Guess again.

38. (a) Evaluate $h(x) = (\tan x - x)/x^3$ for $x = 1$, 0.5, 0.1, 0.05, 0.01, and 0.005.

(b) Guess the value of $\displaystyle\lim_{x \to 0} \frac{\tan x - x}{x^3}$.

(c) Evaluate $h(x)$ for successively smaller values of x until you finally reach a value of 0 for $h(x)$. Are you still confident that your guess in part (b) is correct? Explain why you eventually obtained a value of 0. (In Section 7.8 a method for evaluating the limit will be explained.)

(d) Graph the function h in the viewing rectangle $[-1, 1]$ by $[0, 1]$. Then zoom in toward the point where the graph crosses the y-axis to estimate the limit of $h(x)$ as x approaches 0. Continue to zoom in until you observe distortions in the graph of h. Compare with the results of part (c).

39. Graph the function $f(x) = \sin(\pi/x)$ of Example 4 in the viewing rectangle $[-1, 1]$ by $[-1, 1]$. Then zoom in toward the origin several times. Comment on the behavior of this function.

40. In the theory of relativity, the mass of a particle with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the mass of the particle at rest and c is the speed of light. What happens as $v \to c^-$?

41. Use a graph to estimate the equations of all the vertical asymptotes of the curve

$$y = \tan(2 \sin x) \qquad -\pi \leqslant x \leqslant \pi$$

Then find the exact equations of these asymptotes.

42. (a) Use numerical and graphical evidence to guess the value of the limit

$$\lim_{x \to 1} \frac{x^3 - 1}{\sqrt{x} - 1}$$

(b) How close to 1 does x have to be to ensure that the function in part (a) is within a distance 0.5 of its limit?

2.3 CALCULATING LIMITS USING THE LIMIT LAWS

In Section 2.2 we used calculators and graphs to guess the values of limits, but we saw that such methods don't always lead to the correct answer. In this section we use the following properties of limits, called the *Limit Laws,* to calculate limits.

LIMIT LAWS Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \qquad \text{and} \qquad \lim_{x \to a} g(x)$$

exist. Then

1. $\displaystyle\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. $\displaystyle\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3. $\displaystyle\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$

4. $\displaystyle\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

5. $\displaystyle\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\displaystyle\lim_{x \to a} f(x)}{\displaystyle\lim_{x \to a} g(x)} \quad \text{if } \lim_{x \to a} g(x) \neq 0$

EXAMPLE 11 Show that $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$.

SOLUTION First note that we **cannot** use

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = \lim_{x \to 0} x^2 \cdot \lim_{x \to 0} \sin \frac{1}{x}$$

because $\lim_{x \to 0} \sin(1/x)$ does not exist (see Example 4 in Section 2.2). However, since

$$-1 \leqslant \sin \frac{1}{x} \leqslant 1$$

we have, as illustrated by Figure 8,

$$-x^2 \leqslant x^2 \sin \frac{1}{x} \leqslant x^2$$

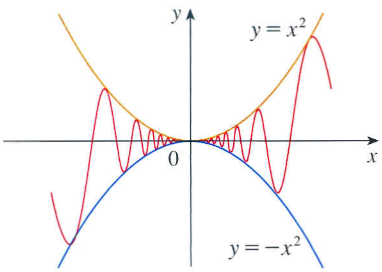


FIGURE 8
$y = x^2 \sin(1/x)$

We know that

$$\lim_{x \to 0} x^2 = 0 \qquad \text{and} \qquad \lim_{x \to 0} (-x^2) = 0$$

Taking $f(x) = -x^2$, $g(x) = x^2 \sin(1/x)$, and $h(x) = x^2$ in the Squeeze Theorem, we obtain

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$$

□

2.3 EXERCISES

1. Given that

$$\lim_{x \to 2} f(x) = 4 \qquad \lim_{x \to 2} g(x) = -2 \qquad \lim_{x \to 2} h(x) = 0$$

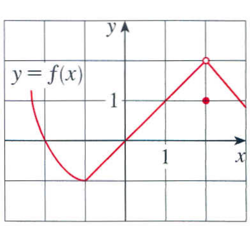
find the limits that exist. If the limit does not exist, explain why.

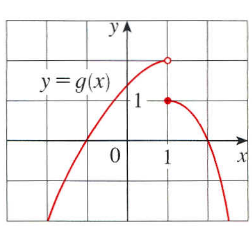
(a) $\lim_{x \to 2} [f(x) + 5g(x)]$ (b) $\lim_{x \to 2} [g(x)]^3$

(c) $\lim_{x \to 2} \sqrt{f(x)}$ (d) $\lim_{x \to 2} \dfrac{3f(x)}{g(x)}$

(e) $\lim_{x \to 2} \dfrac{g(x)}{h(x)}$ (f) $\lim_{x \to 2} \dfrac{g(x)h(x)}{f(x)}$

2. The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.





(a) $\lim_{x \to 2} [f(x) + g(x)]$ (b) $\lim_{x \to 1} [f(x) + g(x)]$

(c) $\lim_{x \to 0} [f(x)g(x)]$ (d) $\lim_{x \to -1} \dfrac{f(x)}{g(x)}$

(e) $\lim_{x \to 2} [x^3 f(x)]$ (f) $\lim_{x \to 1} \sqrt{3 + f(x)}$

3–9 Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

3. $\lim_{x \to 4} (5x^2 - 2x + 3)$

4. $\lim_{x \to -1} \dfrac{x - 2}{x^2 + 4x - 3}$

5. $\lim_{x \to 8} (1 + \sqrt[3]{x})(2 - 6x^2 + x^3)$

6. $\lim_{t \to -1} (t^2 + 1)^3 (t + 3)^5$

7. $\lim_{x \to 1} \left(\dfrac{1 + 3x}{1 + 4x^2 + 3x^4} \right)^3$

8. $\lim_{u \to -2} \sqrt{u^4 + 3u + 6}$

9. $\lim_{x \to 4^-} \sqrt{16 - x^2}$

10. (a) What is wrong with the following equation?

$$\frac{x^2 + x - 6}{x - 2} = x + 3$$

(b) In view of part (a), explain why the equation

$$\lim_{x\to 2}\frac{x^2+x-6}{x-2} = \lim_{x\to 2}(x+3)$$

is correct.

11–30 Evaluate the limit, if it exists.

11. $\displaystyle\lim_{x\to 2}\frac{x^2+x-6}{x-2}$

12. $\displaystyle\lim_{x\to -4}\frac{x^2+5x+4}{x^2+3x-4}$

13. $\displaystyle\lim_{x\to 2}\frac{x^2-x+6}{x-2}$

14. $\displaystyle\lim_{x\to 4}\frac{x^2-4x}{x^2-3x-4}$

15. $\displaystyle\lim_{t\to -3}\frac{t^2-9}{2t^2+7t+3}$

16. $\displaystyle\lim_{x\to -1}\frac{x^2-4x}{x^2-3x-4}$

17. $\displaystyle\lim_{h\to 0}\frac{(4+h)^2-16}{h}$

18. $\displaystyle\lim_{x\to 1}\frac{x^3-1}{x^2-1}$

19. $\displaystyle\lim_{x\to -2}\frac{x+2}{x^3+8}$

20. $\displaystyle\lim_{h\to 0}\frac{(2+h)^3-8}{h}$

21. $\displaystyle\lim_{t\to 9}\frac{9-t}{3-\sqrt{t}}$

22. $\displaystyle\lim_{h\to 0}\frac{\sqrt{1+h}-1}{h}$

23. $\displaystyle\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7}$

24. $\displaystyle\lim_{x\to 2}\frac{x^4-16}{x-2}$

25. $\displaystyle\lim_{x\to -4}\frac{\frac{1}{4}+\frac{1}{x}}{4+x}$

26. $\displaystyle\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right)$

27. $\displaystyle\lim_{x\to 9}\frac{x^2-81}{\sqrt{x}-3}$

28. $\displaystyle\lim_{h\to 0}\frac{(3+h)^{-1}-3^{-1}}{h}$

29. $\displaystyle\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{1}{t}\right)$

30. $\displaystyle\lim_{x\to -4}\frac{\sqrt{x^2+9}-5}{x+4}$

31. (a) Estimate the value of

$$\lim_{x\to 0}\frac{x}{\sqrt{1+3x}-1}$$

by graphing the function $f(x) = x/(\sqrt{1+3x}-1)$.

(b) Make a table of values of $f(x)$ for x close to 0 and guess the value of the limit.

(c) Use the Limit Laws to prove that your guess is correct.

32. (a) Use a graph of

$$f(x) = \frac{\sqrt{3+x}-\sqrt{3}}{x}$$

to estimate the value of $\lim_{x\to 0} f(x)$ to two decimal places.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Use the Limit Laws to find the exact value of the limit.

33. Use the Squeeze Theorem to show that $\lim_{x\to 0}(x^2\cos 20\pi x) = 0$. Illustrate by graphing the functions $f(x) = -x^2$, $g(x) = x^2\cos 20\pi x$, and $h(x) = x^2$ on the same screen.

34. Use the Squeeze Theorem to show that

$$\lim_{x\to 0}\sqrt{x^3+x^2}\,\sin\frac{\pi}{x} = 0$$

Illustrate by graphing the functions f, g, and h (in the notation of the Squeeze Theorem) on the same screen.

35. If $4x-9 \leqslant f(x) \leqslant x^2-4x+7$ for $x \geqslant 0$, find $\lim_{x\to 4} f(x)$.

36. If $2x \leqslant g(x) \leqslant x^4-x^2+2$ for all x, evaluate $\lim_{x\to 1} g(x)$.

37. Prove that $\displaystyle\lim_{x\to 0} x^4\cos\frac{2}{x} = 0$.

38. Prove that $\displaystyle\lim_{x\to 0^+}\sqrt{x}\,[1+\sin^2(2\pi/x)] = 0$.

39–44 Find the limit, if it exists. If the limit does not exist, explain why.

39. $\displaystyle\lim_{x\to 3}(2x+|x-3|)$

40. $\displaystyle\lim_{x\to -6}\frac{2x+12}{|x+6|}$

41. $\displaystyle\lim_{x\to 0.5^-}\frac{2x-1}{|2x^3-x^2|}$

42. $\displaystyle\lim_{x\to -2}\frac{2-|x|}{2+x}$

43. $\displaystyle\lim_{x\to 0^-}\left(\frac{1}{x}-\frac{1}{|x|}\right)$

44. $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{|x|}\right)$

45. The *signum* (or sign) *function*, denoted by sgn, is defined by

$$\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

(a) Sketch the graph of this function.

(b) Find each of the following limits or explain why it does not exist.

(i) $\displaystyle\lim_{x\to 0^+}\operatorname{sgn} x$ (ii) $\displaystyle\lim_{x\to 0^-}\operatorname{sgn} x$

(iii) $\displaystyle\lim_{x\to 0}\operatorname{sgn} x$ (iv) $\displaystyle\lim_{x\to 0}|\operatorname{sgn} x|$

46. Let

$$f(x) = \begin{cases} 4-x^2 & \text{if } x \leqslant 2 \\ x-1 & \text{if } x > 2 \end{cases}$$

(a) Find $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$.

(b) Does $\lim_{x\to 2} f(x)$ exist?

(c) Sketch the graph of f.

47. Let $F(x) = \dfrac{x^2-1}{|x-1|}$.

(a) Find

(i) $\displaystyle\lim_{x\to 1^+} F(x)$ (ii) $\displaystyle\lim_{x\to 1^-} F(x)$

(b) Does $\lim_{x\to 1} F(x)$ exist?
(c) Sketch the graph of F.

48. Let

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \leqslant 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$

(a) Evaluate each of the following limits, if it exists.
(i) $\lim_{x\to 1^-} g(x)$ (ii) $\lim_{x\to 1} g(x)$ (iii) $g(1)$
(iv) $\lim_{x\to 2^-} g(x)$ (v) $\lim_{x\to 2^+} g(x)$ (vi) $\lim_{x\to 2} g(x)$
(b) Sketch the graph of g.

49. (a) If the symbol $[\![\ ]\!]$ denotes the greatest integer function defined in Example 10, evaluate
(i) $\lim_{x\to -2^+} [\![x]\!]$ (ii) $\lim_{x\to -2} [\![x]\!]$ (iii) $\lim_{x\to -2.4} [\![x]\!]$
(b) If n is an integer, evaluate
(i) $\lim_{x\to n^-} [\![x]\!]$ (ii) $\lim_{x\to n^+} [\![x]\!]$
(c) For what values of a does $\lim_{x\to a} [\![x]\!]$ exist?

50. Let $f(x) = [\![\cos x]\!]$, $-\pi \leqslant x \leqslant \pi$.
(a) Sketch the graph of f.
(b) Evaluate each limit, if it exists.
(i) $\lim_{x\to 0} f(x)$ (ii) $\lim_{x\to (\pi/2)^-} f(x)$
(iii) $\lim_{x\to (\pi/2)^+} f(x)$ (iv) $\lim_{x\to \pi/2} f(x)$
(c) For what values of a does $\lim_{x\to a} f(x)$ exist?

51. If $f(x) = [\![x]\!] + [\![-x]\!]$, show that $\lim_{x\to 2} f(x)$ exists but is not equal to $f(2)$.

52. In the theory of relativity, the Lorentz contraction formula

$$L = L_0\sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $\lim_{v\to c^-} L$ and interpret the result. Why is a left-hand limit necessary?

53. If p is a polynomial, show that $\lim_{x\to a} p(x) = p(a)$.

54. If r is a rational function, use Exercise 53 to show that $\lim_{x\to a} r(x) = r(a)$ for every number a in the domain of r.

55. If $\lim_{x\to 1} \dfrac{f(x) - 8}{x - 1} = 10$, find $\lim_{x\to 1} f(x)$.

56. If $\lim_{x\to 0} \dfrac{f(x)}{x^2} = 5$, find the following limits.
(a) $\lim_{x\to 0} f(x)$ (b) $\lim_{x\to 0} \dfrac{f(x)}{x}$

57. If

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x\to 0} f(x) = 0$.

58. Show by means of an example that $\lim_{x\to a} [f(x) + g(x)]$ may exist even though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists.

59. Show by means of an example that $\lim_{x\to a} [f(x)g(x)]$ may exist even though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists.

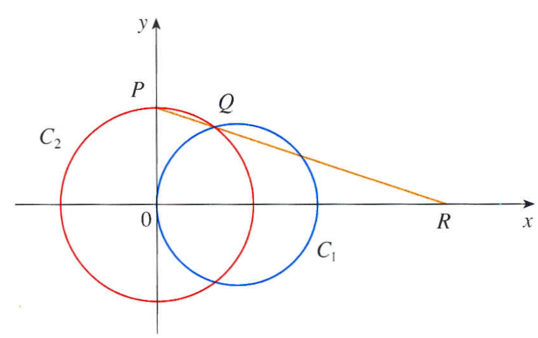
60. Evaluate $\lim_{x\to 2} \dfrac{\sqrt{6 - x} - 2}{\sqrt{3 - x} - 1}$.

61. Is there a number a such that

$$\lim_{x\to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists? If so, find the value of a and the value of the limit.

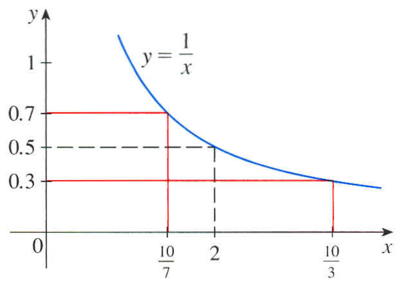
62. The figure shows a fixed circle C_1 with equation $(x - 1)^2 + y^2 = 1$ and a shrinking circle C_2 with radius r and center the origin. P is the point $(0, r)$, Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x-axis. What happens to R as C_2 shrinks, that is, as $r \to 0^+$?



2.4 EXERCISES

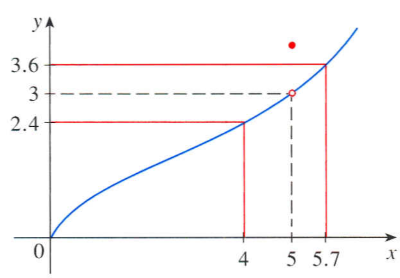
1. Use the given graph of $f(x) = 1/x$ to find a number δ such that

$$\text{if} \quad |x - 2| < \delta \quad \text{then} \quad \left|\frac{1}{x} - 0.5\right| < 0.2$$



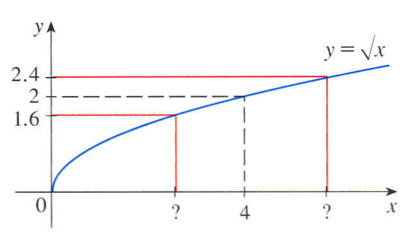
2. Use the given graph of f to find a number δ such that

$$\text{if} \quad 0 < |x - 5| < \delta \quad \text{then} \quad |f(x) - 3| < 0.6$$



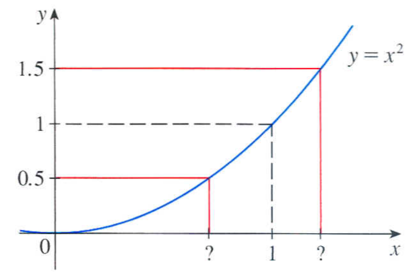
3. Use the given graph of $f(x) = \sqrt{x}$ to find a number δ such that

$$\text{if} \quad |x - 4| < \delta \quad \text{then} \quad |\sqrt{x} - 2| < 0.4$$



4. Use the given graph of $f(x) = x^2$ to find a number δ such that

$$\text{if} \quad |x - 1| < \delta \quad \text{then} \quad |x^2 - 1| < \tfrac{1}{2}$$



5. Use a graph to find a number δ such that

$$\text{if} \quad \left|x - \frac{\pi}{4}\right| < \delta \quad \text{then} \quad |\tan x - 1| < 0.2$$

6. Use a graph to find a number δ such that

$$\text{if} \quad |x - 1| < \delta \quad \text{then} \quad \left|\frac{2x}{x^2 + 4} - 0.4\right| < 0.1$$

7. For the limit

$$\lim_{x \to 1} (4 + x - 3x^3) = 2$$

illustrate Definition 2 by finding values of δ that correspond to $\varepsilon = 1$ and $\varepsilon = 0.1$.

8. For the limit

$$\lim_{x \to 2} \frac{4x + 1}{3x - 4} = 4.5$$

illustrate Definition 2 by finding values of δ that correspond to $\varepsilon = 0.5$ and $\varepsilon = 0.1$.

9. Given that $\lim_{x \to \pi/2} \tan^2 x = \infty$, illustrate Definition 6 by finding values of δ that correspond to (a) $M = 1000$ and (b) $M = 10{,}000$.

10. Use a graph to find a number δ such that

$$\text{if} \quad 5 < x < 5 + \delta \quad \text{then} \quad \frac{x^2}{\sqrt{x - 5}} > 100$$

11. A machinist is required to manufacture a circular metal disk with area 1000 cm^2.
(a) What radius produces such a disk?
(b) If the machinist is allowed an error tolerance of $\pm 5\ \text{cm}^2$ in the area of the disk, how close to the ideal radius in part (a) must the machinist control the radius?
(c) In terms of the ε, δ definition of $\lim_{x \to a} f(x) = L$, what is x? What is $f(x)$? What is a? What is L? What value of ε is given? What is the corresponding value of δ?

12. A crystal growth furnace is used in research to determine how best to manufacture crystals used in electronic components for the space shuttle. For proper growth of the crystal, the temperature must be controlled accurately by adjusting the input power. Suppose the relationship is given by

$$T(w) = 0.1w^2 + 2.155w + 20$$

where T is the temperature in degrees Celsius and w is the power input in watts.

(a) How much power is needed to maintain the temperature at 200°C?

(b) If the temperature is allowed to vary from 200°C by up to ±1°C, what range of wattage is allowed for the input power?

(c) In terms of the ε, δ definition of $\lim_{x\to a} f(x) = L$, what is x? What is $f(x)$? What is a? What is L? What value of ε is given? What is the corresponding value of δ?

13. (a) Find a number δ such that if $|x - 2| < \delta$, then $|4x - 8| < \varepsilon$, where $\varepsilon = 0.1$.

(b) Repeat part (a) with $\varepsilon = 0.01$.

14. Given that $\lim_{x\to 2}(5x - 7) = 3$, illustrate Definition 2 by finding values of δ that correspond to $\varepsilon = 0.1$, $\varepsilon = 0.05$, and $\varepsilon = 0.01$.

15–18 Prove the statement using the ε, δ definition of limit and illustrate with a diagram like Figure 9.

15. $\lim\limits_{x\to 1}(2x + 3) = 5$

16. $\lim\limits_{x\to -2}\left(\frac{1}{2}x + 3\right) = 2$

17. $\lim\limits_{x\to -3}(1 - 4x) = 13$

18. $\lim\limits_{x\to 4}(7 - 3x) = -5$

19–32 Prove the statement using the ε, δ definition of limit.

19. $\lim\limits_{x\to 3}\frac{x}{5} = \frac{3}{5}$

20. $\lim\limits_{x\to 6}\left(\frac{x}{4} + 3\right) = \frac{9}{2}$

21. $\lim\limits_{x\to -5}\left(4 - \frac{3x}{5}\right) = 7$

22. $\lim\limits_{x\to 3}\frac{x^2 + x - 12}{x - 3} = 7$

23. $\lim\limits_{x\to a} x = a$

24. $\lim\limits_{x\to a} c = c$

25. $\lim\limits_{x\to 0} x^2 = 0$

26. $\lim\limits_{x\to 0} x^3 = 0$

27. $\lim\limits_{x\to 0} |x| = 0$

28. $\lim\limits_{x\to 9^-} \sqrt[4]{9 - x} = 0$

29. $\lim\limits_{x\to 2}(x^2 - 4x + 5) = 1$

30. $\lim\limits_{x\to 3}(x^2 + x - 4) = 8$

31. $\lim\limits_{x\to -2}(x^2 - 1) = 3$

32. $\lim\limits_{x\to 2} x^3 = 8$

33. Verify that another possible choice of δ for showing that $\lim_{x\to 3} x^2 = 9$ in Example 4 is $\delta = \min\{2, \varepsilon/8\}$.

34. Verify, by a geometric argument, that the largest possible choice of δ for showing that $\lim_{x\to 3} x^2 = 9$ is $\delta = \sqrt{9 + \varepsilon} - 3$.

CAS **35.** (a) For the limit $\lim_{x\to 1}(x^3 + x + 1) = 3$, use a graph to find a value of δ that corresponds to $\varepsilon = 0.4$.

(b) By using a computer algebra system to solve the cubic equation $x^3 + x + 1 = 3 + \varepsilon$, find the largest possible value of δ that works for any given $\varepsilon > 0$.

(c) Put $\varepsilon = 0.4$ in your answer to part (b) and compare with your answer to part (a).

36. Prove that $\lim\limits_{x\to 2}\frac{1}{x} = \frac{1}{2}$.

37. Prove that $\lim\limits_{x\to a}\sqrt{x} = \sqrt{a}$ if $a > 0$.

$\left[\textit{Hint: } \text{Use } \left|\sqrt{x} - \sqrt{a}\right| = \dfrac{|x - a|}{\sqrt{x} + \sqrt{a}}.\right]$

38. If H is the Heaviside function defined in Example 6 in Section 2.2, prove, using Definition 2, that $\lim_{t\to 0} H(t)$ does not exist. [*Hint:* Use an indirect proof as follows. Suppose that the limit is L. Take $\varepsilon = \frac{1}{2}$ in the definition of a limit and try to arrive at a contradiction.]

39. If the function f is defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x\to 0} f(x)$ does not exist.

40. By comparing Definitions 2, 3, and 4, prove Theorem 1 in Section 2.3.

41. How close to -3 do we have to take x so that

$$\frac{1}{(x + 3)^4} > 10{,}000$$

42. Prove, using Definition 6, that $\lim\limits_{x\to -3}\frac{1}{(x + 3)^4} = \infty$.

43. Prove that $\lim\limits_{x\to -1^-}\frac{5}{(x + 1)^3} = -\infty$.

44. Suppose that $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = c$, where c is a real number. Prove each statement.

(a) $\lim\limits_{x\to a}[f(x) + g(x)] = \infty$

(b) $\lim\limits_{x\to a}[f(x)g(x)] = \infty$ if $c > 0$

(c) $\lim\limits_{x\to a}[f(x)g(x)] = -\infty$ if $c < 0$

Thus $f(1) < 0 < f(2)$; that is, $N = 0$ is a number between $f(1)$ and $f(2)$. Now f is continuous since it is a polynomial, so the Intermediate Value Theorem says there is a number c between 1 and 2 such that $f(c) = 0$. In other words, the equation $4x^3 - 6x^2 + 3x - 2 = 0$ has at least one root c in the interval $(1, 2)$.

In fact, we can locate a root more precisely by using the Intermediate Value Theorem again. Since

$$f(1.2) = -0.128 < 0 \quad \text{and} \quad f(1.3) = 0.548 > 0$$

a root must lie between 1.2 and 1.3. A calculator gives, by trial and error,

$$f(1.22) = -0.007008 < 0 \quad \text{and} \quad f(1.23) = 0.056068 > 0$$

so a root lies in the interval $(1.22, 1.23)$. □

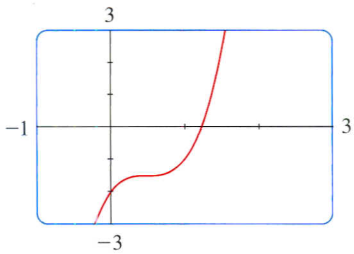


FIGURE 9

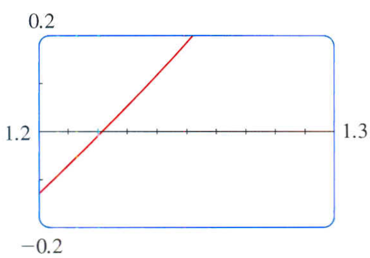


FIGURE 10

We can use a graphing calculator or computer to illustrate the use of the Intermediate Value Theorem in Example 9. Figure 9 shows the graph of f in the viewing rectangle $[-1, 3]$ by $[-3, 3]$ and you can see that the graph crosses the x-axis between 1 and 2. Figure 10 shows the result of zooming in to the viewing rectangle $[1.2, 1.3]$ by $[-0.2, 0.2]$.

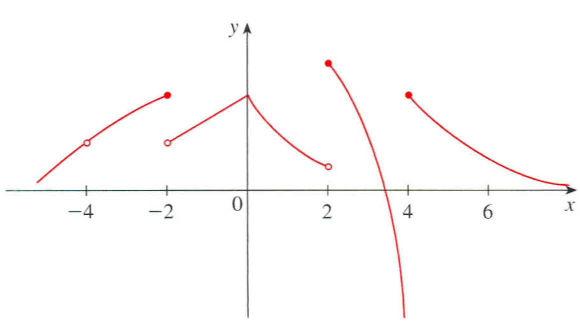
In fact, the Intermediate Value Theorem plays a role in the very way these graphing devices work. A computer calculates a finite number of points on the graph and turns on the pixels that contain these calculated points. It assumes that the function is continuous and takes on all the intermediate values between two consecutive points. The computer therefore connects the pixels by turning on the intermediate pixels.

2.5 EXERCISES

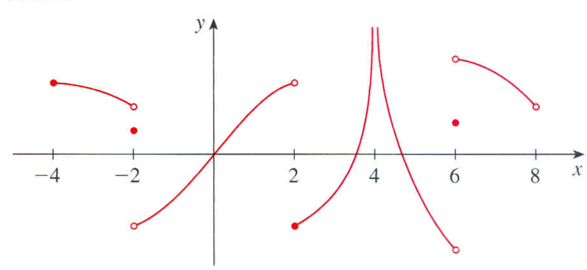
1. Write an equation that expresses the fact that a function f is continuous at the number 4.

2. If f is continuous on $(-\infty, \infty)$, what can you say about its graph?

3. (a) From the graph of f, state the numbers at which f is discontinuous and explain why.
(b) For each of the numbers stated in part (a), determine whether f is continuous from the right, or from the left, or neither.



4. From the graph of g, state the intervals on which g is continuous.



5. Sketch the graph of a function that is continuous everywhere except at $x = 3$ and is continuous from the left at 3.

6. Sketch the graph of a function that has a jump discontinuity at $x = 2$ and a removable discontinuity at $x = 4$, but is continuous elsewhere.

7. A parking lot charges \$3 for the first hour (or part of an hour) and \$2 for each succeeding hour (or part), up to a daily maximum of \$10.
(a) Sketch a graph of the cost of parking at this lot as a function of the time parked there.

(b) Discuss the discontinuities of this function and their significance to someone who parks in the lot.

8. Explain why each function is continuous or discontinuous.
(a) The temperature at a specific location as a function of time
(b) The temperature at a specific time as a function of the distance due west from Paris
(c) The altitude above sea level as a function of the distance due west from Paris
(d) The cost of a taxi ride as a function of the distance traveled
(e) The current in the circuit for the lights in a room as a function of time

9. If f and g are continuous functions with $f(3) = 5$ and $\lim_{x\to 3}[2f(x) - g(x)] = 4$, find $g(3)$.

10–12 Use the definition of continuity and the properties of limits to show that the function is continuous at the given number a.

10. $f(x) = x^2 + \sqrt{7 - x}, \quad a = 4$

11. $f(x) = (x + 2x^3)^4, \quad a = -1$

12. $h(t) = \dfrac{2t - 3t^2}{1 + t^3}, \quad a = 1$

13–14 Use the definition of continuity and the properties of limits to show that the function is continuous on the given interval.

13. $f(x) = \dfrac{2x + 3}{x - 2}, \quad (2, \infty)$

14. $g(x) = 2\sqrt{3 - x}, \quad (-\infty, 3]$

15–20 Explain why the function is discontinuous at the given number a. Sketch the graph of the function.

15. $f(x) = -\dfrac{1}{(x - 1)^2} \qquad a = 1$

16. $f(x) = \begin{cases} \dfrac{1}{x - 1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases} \qquad a = 1$

17. $f(x) = \begin{cases} 1 - x^2 & \text{if } x < 1 \\ 1/x & \text{if } x \geqslant 1 \end{cases} \qquad a = 1$

18. $f(x) = \begin{cases} \dfrac{x^2 - x}{x^2 - 1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases} \qquad a = 1$

19. $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases} \qquad a = 0$

20. $f(x) = \begin{cases} \dfrac{2x^2 - 5x - 3}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases} \qquad a = 3$

21–28 Explain, using Theorems 4, 5, 7, and 9, why the function is continuous at every number in its domain. State the domain.

21. $G(x) = \dfrac{x^4 + 17}{6x^2 + x - 1}$

22. $G(x) = \sqrt[3]{x}\,(1 + x^3)$

23. $R(x) = x^2 + \sqrt{2x - 1}$

24. $h(x) = \dfrac{\sin x}{x + 1}$

25. $h(x) = \cos(1 - x^2)$

26. $h(x) = \tan 2x$

27. $F(x) = \sqrt{x}\,\sin x$

28. $F(x) = \sin(\cos(\sin x))$

29–30 Locate the discontinuities of the function and illustrate by graphing.

29. $y = \dfrac{1}{1 + \sin x}$

30. $y = \tan\sqrt{x}$

31–34 Use continuity to evaluate the limit.

31. $\displaystyle\lim_{x\to 4} \frac{5 + \sqrt{x}}{\sqrt{5 + x}}$

32. $\displaystyle\lim_{x\to \pi} \sin(x + \sin x)$

33. $\displaystyle\lim_{x\to \pi/4} x\cos^2 x$

34. $\displaystyle\lim_{x\to 2} (x^3 - 3x + 1)^{-3}$

35–36 Show that f is continuous on $(-\infty, \infty)$.

35. $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geqslant 1 \end{cases}$

36. $f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geqslant \pi/4 \end{cases}$

37–39 Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither? Sketch the graph of f.

37. $f(x) = \begin{cases} 1 + x^2 & \text{if } x \leqslant 0 \\ 2 - x & \text{if } 0 < x \leqslant 2 \\ (x - 2)^2 & \text{if } x > 2 \end{cases}$

38. $f(x) = \begin{cases} x + 1 & \text{if } x \leqslant 1 \\ 1/x & \text{if } 1 < x < 3 \\ \sqrt{x - 3} & \text{if } x \geqslant 3 \end{cases}$

39. $f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ 2x^2 & \text{if } 0 \leqslant x \leqslant 1 \\ 2 - x & \text{if } x > 1 \end{cases}$

40. The gravitational force exerted by the earth on a unit mass at a distance r from the center of the planet is

$$F(r) = \begin{cases} \dfrac{GMr}{R^3} & \text{if } r < R \\ \dfrac{GM}{r^2} & \text{if } r \geqslant R \end{cases}$$

where M is the mass of the earth, R is its radius, and G is the gravitational constant. Is F a continuous function of r?

41. For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geqslant 2 \end{cases}$$

42. Find the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 < x < 3 \\ 2x - a + b & \text{if } x \geqslant 3 \end{cases}$$

43. Which of the following functions f has a removable discontinuity at a? If the discontinuity is removable, find a function g that agrees with f for $x \neq a$ and is continuous at a.

(a) $f(x) = \dfrac{x^4 - 1}{x - 1}, \quad a = 1$

(b) $f(x) = \dfrac{x^3 - x^2 - 2x}{x - 2}, \quad a = 2$

(c) $f(x) = [\![\sin x]\!], \quad a = \pi$

44. Suppose that a function f is continuous on $[0, 1]$ except at 0.25 and that $f(0) = 1$ and $f(1) = 3$. Let $N = 2$. Sketch two possible graphs of f, one showing that f might not satisfy the conclusion of the Intermediate Value Theorem and one showing that f might still satisfy the conclusion of the Intermediate Value Theorem (even though it doesn't satisfy the hypothesis).

45. If $f(x) = x^2 + 10 \sin x$, show that there is a number c such that $f(c) = 1000$.

46. Suppose f is continuous on $[1, 5]$ and the only solutions of the equation $f(x) = 6$ are $x = 1$ and $x = 4$. If $f(2) = 8$, explain why $f(3) > 6$.

47–50 Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

47. $x^4 + x - 3 = 0, \quad (1, 2)$

48. $\sqrt[3]{x} = 1 - x, \quad (0, 1)$

49. $\cos x = x, \quad (0, 1)$

50. $\tan x = 2x, \quad (0, 1.4)$

51–52 (a) Prove that the equation has at least one real root.
(b) Use your calculator to find an interval of length 0.01 that contains a root.

51. $\sin x = 2 - x$

52. $x^5 - x^2 + 2x + 3 = 0$

53–54 (a) Prove that the equation has at least one real root.
(b) Use your graphing device to find the root correct to three decimal places.

53. $x^5 - x^2 - 4 = 0$

54. $\sqrt{x - 5} = \dfrac{1}{x + 3}$

55. Prove that f is continuous at a if and only if

$$\lim_{h \to 0} f(a + h) = f(a)$$

56. To prove that sine is continuous, we need to show that $\lim_{x \to a} \sin x = \sin a$ for every real number a. By Exercise 55 an equivalent statement is that

$$\lim_{h \to 0} \sin(a + h) = \sin a$$

Use (6) to show that this is true.

57. Prove that cosine is a continuous function.

58. (a) Prove Theorem 4, part 3.
(b) Prove Theorem 4, part 5.

59. For what values of x is f continuous?

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

60. For what values of x is g continuous?

$$g(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$$

61. Is there a number that is exactly 1 more than its cube?

62. If a and b are positive numbers, prove that the equation

$$\frac{a}{x^3 + 2x^2 - 1} + \frac{b}{x^3 + x - 2} = 0$$

has at least one solution in the interval $(-1, 1)$.

63. Show that the function

$$f(x) = \begin{cases} x^4 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous on $(-\infty, \infty)$.

64. (a) Show that the absolute value function $F(x) = |x|$ is continuous everywhere.
(b) Prove that if f is a continuous function on an interval, then so is $|f|$.

(c) Is the converse of the statement in part (b) also true? In other words, if $|f|$ is continuous, does it follow that f is continuous? If so, prove it. If not, find a counterexample.

65. A Tibetan monk leaves the monastery at 7:00 AM and takes his usual path to the top of the mountain, arriving at 7:00 PM. The following morning, he starts at 7:00 AM at the top and takes the same path back, arriving at the monastery at 7:00 PM. Use the Intermediate Value Theorem to show that there is a point on the path that the monk will cross at exactly the same time of day on both days.

2 REVIEW

CONCEPT CHECK

1. Explain what each of the following means and illustrate with a sketch.

(a) $\lim_{x \to a} f(x) = L$ (b) $\lim_{x \to a^+} f(x) = L$

(c) $\lim_{x \to a^-} f(x) = L$ (d) $\lim_{x \to a} f(x) = \infty$

(e) $\lim_{x \to a} f(x) = -\infty$

2. Describe several ways in which a limit can fail to exist. Illustrate with sketches.

3. What does it mean to say that the line $x = a$ is a vertical asymptote of the curve $y = f(x)$? Draw curves to illustrate the various possibilities.

4. State the following Limit Laws.

(a) Sum Law (b) Difference Law

(c) Constant Multiple Law (d) Product Law

(e) Quotient Law (f) Power Law

(g) Root Law

5. What does the Squeeze Theorem say?

6. (a) What does it mean for f to be continuous at a?

(b) What does it mean for f to be continuous on the interval $(-\infty, \infty)$? What can you say about the graph of such a function?

7. What does the Intermediate Value Theorem say?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. $\lim_{x \to 4} \left(\dfrac{2x}{x - 4} - \dfrac{8}{x - 4} \right) = \lim_{x \to 4} \dfrac{2x}{x - 4} - \lim_{x \to 4} \dfrac{8}{x - 4}$

2. $\lim_{x \to 1} \dfrac{x^2 + 6x - 7}{x^2 + 5x - 6} = \dfrac{\lim_{x \to 1} (x^2 + 6x - 7)}{\lim_{x \to 1} (x^2 + 5x - 6)}$

3. $\lim_{x \to 1} \dfrac{x - 3}{x^2 + 2x - 4} = \dfrac{\lim_{x \to 1} (x - 3)}{\lim_{x \to 1} (x^2 + 2x - 4)}$

4. If $\lim_{x \to 5} f(x) = 2$ and $\lim_{x \to 5} g(x) = 0$, then $\lim_{x \to 5} [f(x)/g(x)]$ does not exist.

5. If $\lim_{x \to 5} f(x) = 0$ and $\lim_{x \to 5} g(x) = 0$, then $\lim_{x \to 5} [f(x)/g(x)]$ does not exist.

6. If $\lim_{x \to 6} [f(x)g(x)]$ exists, then the limit must be $f(6)g(6)$.

7. If p is a polynomial, then $\lim_{x \to b} p(x) = p(b)$.

8. If $\lim_{x \to 0} f(x) = \infty$ and $\lim_{x \to 0} g(x) = \infty$, then $\lim_{x \to 0} [f(x) - g(x)] = 0$.

9. If the line $x = 1$ is a vertical asymptote of $y = f(x)$, then f is not defined at 1.

10. If $f(1) > 0$ and $f(3) < 0$, then there exists a number c between 1 and 3 such that $f(c) = 0$.

11. If f is continuous at 5 and $f(5) = 2$ and $f(4) = 3$, then $\lim_{x \to 2} f(4x^2 - 11) = 2$.

12. If f is continuous on $[-1, 1]$ and $f(-1) = 4$ and $f(1) = 3$, then there exists a number r such that $|r| < 1$ and $f(r) = \pi$.

13. Let f be a function such that $\lim_{x \to 0} f(x) = 6$. Then there exists a number δ such that if $0 < |x| < \delta$, then $|f(x) - 6| < 1$.

14. If $f(x) > 1$ for all x and $\lim_{x \to 0} f(x)$ exists, then $\lim_{x \to 0} f(x) > 1$.

15. The equation $x^{10} - 10x^2 + 5 = 0$ has a root in the interval $(0, 2)$.

EXERCISES

1. The graph of f is given.

(a) Find each limit, or explain why it does not exist.

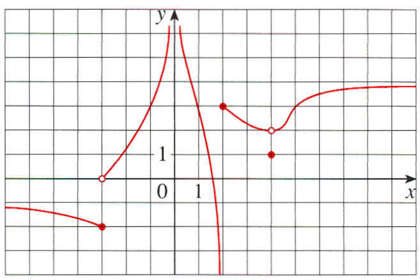
(i) $\lim_{x\to 2^+} f(x)$ (ii) $\lim_{x\to -3^+} f(x)$

(iii) $\lim_{x\to -3} f(x)$ (iv) $\lim_{x\to 4} f(x)$

(v) $\lim_{x\to 0} f(x)$ (vi) $\lim_{x\to 2^-} f(x)$

(b) State the equations of the vertical asymptotes.

(c) At what numbers is f discontinuous? Explain.



2. Sketch the graph of an example of a function f that satisfies all of the following conditions:

$\lim_{x\to 0^+} f(x) = -2,\quad \lim_{x\to 0^-} f(x) = 1,\quad f(0) = -1,$

$\lim_{x\to 2^-} f(x) = \infty,\quad \lim_{x\to 2^+} f(x) = -\infty$

3–16 Find the limit.

3. $\lim_{x\to 0} \cos(x + \sin x)$

4. $\lim_{x\to 3} \dfrac{x^2 - 9}{x^2 + 2x - 3}$

5. $\lim_{x\to -3} \dfrac{x^2 - 9}{x^2 + 2x - 3}$

6. $\lim_{x\to 1^+} \dfrac{x^2 - 9}{x^2 + 2x - 3}$

7. $\lim_{h\to 0} \dfrac{(h-1)^3 + 1}{h}$

8. $\lim_{t\to 2} \dfrac{t^2 - 4}{t^3 - 8}$

9. $\lim_{r\to 9} \dfrac{\sqrt{r}}{(r-9)^4}$

10. $\lim_{v\to 4^+} \dfrac{4 - v}{|4 - v|}$

11. $\lim_{u\to 1} \dfrac{u^4 - 1}{u^3 + 5u^2 - 6u}$

12. $\lim_{x\to 3} \dfrac{\sqrt{x+6} - x}{x^3 - 3x^2}$

13. $\lim_{s\to 16} \dfrac{4 - \sqrt{s}}{s - 16}$

14. $\lim_{v\to 2} \dfrac{v^2 + 2v - 8}{v^4 - 16}$

15. $\lim_{x\to 0} \dfrac{1 - \sqrt{1 - x^2}}{x}$

16. $\lim_{x\to 1} \left(\dfrac{1}{x-1} + \dfrac{1}{x^2 - 3x + 2}\right)$

17. If $2x - 1 \leqslant f(x) \leqslant x^2$ for $0 < x < 3$, find $\lim_{x\to 1} f(x)$.

18. Prove that $\lim_{x\to 0} x^2 \cos(1/x^2) = 0$.

19–22 Prove the statement using the precise definition of a limit.

19. $\lim_{x\to 2} (14 - 5x) = 4$

20. $\lim_{x\to 0} \sqrt[3]{x} = 0$

21. $\lim_{x\to 2} (x^2 - 3x) = -2$

22. $\lim_{x\to 4^+} \dfrac{2}{\sqrt{x - 4}} = \infty$

23. Let

$$f(x) = \begin{cases} \sqrt{-x} & \text{if } x < 0 \\ 3 - x & \text{if } 0 \leqslant x < 3 \\ (x-3)^2 & \text{if } x > 3 \end{cases}$$

(a) Evaluate each limit, if it exists.

(i) $\lim_{x\to 0^+} f(x)$ (ii) $\lim_{x\to 0^-} f(x)$ (iii) $\lim_{x\to 0} f(x)$

(iv) $\lim_{x\to 3^-} f(x)$ (v) $\lim_{x\to 3^+} f(x)$ (vi) $\lim_{x\to 3} f(x)$

(b) Where is f discontinuous?

(c) Sketch the graph of f.

24. Let

$$g(x) = \begin{cases} 2x - x^2 & \text{if } 0 \leqslant x \leqslant 2 \\ 2 - x & \text{if } 2 < x \leqslant 3 \\ x - 4 & \text{if } 3 < x < 4 \\ \pi & \text{if } x \geqslant 4 \end{cases}$$

(a) For each of the numbers 2, 3, and 4, discover whether g is continuous from the left, continuous from the right, or continuous at the number.

(b) Sketch the graph of g.

25–26 Show that each function is continuous on its domain. State the domain.

25. $h(x) = \sqrt[4]{x} + x^3 \cos x$

26. $g(x) = \dfrac{\sqrt{x^2 - 9}}{x^2 - 2}$

27–28 Use the Intermediate Value Theorem to show that there is a root of the equation in the given interval.

27. $2x^3 + x^2 + 2 = 0,\quad (-2, -1)$

28. $2 \sin x = 3 - 2x,\quad (0, 1)$

29. Suppose that $|f(x)| \leqslant g(x)$ for all x, where $\lim_{x\to a} g(x) = 0$. Find $\lim_{x\to a} f(x)$.

30. Let $f(x) = [\![x]\!] + [\![-x]\!]$.

(a) For what values of a does $\lim_{x\to a} f(x)$ exist?

(b) At what numbers is f discontinuous?

PROBLEMS PLUS

PROBLEMS

1. Evaluate $\displaystyle\lim_{x\to 1} \frac{\sqrt[3]{x} - 1}{\sqrt{x} - 1}$.

2. Find numbers a and b such that $\displaystyle\lim_{x\to 0} \frac{\sqrt{ax + b} - 2}{x} = 1$.

3. Evaluate $\displaystyle\lim_{x\to 0} \frac{|2x - 1| - |2x + 1|}{x}$.

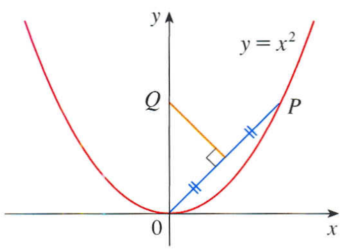


FIGURE FOR PROBLEM 4

4. The figure shows a point P on the parabola $y = x^2$ and the point Q where the perpendicular bisector of OP intersects the y-axis. As P approaches the origin along the parabola, what happens to Q? Does it have a limiting position? If so, find it.

5. Evaluate the following limits, if they exist, where $[\![x]\!]$ denotes the greatest integer function.

(a) $\displaystyle\lim_{x\to 0} \frac{[\![x]\!]}{x}$ (b) $\displaystyle\lim_{x\to 0} x[\![1/x]\!]$

6. Sketch the region in the plane defined by each of the following equations.

(a) $[\![x]\!]^2 + [\![y]\!]^2 = 1$ (b) $[\![x]\!]^2 - [\![y]\!]^2 = 3$ (c) $[\![x + y]\!]^2 = 1$ (d) $[\![x]\!] + [\![y]\!] = 1$

7. Find all values of a such that f is continuous on $\mathbb{R}$:

$$f(x) = \begin{cases} x + 1 & \text{if } x \leqslant a \\ x^2 & \text{if } x > a \end{cases}$$

8. A **fixed point** of a function f is a number c in its domain such that $f(c) = c$. (The function doesn't move c; it stays fixed.)

(a) Sketch the graph of a continuous function with domain $[0, 1]$ whose range also lies in $[0, 1]$. Locate a fixed point of f.

(b) Try to draw the graph of a continuous function with domain $[0, 1]$ and range in $[0, 1]$ that does *not* have a fixed point. What is the obstacle?

(c) Use the Intermediate Value Theorem to prove that any continuous function with domain $[0, 1]$ and range a subset of $[0, 1]$ must have a fixed point.

9. If $\lim_{x\to a}[f(x) + g(x)] = 2$ and $\lim_{x\to a}[f(x) - g(x)] = 1$, find $\lim_{x\to a}[f(x)g(x)]$.

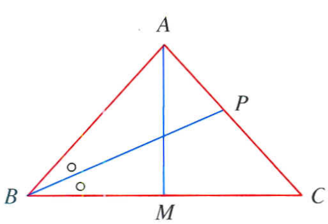


FIGURE FOR PROBLEM 10

10. (a) The figure shows an isosceles triangle ABC with $\angle B = \angle C$. The bisector of angle B intersects the side AC at the point P. Suppose that the base BC remains fixed but the altitude $|AM|$ of the triangle approaches 0, so A approaches the midpoint M of BC. What happens to P during this process? Does it have a limiting position? If so, find it.

(b) Try to sketch the path traced out by P during this process. Then find an equation of this curve and use this equation to sketch the curve.

11. (a) If we start from 0° latitude and proceed in a westerly direction, we can let $T(x)$ denote the temperature at the point x at any given time. Assuming that T is a continuous function of x, show that at any fixed time there are at least two diametrically opposite points on the equator that have exactly the same temperature.

(b) Does the result in part (a) hold for points lying on any circle on the earth's surface?

(c) Does the result in part (a) hold for barometric pressure and for altitude above sea level?