8.1

1. Let $u = \ln x$, $dv = x^2\,dx \quad \Rightarrow \quad du = \frac{1}{x}\,dx$, $v = \frac{1}{3}x^3$. Then by Equation 2,

$$\int x^2 \ln x\,dx = (\ln x)\left(\tfrac{1}{3}x^3\right) - \int \left(\tfrac{1}{3}x^3\right)\left(\tfrac{1}{x}\right)\,dx = \tfrac{1}{3}x^3\ln x - \tfrac{1}{3}\int x^2\,dx = \tfrac{1}{3}x^3\ln x - \tfrac{1}{3}\left(\tfrac{1}{3}x^3\right) + C$$
$$= \tfrac{1}{3}x^3\ln x - \tfrac{1}{9}x^3 + C \quad \left[\text{or } \tfrac{1}{3}x^3\left(\ln x - \tfrac{1}{3}\right) + C\right]$$

15. First let $u = (\ln x)^2$, $dv = dx \quad \Rightarrow \quad du = 2\ln x \cdot \frac{1}{x}\,dx$, $v = x$. Then by Equation 2,
$I = \int (\ln x)^2\,dx = x(\ln x)^2 - 2\int x\ln x \cdot \frac{1}{x}\,dx = x(\ln x)^2 - 2\int \ln x\,dx$. Next let $U = \ln x$, $dV = dx \quad \Rightarrow$
$dU = 1/x\,dx$, $V = x$ to get $\int \ln x\,dx = x\ln x - \int x\cdot(1/x)\,dx = x\ln x - \int dx = x\ln x - x + C_1$. Thus,
$I = x(\ln x)^2 - 2(x\ln x - x + C_1) = x(\ln x)^2 - 2x\ln x + 2x + C$, where $C = -2C_1$.

20. First let $u = x^2+1$, $dv = e^{-x}\,dx \quad \Rightarrow \quad du = 2x\,dx$, $v = -e^{-x}$. By (6),

$$\int_0^1 (x^2+1)e^{-x}\,dx = \left[-(x^2+1)e^{-x}\right]_0^1 + \int_0^1 2xe^{-x}\,dx = -2e^{-1} + 1 + 2\int_0^1 xe^{-x}\,dx.$$

Next let $U = x$, $dV = e^{-x}\,dx \quad \Rightarrow \quad dU = dx$, $V = -e^{-x}$. By (6) again,

$$\int_0^1 xe^{-x}\,dx = \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\,dx = -e^{-1} + \left[-e^{-x}\right]_0^1 = -e^{-1} - e^{-1} + 1 = -2e^{-1} + 1.$$ So

$$\int_0^1 (x^2+1)e^{-x}\,dx = -2e^{-1} + 1 + 2(-2e^{-1}+1) = -2e^{-1} + 1 - 4e^{-1} + 2 = -6e^{-1} + 3.$$

23. Let $u = \ln x$, $dv = x^{-2}\,dx \quad \Rightarrow \quad du = \dfrac{1}{x}\,dx$, $v = -x^{-1}$. By (6),

$$\int_1^2 \frac{\ln x}{x^2}\,dx = \left[-\frac{\ln x}{x}\right]_1^2 + \int_1^2 x^{-2}\,dx = -\tfrac{1}{2}\ln 2 + \ln 1 + \left[-\frac{1}{x}\right]_1^2 = -\tfrac{1}{2}\ln 2 + 0 - \tfrac{1}{2} + 1 = \tfrac{1}{2} - \tfrac{1}{2}\ln 2.$$

34. Let $x = -t^2$, so that $dx = -2t\,dt$. Thus, $\int t^3 e^{-t^2}\,dt = \int (-t^2)e^{-t^2}\left(\frac{1}{2}\right)(-2t\,dt) = \frac{1}{2}\int xe^x\,dx$. Now use parts with
$u = x$, $dv = e^x\,dx$, $du = dx$, $v = e^x$ to get

$$\tfrac{1}{2}\int xe^x\,dx = \tfrac{1}{2}\left(xe^x - \int e^x\,dx\right) = \tfrac{1}{2}xe^x - \tfrac{1}{2}e^x + C = -\tfrac{1}{2}(1-x)e^x + C = -\tfrac{1}{2}(1+t^2)e^{-t^2} + C.$$

46. Using Exercise 45(a), we see that the formula holds for $n = 1$, because $\int_0^{\pi/2} \sin^2 x\,dx = \frac{1}{2}\int_0^{\pi/2} 1\,dx = \frac{1}{2}\left[x\right]_0^{\pi/2} = \frac{1}{2}\cdot\frac{\pi}{2}$.

Now assume it holds for some $k \ge 1$. Then $\displaystyle\int_0^{\pi/2} \sin^{2k} x\,dx = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}\,\frac{\pi}{2}$. By Exercise 45(a),

$$\int_0^{\pi/2} \sin^{2(k+1)} x\,dx = \frac{2k+1}{2k+2}\int_0^{\pi/2} \sin^{2k} x\,dx = \frac{2k+1}{2k+2}\cdot\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}\,\frac{\pi}{2}$$
$$= \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)(2k+1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)(2k+2)}\cdot\frac{\pi}{2},$$

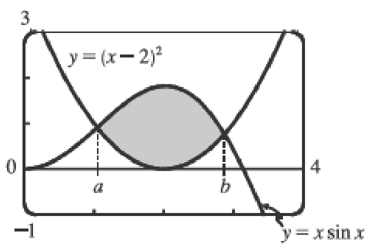
so the formula holds for $n = k+1$. By induction, the formula holds for all $n \ge 1$.

55. The curves $y = x\sin x$ and $y = (x-2)^2$ intersect at $a \approx 1.04748$ and $b \approx 2.87307$, so

area $= \int_a^b \left[x\sin x - (x-2)^2\right]dx$

$= \left[-x\cos x + \sin x - \frac{1}{3}(x-2)^3\right]_a^b \qquad$ [by Example 1]

$\approx 2.81358 - 0.63075 = 2.18283$



62. The rocket will have height $H = \int_0^{60} v(t)\,dt$ after 60 seconds.

$$H = \int_0^{60}\left[-gt - v_e \ln\left(\frac{m-rt}{m}\right)\right]dt = -g\left[\tfrac{1}{2}t^2\right]_0^{60} - v_e\left[\int_0^{60}\ln(m-rt)\,dt - \int_0^{60}\ln m\,dt\right]$$
$$= -g(1800) + v_e(\ln m)(60) - v_e\int_0^{60}\ln(m-rt)\,dt$$

Let $u = \ln(m-rt)$, $dv = dt \quad\Rightarrow\quad du = \dfrac{1}{m-rt}(-r)\,dt$, $v = t$. Then

$$\int_0^{60}\ln(m-rt)\,dt = \Big[t\ln(m-rt)\Big]_0^{60} + \int_0^{60}\frac{rt}{m-rt}\,dt = 60\ln(m-60r) + \int_0^{60}\left(-1+\frac{m}{m-rt}\right)dt$$
$$= 60\ln(m-60r) + \left[-t - \frac{m}{r}\ln(m-rt)\right]_0^{60} = 60\ln(m-60r) - 60 - \frac{m}{r}\ln(m-60r) + \frac{m}{r}\ln m$$

So $H = -1800g + 60v_e\ln m - 60v_e\ln(m-60r) + 60v_e + \dfrac{m}{r}v_e\ln(m-60r) - \dfrac{m}{r}v_e\ln m$. Substituting $g = 9.8$, $m = 30{,}000$, $r = 160$, and $v_e = 3000$ gives us $H \approx 14{,}844$ m.

8.2

3. $\int_{\pi/2}^{3\pi/4}\sin^5 x\,\cos^3 x\,dx = \int_{\pi/2}^{3\pi/4}\sin^5 x\,\cos^2 x\,\cos x\,dx = \int_{\pi/2}^{3\pi/4}\sin^5 x\,(1-\sin^2 x)\cos x\,dx \overset{s}{=} \int_1^{\sqrt{2}/2}u^5(1-u^2)\,du$

$$= \int_1^{\sqrt{2}/2}(u^5-u^7)\,du = \left[\tfrac{1}{6}u^6 - \tfrac{1}{8}u^8\right]_1^{\sqrt{2}/2} = \left(\frac{1/8}{6} - \frac{1/16}{8}\right) - \left(\tfrac{1}{6}-\tfrac{1}{8}\right) = -\tfrac{11}{384}$$

8. $\int_0^{\pi/2}\sin^2(2\theta)\,d\theta = \int_0^{\pi/2}\tfrac{1}{2}(1-\cos 4\theta)\,d\theta = \tfrac{1}{2}\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \tfrac{1}{2}\left[\left(\tfrac{\pi}{2}-0\right)-(0-0)\right] = \tfrac{\pi}{4}$

13. $\int_0^{\pi/2}\sin^2 x\,\cos^2 x\,dx = \int_0^{\pi/2}\tfrac{1}{4}(4\sin^2 x\,\cos^2 x)\,dx = \int_0^{\pi/2}\tfrac{1}{4}(2\sin x\,\cos x)^2 dx = \tfrac{1}{4}\int_0^{\pi/2}\sin^2 2x\,dx$

$$= \tfrac{1}{4}\int_0^{\pi/2}\tfrac{1}{2}(1-\cos 4x)\,dx = \tfrac{1}{8}\int_0^{\pi/2}(1-\cos 4x)\,dx = \tfrac{1}{8}\left[x - \tfrac{1}{4}\sin 4x\right]_0^{\pi/2} = \tfrac{1}{8}\left(\tfrac{\pi}{2}\right) = \tfrac{\pi}{16}$$

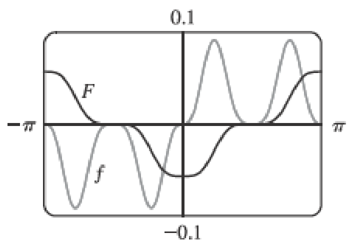
31. $\int\tan^5 x\,dx = \int(\sec^2 x - 1)^2\,\tan x\,dx = \int\sec^4 x\,\tan x\,dx - 2\int\sec^2 x\,\tan x\,dx + \int\tan x\,dx$

$$= \int\sec^3 x\,\sec x\,\tan x\,dx - 2\int\tan x\,\sec^2 x\,dx + \int\tan x\,dx$$
$$= \tfrac{1}{4}\sec^4 x - \tan^2 x + \ln|\sec x| + C \quad \left[\text{or } \tfrac{1}{4}\sec^4 x - \sec^2 x + \ln|\sec x| + C\right]$$

52. $\int\sin^3 x\,\cos^4 x\,dx = \int\cos^4 x\,(1-\cos^2 x)\sin x\,dx$

$$\overset{c}{=} \int u^4(1-u^2)(-du) = \int(u^6-u^4)\,du$$
$$= \tfrac{1}{7}u^7 - \tfrac{1}{5}u^5 + C = \tfrac{1}{7}\cos^7 x - \tfrac{1}{5}\cos^5 x + C$$

We see from the graph that this is reasonable, since F increases where f is positive and F decreases where f is negative. Note also that f is an odd function and F is an even function.

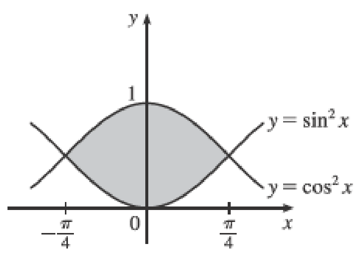


55. $f_{\text{ave}} = \tfrac{1}{2\pi}\int_{-\pi}^{\pi}\sin^2 x\,\cos^3 x\,dx = \tfrac{1}{2\pi}\int_{-\pi}^{\pi}\sin^2 x\,(1-\sin^2 x)\,\cos x\,dx$

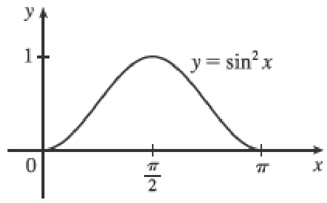
$$= \tfrac{1}{2\pi}\int_0^0 u^2(1-u^2)\,du \qquad [\text{where } u = \sin x]$$
$$= 0$$

57. $A = \displaystyle\int_{-\pi/4}^{\pi/4}(\cos^2 x - \sin^2 x)\,dx = \int_{-\pi/4}^{\pi/4}\cos 2x\,dx$

$$= 2\int_0^{\pi/4}\cos 2x\,dx = 2\left[\frac{1}{2}\sin 2x\right]_0^{\pi/4} = \big[\sin 2x\big]_0^{\pi/4}$$
$$= 1 - 0 = 1$$



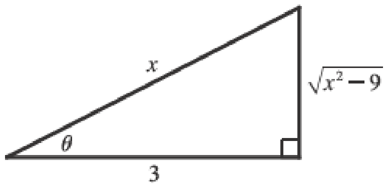
62. Using disks,



$$V = \int_0^{\pi} \pi(\sin^2 x)^2\,dx = 2\pi\int_0^{\pi/2}\left[\tfrac{1}{2}(1-\cos 2x)\right]^2 dx$$
$$= \tfrac{\pi}{2}\int_0^{\pi/2}(1-2\cos 2x+\cos^2 2x)\,dx$$
$$= \tfrac{\pi}{2}\int_0^{\pi/2}\left[1-2\cos 2x+\tfrac{1}{2}(1-\cos 4x)\right]dx$$
$$= \tfrac{\pi}{2}\int_0^{\pi/2}\left(\tfrac{3}{2}-2\cos 2x-\tfrac{1}{2}\cos 4x\right)dx = \tfrac{\pi}{2}\left[\tfrac{3}{2}x-\sin 2x+\tfrac{1}{8}\sin 4x\right]_0^{\pi/2}$$
$$= \tfrac{\pi}{2}\left[\left(\tfrac{3\pi}{4}-0+0\right)-(0-0+0)\right] = \tfrac{3}{8}\pi^2$$

8.3

1. Let $x = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = 3\sec\theta\,\tan\theta\,d\theta$ and



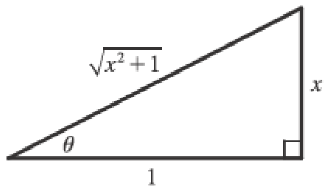
$$\sqrt{x^2-9} = \sqrt{9\sec^2\theta-9} = \sqrt{9(\sec^2\theta-1)} = \sqrt{9\tan^2\theta}$$
$$= 3\,|\tan\theta| = 3\tan\theta \text{ for the relevant values of } \theta.$$

$$\int \frac{1}{x^2\sqrt{x^2-9}}\,dx = \int \frac{1}{9\sec^2\theta\cdot 3\tan\theta}\,3\sec\theta\,\tan\theta\,d\theta = \tfrac{1}{9}\int\cos\theta\,d\theta = \tfrac{1}{9}\sin\theta + C = \frac{1}{9}\frac{\sqrt{x^2-9}}{x}+C$$

Note that $-\sec(\theta+\pi) = \sec\theta$, so the figure is sufficient for the case $\pi \le \theta < \frac{3\pi}{2}$.

12. $\int_0^1 x\sqrt{x^2+4}\,dx = \int_4^5 \sqrt{u}\left(\tfrac{1}{2}\,du\right)$ $\quad [u = x^2+4,\ du = 2x\,dx]$ $\quad = \tfrac{1}{2}\cdot\tfrac{2}{3}\left[u^{3/2}\right]_4^5 = \tfrac{1}{3}\left(5\sqrt{5}-8\right)$

22. Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta\,d\theta$, $\sqrt{x^2+1} = \sec\theta$ and $x = 0 \;\Rightarrow\; \theta = 0$, $x = 1 \;\Rightarrow\; \theta = \frac{\pi}{4}$, so



$$\int_0^1 \sqrt{x^2+1}\,dx = \int_0^{\pi/4}\sec\theta\,\sec^2\theta\,d\theta = \int_0^{\pi/4}\sec^3\theta\,d\theta$$
$$= \tfrac{1}{2}\Big[\sec\theta\,\tan\theta + \ln|\sec\theta+\tan\theta|\Big]_0^{\pi/4} \quad \text{[by Example 7.2.8]}$$
$$= \tfrac{1}{2}\left[\sqrt{2}\cdot 1+\ln\left(1+\sqrt{2}\right)-0-\ln(1+0)\right] = \tfrac{1}{2}\left[\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right]$$

32. (a) Let $x = a\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then

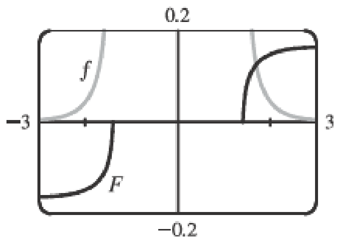
$$I = \int \frac{x^2}{(x^2+a^2)^{3/2}}\,dx = \int \frac{a^2\tan^2\theta}{a^3\sec^3\theta}\,a\sec^2\theta\,d\theta = \int\frac{\tan^2\theta}{\sec\theta}\,d\theta = \int\frac{\sec^2\theta-1}{\sec\theta}\,d\theta$$
$$= \int(\sec\theta-\cos\theta)\,d\theta = \ln|\sec\theta+\tan\theta| - \sin\theta + C$$
$$= \ln\left|\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right| - \frac{x}{\sqrt{x^2+a^2}} + C = \ln\left(x+\sqrt{x^2+a^2}\right) - \frac{x}{\sqrt{x^2+a^2}} + C_1$$

(b) Let $x = a\sinh t$. Then

$$I = \int \frac{a^2\sinh^2 t}{a^3\cosh^3 t}\,a\cosh t\,dt = \int\tanh^2 t\,dt = \int(1-\operatorname{sech}^2 t)\,dt = t-\tanh t + C$$
$$= \sinh^{-1}\frac{x}{a} - \frac{x}{\sqrt{a^2+x^2}} + C$$

36. Let $x = \sqrt{2}\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$, so $dx = \sqrt{2}\,\sec\theta\tan\theta\,d\theta$. Then

$$\int \frac{dx}{x^4\sqrt{x^2-2}} = \int \frac{\sqrt{2}\sec\theta\,\tan\theta\,d\theta}{4\sec^4\theta\,\sqrt{2}\,\tan\theta}$$
$$= \tfrac{1}{4}\int \cos^3\theta\,d\theta = \tfrac{1}{4}\int\left(1-\sin^2\theta\right)\cos\theta\,d\theta$$
$$= \tfrac{1}{4}\left[\sin\theta - \tfrac{1}{3}\sin^3\theta\right] + C \qquad \text{[substitute } u = \sin\theta\text{]}$$
$$= \frac{1}{4}\left[\frac{\sqrt{x^2-2}}{x} - \frac{\left(x^2-2\right)^{3/2}}{3x^3}\right] + C$$

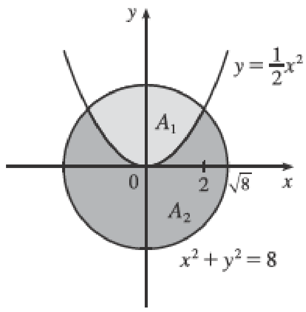


From the graph, it appears that our answer is reasonable. [Notice that $f(x)$ is large when F increases rapidly and small when F levels out.]

40. The curves intersect when $x^2 + \left(\frac{1}{2}x^2\right)^2 = 8 \quad\Leftrightarrow\quad x^2 + \frac{1}{4}x^4 = 8 \quad\Leftrightarrow\quad x^4 + 4x^2 - 32 = 0 \quad\Leftrightarrow$

$(x^2+8)(x^2-4) = 0 \quad\Leftrightarrow\quad x = \pm 2$. The area inside the circle and above the parabola is given by

$$A_1 = \int_{-2}^{2}\left(\sqrt{8-x^2} - \tfrac{1}{2}x^2\right)dx = 2\int_0^2 \sqrt{8-x^2}\,dx - 2\int_0^2 \tfrac{1}{2}x^2\,dx$$
$$= 2\left[\tfrac{1}{2}(8)\sin^{-1}\left(\tfrac{2}{\sqrt{8}}\right) + \tfrac{1}{2}(2)\sqrt{8-2^2} - \tfrac{1}{2}\left[\tfrac{1}{3}x^3\right]_0^2\right] \qquad \text{[by Exercise 39]}$$
$$= 8\sin^{-1}\left(\tfrac{1}{\sqrt{2}}\right) + 2\sqrt{4} - \tfrac{8}{3} = 8\left(\tfrac{\pi}{4}\right) + 4 - \tfrac{8}{3} = 2\pi + \tfrac{4}{3}$$



Since the area of the disk is $\pi\left(\sqrt{8}\right)^2 = 8\pi$, the area inside the circle and below the parabola ia $A_2 = 8\pi - \left(2\pi + \frac{4}{3}\right) = 6\pi - \frac{4}{3}$.

8.4

2. (a) $\dfrac{x}{x^2+x-2} = \dfrac{x}{(x+2)(x-1)} = \dfrac{A}{x+2} + \dfrac{B}{x-1}$

(b) $\dfrac{x^2}{x^2+x+2} = \dfrac{\left(x^2+x+2\right) - (x+2)}{x^2+x+2} = 1 - \dfrac{x+2}{x^2+x+2}$

Notice that x^2+x+2 can't be factored because its discriminant is $b^2 - 4ac = -7 < 0$.

7. $\displaystyle\int \frac{x}{x-6}\,dx = \int \frac{(x-6)+6}{x-6}dx = \int\left(1 + \frac{6}{x-6}\right)dx = x + 6\ln|x-6| + C$

23. $\dfrac{5x^2+3x-2}{x^3+2x^2} = \dfrac{5x^2+3x-2}{x^2\,(x+2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+2}$. Multiply by $x^2(x+2)$ to get $5x^2+3x-2 = Ax(x+2) + B(x+2) + Cx^2$. Set $x = -2$ to get $C = 3$, and take $x = 0$ to get $B = -1$. Equating the coefficients of x^2 gives $5 = A + C \quad\Rightarrow\quad A = 2$. So

$$\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx = \int\left(\frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2}\right)dx = 2\ln|x| + \frac{1}{x} + 3\ln|x+2| + C.$$

35. $\dfrac{1}{x(x^2+4)^2} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+4} + \dfrac{Dx+E}{(x^2+4)^2} \quad \Rightarrow \quad 1 = A(x^2+4)^2 + (Bx+C)x(x^2+4) + (Dx+E)x$. Setting $x = 0$ gives $1 = 16A$, so $A = \frac{1}{16}$. Now compare coefficients.

$$\begin{aligned} 1 &= \tfrac{1}{16}(x^4 + 8x^2 + 16) + (Bx^2 + Cx)(x^2+4) + Dx^2 + Ex \\ 1 &= \tfrac{1}{16}x^4 + \tfrac{1}{2}x^2 + 1 + Bx^4 + Cx^3 + 4Bx^2 + 4Cx + Dx^2 + Ex \\ 1 &= \left(\tfrac{1}{16} + B\right)x^4 + Cx^3 + \left(\tfrac{1}{2} + 4B + D\right)x^2 + (4C+E)x + 1 \end{aligned}$$

So $B + \frac{1}{16} = 0 \quad \Rightarrow \quad B = -\frac{1}{16}$, $C = 0$, $\frac{1}{2} + 4B + D = 0 \quad \Rightarrow \quad D = -\frac{1}{4}$, and $4C + E = 0 \quad \Rightarrow \quad E = 0$. Thus,

$$\begin{aligned} \int \frac{dx}{x(x^2+4)^2} &= \int \left(\frac{\frac{1}{16}}{x} + \frac{-\frac{1}{16}x}{x^2+4} + \frac{-\frac{1}{4}x}{(x^2+4)^2} \right) dx = \frac{1}{16}\ln|x| - \frac{1}{16}\cdot\frac{1}{2}\ln|x^2+4| - \frac{1}{4}\left(-\frac{1}{2}\right)\frac{1}{x^2+4} + C \\ &= \frac{1}{16}\ln|x| - \frac{1}{32}\ln(x^2+4) + \frac{1}{8(x^2+4)} + C \end{aligned}$$

44. Let $u = \sqrt{x}$. Then $x = u^2$, $dx = 2u\,du \quad \Rightarrow$

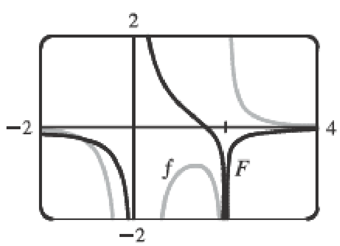
$$\int_{1/3}^{3} \frac{\sqrt{x}}{x^2+x}\,dx = \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{u \cdot 2u\,du}{u^4+u^2} = 2\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{du}{u^2+1} = 2\big[\tan^{-1} u\big]_{1/\sqrt{3}}^{\sqrt{3}} = 2\left(\tfrac{\pi}{3} - \tfrac{\pi}{6}\right) = \tfrac{\pi}{3}.$$

51. Let $u = \ln(x^2 - x + 2)$, $dv = dx$. Then $du = \dfrac{2x-1}{x^2-x+2}\,dx$, $v = x$, and (by integration by parts)

$$\begin{aligned} \int \ln(x^2-x+2)\,dx &= x\ln(x^2-x+2) - \int \frac{2x^2-x}{x^2-x+2}\,dx = x\ln(x^2-x+2) - \int \left(2 + \frac{x-4}{x^2-x+2}\right) dx \\ &= x\ln(x^2-x+2) - 2x - \int \frac{\frac{1}{2}(2x-1)}{x^2-x+2}\,dx + \frac{7}{2}\int \frac{dx}{(x-\frac{1}{2})^2 + \frac{7}{4}} \\ &= x\ln(x^2-x+2) - 2x - \frac{1}{2}\ln(x^2-x+2) + \frac{7}{2}\int \frac{\frac{\sqrt{7}}{2}\,du}{\frac{7}{4}(u^2+1)} \qquad \left[\begin{array}{c} \text{where } x - \frac{1}{2} = \frac{\sqrt{7}}{2}u, \\ dx = \frac{\sqrt{7}}{2}\,du, \\ (x-\frac{1}{2})^2 + \frac{7}{4} = \frac{7}{4}(u^2+1) \end{array}\right] \\ &= \left(x - \tfrac{1}{2}\right)\ln(x^2-x+2) - 2x + \sqrt{7}\tan^{-1} u + C \\ &= \left(x - \tfrac{1}{2}\right)\ln(x^2-x+2) - 2x + \sqrt{7}\tan^{-1}\frac{2x-1}{\sqrt{7}} + C \end{aligned}$$

54. $\dfrac{1}{x^3-2x^2} = \dfrac{1}{x^2(x-2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-2} \quad \Rightarrow \quad 1 = (A+C)x^2 + (B-2A)x - 2B$, so $A + C = B - 2A = 0$ and $-2B = 1 \quad \Rightarrow \quad B = -\frac{1}{2}$, $A = -\frac{1}{4}$, and $C = \frac{1}{4}$. So the general antiderivative of $\dfrac{1}{x^3-2x^2}$ is

$$\begin{aligned} \int \frac{dx}{x^3-2x^2} &= -\frac{1}{4}\int \frac{dx}{x} - \frac{1}{2}\int \frac{dx}{x^2} + \frac{1}{4}\int \frac{dx}{x-2} \\ &= -\tfrac{1}{4}\ln|x| - \tfrac{1}{2}(-1/x) + \tfrac{1}{4}\ln|x-2| + C \\ &= \frac{1}{4}\ln\left|\frac{x-2}{x}\right| + \frac{1}{2x} + C \end{aligned}$$



We plot this function with $C = 0$ on the same screen as $y = \dfrac{1}{x^3-2x^2}$.

55. $\displaystyle\int \frac{dx}{x^2-2x} = \int \frac{dx}{(x-1)^2-1} = \int \frac{du}{u^2-1}$ [put $u = x - 1$]

$$= \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C \quad \text{[by Equation 6]} \quad = \frac{1}{2}\ln\left|\frac{x-2}{x}\right| + C$$

62. $\dfrac{1}{x^3+x} = \dfrac{1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1} \quad\Rightarrow\quad 1 = A(x^2+1) + (Bx+C)x$. Set $x = 0$ to get $1 = A$. So

$1 = (1+B)x^2 + Cx + 1 \quad\Rightarrow\quad B+1=0 \quad [B=-1]$ and $C = 0$. Thus, the area is

$$\int_1^2 \frac{1}{x^3+x}\,dx = \int_1^2 \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx = \left[\ln|x| - \tfrac{1}{2}\ln|x^2+1|\right]_1^2 = \left(\ln 2 - \tfrac{1}{2}\ln 5\right) - \left(0 - \tfrac{1}{2}\ln 2\right)$$
$$= \tfrac{3}{2}\ln 2 - \tfrac{1}{2}\ln 5 \quad \left[\text{or } \tfrac{1}{2}\ln\tfrac{8}{5}\right]$$

8.5

2. $\displaystyle\int \frac{\sin^3 x}{\cos x}\,dx = \int \frac{\sin^2 x\,\sin x}{\cos x}\,dx = \int \frac{(1-\cos^2 x)\sin x}{\cos x}\,dx = \int \frac{1-u^2}{u}(-du) \qquad \begin{bmatrix} u = \cos x \\ du = -\sin x\,dx \end{bmatrix}$

$$= \int \left(u - \tfrac{1}{u}\right) du = \tfrac{1}{2}u^2 - \ln|u| + C = \tfrac{1}{2}\cos^2 x - \ln|\cos x| + C$$

9. $\int_1^3 r^4 \ln r\,dr \quad \begin{bmatrix} u = \ln r, & dv = r^4\,dr, \\ du = \dfrac{dr}{r} & v = \tfrac{1}{5}r^5 \end{bmatrix} = \left[\tfrac{1}{5}r^5 \ln r\right]_1^3 - \int_1^3 \tfrac{1}{5}r^4\,dr = \tfrac{243}{5}\ln 3 - 0 - \left[\tfrac{1}{25}r^5\right]_1^3$

$$= \tfrac{243}{5}\ln 3 - \left(\tfrac{243}{25} - \tfrac{1}{25}\right) = \tfrac{243}{5}\ln 3 - \tfrac{242}{25}$$

25. $\dfrac{3x^2-2}{x^2-2x-8} = 3 + \dfrac{6x+22}{(x-4)(x+2)} = 3 + \dfrac{A}{x-4} + \dfrac{B}{x+2} \quad\Rightarrow\quad 6x+22 = A(x+2) + B(x-4)$. Setting

$x = 4$ gives $46 = 6A$, so $A = \frac{23}{3}$. Setting $x = -2$ gives $10 = -6B$, so $B = -\frac{5}{3}$. Now

$$\int \frac{3x^2-2}{x^2-2x-8}\,dx = \int \left(3 + \frac{23/3}{x-4} - \frac{5/3}{x+2}\right) dx = 3x + \tfrac{23}{3}\ln|x-4| - \tfrac{5}{3}\ln|x+2| + C.$$

30. $x^2 - 4x < 0$ on $[0, 4]$, so

$$\int_{-2}^{2} |x^2-4x|\,dx = \int_{-2}^{0}(x^2-4x)\,dx + \int_0^2 (4x-x^2)\,dx = \left[\tfrac{1}{3}x^3 - 2x^2\right]_{-2}^{0} + \left[2x^2 - \tfrac{1}{3}x^3\right]_0^2$$
$$= 0 - \left(-\tfrac{8}{3} - 8\right) + \left(8 - \tfrac{8}{3}\right) - 0 = 16$$

41. Let $u = \theta$, $dv = \tan^2\theta\,d\theta = \left(\sec^2\theta - 1\right)d\theta \quad\Rightarrow\quad du = d\theta$ and $v = \tan\theta - \theta$. So

$$\int \theta\tan^2\theta\,d\theta = \theta(\tan\theta - \theta) - \int(\tan\theta - \theta)\,d\theta = \theta\tan\theta - \theta^2 - \ln|\sec\theta| + \tfrac{1}{2}\theta^2 + C$$
$$= \theta\tan\theta - \tfrac{1}{2}\theta^2 - \ln|\sec\theta| + C$$

48. Let $u = x^2$. Then $du = 2x\,dx \quad\Rightarrow\quad \displaystyle\int \frac{x\,dx}{x^4-a^4} = \int \frac{\frac{1}{2}\,du}{u^2-(a^2)^2} = \frac{1}{4a^2}\ln\left|\frac{u-a^2}{u+a^2}\right| + C = \frac{1}{4a^2}\ln\left|\frac{x^2-a^2}{x^2+a^2}\right| + C.$

52. Let $u = x^2$. Then $du = 2x\,dx \quad\Rightarrow$

$$\int \frac{dx}{x(x^4+1)} = \int \frac{x\,dx}{x^2(x^4+1)} = \frac{1}{2}\int \frac{du}{u(u^2+1)} = \frac{1}{2}\int\left[\frac{1}{u} - \frac{u}{u^2+1}\right] du = \tfrac{1}{2}\ln|u| - \tfrac{1}{4}\ln(u^2+1) + C$$
$$= \tfrac{1}{2}\ln(x^2) - \tfrac{1}{4}\ln(x^4+1) + C = \tfrac{1}{4}\left[\ln(x^4) - \ln(x^4+1)\right] + C = \frac{1}{4}\ln\left(\frac{x^4}{x^4+1}\right) + C$$

Or: Write $I = \displaystyle\int \frac{x^3\,dx}{x^4(x^4+1)}$ and let $u = x^4$.

8.6

1. We could make the substitution $u = \sqrt{2}\,x$ to obtain the radical $\sqrt{7-u^2}$ and then use Formula 33 with $a = \sqrt{7}$. Alternatively, we will factor $\sqrt{2}$ out of the radical and use $a = \sqrt{\frac{7}{2}}$.

$$\int \frac{\sqrt{7-2x^2}}{x^2}\,dx = \sqrt{2}\int \frac{\sqrt{\frac{7}{2}-x^2}}{x^2}\,dx \overset{33}{=} \sqrt{2}\left[-\frac{1}{x}\sqrt{\tfrac{7}{2}-x^2} - \sin^{-1}\frac{x}{\sqrt{\frac{7}{2}}}\right] + C$$

$$= -\frac{1}{x}\sqrt{7-2x^2} - \sqrt{2}\,\sin^{-1}\left(\sqrt{\tfrac{2}{7}}\,x\right) + C$$

7. Let $u = \pi x$, so that $du = \pi\,dx$. Then

$$\int \tan^3(\pi x)\,dx = \int \tan^3 u\left(\tfrac{1}{\pi}\,du\right) = \tfrac{1}{\pi}\int \tan^3 u\,du \overset{69}{=} \tfrac{1}{\pi}\left[\tfrac{1}{2}\tan^2 u + \ln|\cos u|\right] + C$$

$$= \tfrac{1}{2\pi}\tan^2(\pi x) + \tfrac{1}{\pi}\ln|\cos(\pi x)| + C$$

14. Let $u = \sqrt{x}$. Then $u^2 = x$ and $2u\,du = dx$, so

$$\int \sin^{-1}\sqrt{x}\,dx = 2\int u\sin^{-1}u\,du \overset{90}{=} \frac{2u^2-1}{2}\sin^{-1}u + \frac{u\sqrt{1-u^2}}{2} + C = \frac{2x-1}{2}\sin^{-1}\sqrt{x} + \frac{\sqrt{x(1-x)}}{2} + C.$$

31. Using cylindrical shells, we get

$$V = 2\pi\int_0^2 x\cdot x\sqrt{4-x^2}\,dx = 2\pi\int_0^2 x^2\sqrt{4-x^2}\,dx \overset{31}{=} 2\pi\left[\frac{x}{8}(2x^2-4)\sqrt{4-x^2} + \frac{16}{8}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 2\pi[(0+2\sin^{-1}1) - (0+2\sin^{-1}0] = 2\pi\left(2\cdot\frac{\pi}{2}\right) = 2\pi^2$$

37. Derive gives $\int x^2\sqrt{x^2+4}\,dx = \frac{1}{4}x(x^2+2)\sqrt{x^2+4} - 2\ln\left(\sqrt{x^2+4}+x\right)$. Maple gives $\frac{1}{4}x(x^2+4)^{3/2} - \frac{1}{2}x\sqrt{x^2+4} - 2\,\mathrm{arcsinh}\left(\frac{1}{2}x\right)$. Applying the command `convert(%,ln);` yields

$$\tfrac{1}{4}x(x^2+4)^{3/2} - \tfrac{1}{2}x\sqrt{x^2+4} - 2\ln\left(\tfrac{1}{2}x + \tfrac{1}{2}\sqrt{x^2+4}\right) = \tfrac{1}{4}x(x^2+4)^{1/2}\left[(x^2+4)-2\right] - 2\ln\left[\left(x+\sqrt{x^2+4}\right)/2\right]$$

$$= \tfrac{1}{4}x(x^2+2)\sqrt{x^2+4} - 2\ln\left(\sqrt{x^2+4}+x\right) + 2\ln 2$$

Mathematica gives $\frac{1}{4}x(2+x^2)\sqrt{3+x^2} - 2\,\mathrm{arcsinh}(x/2)$. Applying the `TrigToExp` and `Simplify` commands gives $\frac{1}{4}\left[x(2+x^2)\sqrt{4+x^2} - 8\log\left(\frac{1}{2}\left(x+\sqrt{4+x^2}\right)\right)\right] = \frac{1}{4}x(x^2+2)\sqrt{x^2+4} - 2\ln\left(x+\sqrt{4+x^2}\right) + 2\ln 2$, so all are equivalent (without constant).

Now use Formula 22 to get

$$\int x^2\sqrt{2^2+x^2}\,dx = \frac{x}{8}(2^2+2x^2)\sqrt{2^2+x^2} - \frac{2^4}{8}\ln\left(x+\sqrt{2^2+x^2}\right) + C$$

$$= \frac{x}{8}(2)(2+x^2)\sqrt{4+x^2} - 2\ln\left(x+\sqrt{4+x^2}\right) + C$$

$$= \tfrac{1}{4}x(x^2+2)\sqrt{x^2+4} - 2\ln\left(\sqrt{x^2+4}+x\right) + C$$

8.7

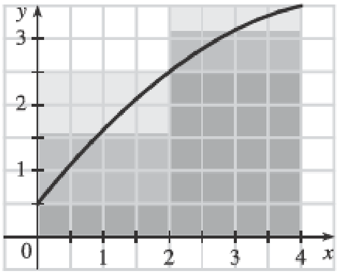
1. (a) $\Delta x = (b-a)/n = (4-0)/2 = 2$

$$L_2 = \sum_{i=1}^{2} f(x_{i-1})\,\Delta x = f(x_0)\cdot 2 + f(x_1)\cdot 2 = 2\,[f(0)+f(2)] = 2(0.5+2.5) = 6$$

$$R_2 = \sum_{i=1}^{2} f(x_i)\,\Delta x = f(x_1)\cdot 2 + f(x_2)\cdot 2 = 2\,[f(2)+f(4)] = 2(2.5+3.5) = 12$$

$$M_2 = \sum_{i=1}^{2} f(\overline{x}_i)\Delta x = f(\overline{x}_1)\cdot 2 + f(\overline{x}_2)\cdot 2 = 2\,[f(1)+f(3)] \approx 2(1.6+3.2) = 9.6$$

(b)



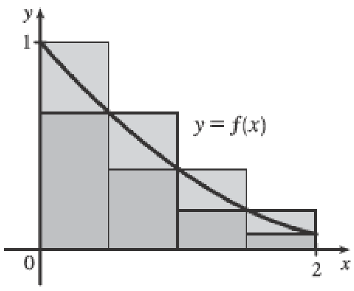
L_2 is an underestimate, since the area under the small rectangles is less than the area under the curve, and R_2 is an overestimate, since the area under the large rectangles is greater than the area under the curve. It appears that M_2 is an overestimate, though it is fairly close to I. See the solution to Exercise 45 for a proof of the fact that if f is concave down on $[a,b]$, then the Midpoint Rule is an overestimate of $\int_a^b f(x)\,dx$.

(c) $T_2 = \left(\frac{1}{2}\Delta x\right)[f(x_0)+2f(x_1)+f(x_2)] = \frac{2}{2}[f(0)+2f(2)+f(4)] = 0.5+2(2.5)+3.5 = 9.$

This approximation is an underestimate, since the graph is concave down. Thus, $T_2 = 9 < I$. See the solution to Exercise 45 for a general proof of this conclusion.

(d) For any n, we will have $L_n < T_n < I < M_n < R_n$.

2.



The diagram shows that $L_4 > T_4 > \int_0^2 f(x)\,dx > R_4$, and it appears that M_4 is a bit less than $\int_0^2 f(x)\,dx$. In fact, for any function that is concave upward, it can be shown that $L_n > T_n > \int_0^2 f(x)\,dx > M_n > R_n$.

(a) Since $0.9540 > 0.8675 > 0.8632 > 0.7811$, it follows that $L_n = 0.9540$, $T_n = 0.8675$, $M_n = 0.8632$, and $R_n = 0.7811$.

(b) Since $M_n < \int_0^2 f(x)\,dx < T_n$, we have $0.8632 < \int_0^2 f(x)\,dx < 0.8675$.

10. $f(t) = \dfrac{1}{1+t^2+t^4}$, $\Delta t = \dfrac{3-0}{6} = \dfrac{1}{2}$

(a) $T_6 = \frac{1}{2\cdot 2}\left[f(0)+2f\left(\frac{1}{2}\right)+2f(1)+2f\left(\frac{3}{2}\right)+2f(2)+2f\left(\frac{5}{2}\right)+f(3)\right] \approx 0.895122$

(b) $M_6 = \frac{1}{2}\left[f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)+f\left(\frac{5}{4}\right)+f\left(\frac{7}{4}\right)+f\left(\frac{9}{4}\right)+f\left(\frac{11}{4}\right)\right] \approx 0.895478$

(c) $S_6 = \frac{1}{2\cdot 3}\left[f(0)+4f\left(\frac{1}{2}\right)+2f(1)+4f\left(\frac{3}{2}\right)+2f(2)+4f\left(\frac{5}{2}\right)+f(3)\right] \approx 0.898014$

19. $f(x) = \cos(x^2)$, $\Delta x = \frac{1-0}{8} = \frac{1}{8}$

(a) $T_8 = \frac{1}{8 \cdot 2}\{f(0) + 2[f(\frac{1}{8}) + f(\frac{2}{8}) + \cdots + f(\frac{7}{8})] + f(1)\} \approx 0.902333$

$M_8 = \frac{1}{8}[f(\frac{1}{16}) + f(\frac{3}{16}) + f(\frac{5}{16}) + \cdots + f(\frac{15}{16})] = 0.905620$

(b) $f(x) = \cos(x^2)$, $f'(x) = -2x\sin(x^2)$, $f''(x) = -2\sin(x^2) - 4x^2\cos(x^2)$. For $0 \le x \le 1$, sin and cos are positive, so $|f''(x)| = 2\sin(x^2) + 4x^2\cos(x^2) \le 2 \cdot 1 + 4 \cdot 1 \cdot 1 = 6$ since $\sin(x^2) \le 1$ and $\cos(x^2) \le 1$ for all x, and $x^2 \le 1$ for $0 \le x \le 1$. So for $n = 8$, we take $K = 6$, $a = 0$, and $b = 1$ in Theorem 3, to get $|E_T| \le 6 \cdot 1^3/(12 \cdot 8^2) = \frac{1}{128} = 0.0078125$ and $|E_M| \le \frac{1}{256} = 0.00390625$. [A better estimate is obtained by noting from a graph of f'' that $|f''(x)| \le 4$ for $0 \le x \le 1$.]

(c) Take $K = 6$ [as in part (b)] in Theorem 3. $|E_T| \le \dfrac{K(b-a)^3}{12n^2} \le 0.0001 \quad\Leftrightarrow\quad \dfrac{6(1-0)^3}{12n^2} \le 10^{-4} \quad\Leftrightarrow$

$\dfrac{1}{2n^2} \le \dfrac{1}{10^4} \quad\Leftrightarrow\quad 2n^2 \ge 10^4 \quad\Leftrightarrow\quad n^2 \ge 5000 \quad\Leftrightarrow\quad n \ge 71$. Take $n = 71$ for T_n. For E_M, again take $K = 6$ in Theorem 3 to get $|E_M| \le 10^{-4} \quad\Leftrightarrow\quad 4n^2 \ge 10^4 \quad\Leftrightarrow\quad n^2 \ge 2500 \quad\Leftrightarrow\quad n \ge 50$. Take $n = 50$ for M_n.

26. $I = \int_1^2 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^2 = -\frac{1}{2} - (-1) = \frac{1}{2}$, $f(x) = \frac{1}{x^2}$, $\Delta x = \frac{1}{n}$

$n = 5$:

$$L_5 = \tfrac{1}{5}[f(1) + f(1.2) + f(1.4) + f(1.6) + f(1.8)] \approx 0.580783$$
$$R_5 = \tfrac{1}{5}[f(1.2) + f(1.4) + f(1.6) + f(1.8) + f(2)] \approx 0.430783$$
$$T_5 = \tfrac{1}{5\cdot 2}[f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)] \approx 0.505783$$
$$M_5 = \tfrac{1}{5}[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)] \approx 0.497127$$

$$E_L = I - L_5 \approx \tfrac{1}{2} - 0.580783 = -0.080783$$
$$E_R \approx \tfrac{1}{2} - 0.430783 = 0.069217$$
$$E_T \approx \tfrac{1}{2} - 0.505783 = -0.005783$$
$$E_M \approx \tfrac{1}{2} - 0.497127 = 0.002873$$

$n = 10$:

$$L_{10} = \tfrac{1}{10}[f(1) + f(1.1) + f(1.2) + \cdots + f(1.9)] \approx 0.538955$$
$$R_{10} = \tfrac{1}{10}[f(1.1) + f(1.2) + \cdots + f(1.9) + f(2)] \approx 0.463955$$
$$T_{10} = \tfrac{1}{10\cdot 2}\{f(1) + 2[f(1.1) + f(1.2) + \cdots + f(1.9)] + f(2)\} \approx 0.501455$$
$$M_{10} = \tfrac{1}{10}[f(1.05) + f(1.15) + \cdots + f(1.85) + f(1.95)] \approx 0.499274$$

$$E_L = I - L_{10} \approx \tfrac{1}{2} - 0.538955 = -0.038955$$
$$E_R \approx \tfrac{1}{2} - 0.463955 = 0.036049$$
$$E_T \approx \tfrac{1}{2} - 0.501455 = -0.001455$$
$$E_M \approx \tfrac{1}{2} - 0.499274 = 0.000726$$

$n = 20$:

$$L_{20} = \tfrac{1}{20}[f(1) + f(1.05) + f(1.10) + \cdots + f(1.95)] \approx 0.519114$$
$$R_{20} = \tfrac{1}{20}[f(1.05) + f(1.10) + \cdots + f(1.95) + f(2)] \approx 0.481614$$
$$T_{20} = \tfrac{1}{20\cdot 2}\{f(1) + 2[f(1.05) + f(1.10) + \cdots + f(1.95)] + f(2)\} \approx 0.500364$$
$$M_{20} = \tfrac{1}{20}[f(1.025) + f(1.075) + f(1.125) + \cdots + f(1.975)] \approx 0.499818$$

$$E_L = I - L_{20} \approx \tfrac{1}{2} - 0.519114 = -0.019114$$
$$E_R \approx \tfrac{1}{2} - 0.481614 = 0.018386$$
$$E_T \approx \tfrac{1}{2} - 0.500364 = -0.000364$$
$$E_M \approx \tfrac{1}{2} - 0.499818 = 0.000182$$

n	L_n	R_n	T_n	M_n
5	0.580783	0.430783	0.505783	0.497127
10	0.538955	0.463955	0.501455	0.499274
20	0.519114	0.481614	0.500364	0.499818

n	E_L	E_R	E_T	E_M
5	-0.080783	0.069217	-0.005783	0.002873
10	-0.038955	0.036049	-0.001455	0.000726
20	-0.019114	0.018386	-0.000364	0.000182

Observations:

1. E_L and E_R are always opposite in sign, as are E_T and E_M.
2. As n is doubled, E_L and E_R are decreased by about a factor of 2, and E_T and E_M are decreased by a factor of about 4.
3. The Midpoint approximation is about twice as accurate as the Trapezoidal approximation.
4. All the approximations become more accurate as the value of n increases.
5. The Midpoint and Trapezoidal approximations are much more accurate than the endpoint approximations.

29. $\Delta x = (b-a)/n = (6-0)/6 = 1$

(a) $T_6 = \frac{\Delta x}{2}[f(0) + 2f(1) + 2f(2) + 2f(3) + 2f(4) + 2f(5) + f(6)]$

$\approx \frac{1}{2}[3 + 2(5) + 2(4) + 2(2) + 2(2.8) + 2(4) + 1]$

$= \frac{1}{2}(39.6) = 19.8$

(b) $M_6 = \Delta x[f(0.5) + f(1.5) + f(2.5) + f(3.5) + f(4.5) + f(5.5)]$

$\approx 1[4.5 + 4.7 + 2.6 + 2.2 + 3.4 + 3.2]$

$= 20.6$

(c) $S_6 = \frac{\Delta x}{3}[f(0) + 4f(1) + 2f(2) + 4f(3) + 2f(4) + 4f(5) + f(6)]$

$\approx \frac{1}{3}[3 + 4(5) + 2(4) + 4(2) + 2(2.8) + 4(4) + 1]$

$= \frac{1}{3}(61.6) = 20.5\overline{3}$

32. We use Simpson's Rule with $n = 10$ and $\Delta x = \frac{1}{2}$:

$$\begin{aligned}\text{distance} = \int_0^5 v(t)\,dt \approx S_{10} &= \frac{1}{2\cdot 3}[f(0) + 4f(0.5) + 2f(1) + \cdots + 4f(4.5) + f(5)] \\ &= \frac{1}{6}[0 + 4(4.67) + 2(7.34) + 4(8.86) + 2(9.73) + 4(10.22) \\ &\qquad + 2(10.51) + 4(10.67) + 2(10.76) + 4(10.81) + 10.81] \\ &= \frac{1}{6}(268.41) = 44.735 \text{ m}\end{aligned}$$

37. Let $y = f(x)$ denote the curve. Using cylindrical shells, $V = \int_2^{10} 2\pi x f(x)\,dx = 2\pi \int_2^{10} x f(x)\,dx = 2\pi I_1$.

Now use Simpson's Rule to approximate I_1:

$$I_1 \approx S_8 = \frac{10-2}{3(8)}[2f(2) + 4\cdot 3f(3) + 2\cdot 4f(4) + 4\cdot 5f(5) + 2\cdot 6f(6) + 4\cdot 7f(7) + 2\cdot 8f(8) + 4\cdot 9f(9) + 10f(10)]$$

$$\approx \frac{1}{3}[2(0) + 12(1.5) + 8(1.9) + 20(2.2) + 12(3.0) + 28(3.8) + 16(4.0) + 36(3.1) + 10(0)]$$

$$= \frac{1}{3}(395.2)$$

Thus, $V \approx 2\pi \cdot \frac{1}{3}(395.2) \approx 827.7$ or 828 cubic units.

8.8

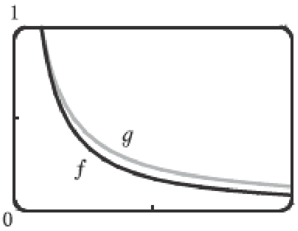
2. (a) Since $y = \dfrac{1}{2x-1}$ is defined and continuous on $[1, 2]$, $\displaystyle\int_1^2 \frac{1}{2x-1}\,dx$ is proper.

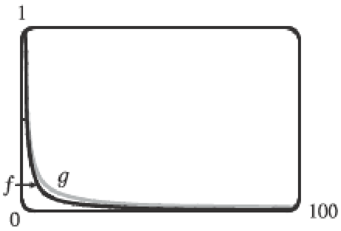
(b) Since $y = \dfrac{1}{2x-1}$ has an infinite discontinuity at $x = \frac{1}{2}$, $\displaystyle\int_0^1 \frac{1}{2x-1}\,dx$ is a Type II improper integral.

(c) Since $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

(d) Since $y = \ln(x-1)$ has an infinite discontinuity at $x = 1$, $\int_1^2 \ln(x-1)dx$ is a Type II improper integral.

4. (a)





(b) The area under the graph of f from $x = 1$ to $x = t$ is

$$F(t) = \int_1^t f(x)\,dx = \int_1^t x^{-1.1}\,dx = \left[-\tfrac{1}{0.1}x^{-0.1}\right]_1^t$$
$$= -10(t^{-0.1} - 1) = 10(1 - t^{-0.1})$$

and the area under the graph of g is

$$G(t) = \int_1^t g(x)\,dx = \int_1^t x^{-0.9}\,dx = \left[\tfrac{1}{0.1}x^{0.1}\right]_1^t = 10(t^{0.1} - 1).$$

t	$F(t)$	$G(t)$
10	2.06	2.59
100	3.69	5.85
10^4	6.02	15.12
10^6	7.49	29.81
10^{10}	9	90
10^{20}	9.9	990

(c) The total area under the graph of f is $\lim_{t\to\infty} F(t) = \lim_{t\to\infty} 10(1 - t^{-0.1}) = 10$.

The total area under the graph of g does not exist, since $\lim_{t\to\infty} G(t) = \lim_{t\to\infty} 10(t^{0.1} - 1) = \infty$.

6. $\displaystyle\int_{-\infty}^{0} \frac{1}{2x-5}\,dx = \lim_{t\to-\infty}\int_t^0 \frac{1}{2x-5}\,dx = \lim_{t\to-\infty}\left[\tfrac{1}{2}\ln|2x-5|\right]_t^0 = \lim_{t\to-\infty}\left[\tfrac{1}{2}\ln 5 - \tfrac{1}{2}\ln|2t-5|\right] = -\infty.$

Divergent

21. $\displaystyle\int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\left[\frac{(\ln x)^2}{2}\right]_1^t \quad \left[\begin{array}{l}\text{by substitution with}\\ u = \ln x,\ du = dx/x\end{array}\right] \quad = \lim_{t\to\infty}\frac{(\ln t)^2}{2} = \infty.$ Divergent

35. $\displaystyle I = \int_0^3 \frac{dx}{x^2 - 6x + 5} = \int_0^3 \frac{dx}{(x-1)(x-5)} = I_1 + I_2 = \int_0^1 \frac{dx}{(x-1)(x-5)} + \int_1^3 \frac{dx}{(x-1)(x-5)}.$

Now $\displaystyle\frac{1}{(x-1)(x-5)} = \frac{A}{x-1} + \frac{B}{x-5} \quad\Rightarrow\quad 1 = A(x-5) + B(x-1).$

Set $x = 5$ to get $1 = 4B$, so $B = \frac{1}{4}$. Set $x = 1$ to get $1 = -4A$, so $A = -\frac{1}{4}$. Thus

$$I_1 = \lim_{t\to1^-}\int_0^t \left(\frac{-\frac{1}{4}}{x-1} + \frac{\frac{1}{4}}{x-5}\right)dx = \lim_{t\to1^-}\left[-\frac{1}{4}\ln|x-1| + \frac{1}{4}\ln|x-5|\right]_0^t$$
$$= \lim_{t\to1^-}\left[\left(-\tfrac{1}{4}\ln|t-1| + \tfrac{1}{4}\ln|t-5|\right) - \left(-\tfrac{1}{4}\ln|-1| + \tfrac{1}{4}\ln|-5|\right)\right]$$
$$= \infty, \quad \text{since } \lim_{t\to1^-}\left(-\tfrac{1}{4}\ln|t-1|\right) = \infty.$$

Since I_1 is divergent, I is divergent.

47. (a)

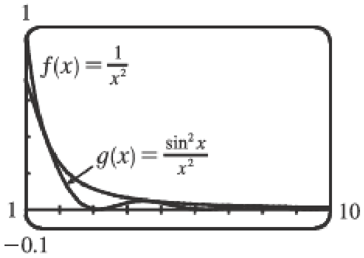
t	$\int_1^t g(x)\,dx$
2	0.447453
5	0.577101
10	0.621306
100	0.668479
1000	0.672957
10,000	0.673407

$g(x) = \dfrac{\sin^2 x}{x^2}$.

It appears that the integral is convergent.

(b) $-1 \le \sin x \le 1 \quad \Rightarrow \quad 0 \le \sin^2 x \le 1 \quad \Rightarrow \quad 0 \le \dfrac{\sin^2 x}{x^2} \le \dfrac{1}{x^2}$. Since $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent [Equation 2 with $p = 2 > 1$], $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ is convergent by the Comparison Theorem.

(c)



Since $\int_1^\infty f(x)\,dx$ is finite and the area under $g(x)$ is less than the area under $f(x)$ on any interval $[1, t]$, $\int_1^\infty g(x)\,dx$ must be finite; that is, the integral is convergent.

50. For $x \ge 1$, $\dfrac{2 + e^{-x}}{x} > \dfrac{2}{x}$ $\quad$ [since $e^{-x} > 0$] $\quad > \dfrac{1}{x}$. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ is divergent by Equation 2 with $p = 1 \le 1$, so $\displaystyle\int_1^\infty \frac{2 + e^{-x}}{x}\,dx$ is divergent by the Comparison Theorem.

57. If $p = 1$, then $\displaystyle\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{x} = \lim_{t\to 0^+} [\ln x]_t^1 = \infty$. $\qquad$ Divergent.

If $p \ne 1$, then $\displaystyle\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{x^p}$ $\qquad$ [note that the integral is not improper if $p < 0$]

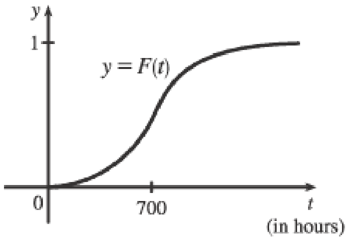
$$= \lim_{t\to 0^+} \left[\frac{x^{-p+1}}{-p+1}\right]_t^1 = \lim_{t\to 0^+} \frac{1}{1-p}\left[1 - \frac{1}{t^{p-1}}\right]$$

If $p > 1$, then $p - 1 > 0$, so $\dfrac{1}{t^{p-1}} \to \infty$ as $t \to 0^+$, and the integral diverges.

If $p < 1$, then $p - 1 < 0$, so $\dfrac{1}{t^{p-1}} \to 0$ as $t \to 0^+$ and $\displaystyle\int_0^1 \frac{dx}{x^p} = \frac{1}{1-p}\left[\lim_{t\to 0^+}\left(1 - t^{1-p}\right)\right] = \frac{1}{1-p}$.

Thus, the integral converges if and only if $p < 1$, and in that case its value is $\dfrac{1}{1-p}$.

67. We would expect a small percentage of bulbs to burn out in the first few hundred hours, most of the bulbs to burn out after close to 700 hours, and a few overachievers to burn on and on.

(a) 

(b) $r(t) = F'(t)$ is the rate at which the fraction $F(t)$ of burnt-out bulbs increases as t increases. This could be interpreted as a fractional burnout rate.

(c) $\int_0^\infty r(t)\,dt = \lim\limits_{x\to\infty} F(x) = 1$, since all of the bulbs will eventually burn out.