15 PARTIAL DERIVATIVES

15.1 FUNCTIONS OF SEVERAL VARIABLES

TRANSPARENCIES AVAILABLE

#35 (Figure 10), #36 (Figure 13), #37 (Figure 17), #38 (Figure 19), #39 (Exercise 30), #40 (Exercises 51–56)

SUGGESTED TIME AND EMPHASIS

1 class Essential material

POINTS TO STRESS

1. A function of two or three variables is a rule assigning a real number to every point in its domain.
2. Functions of two variables can be represented as surfaces, and can be described in two dimensions by contour maps and horizontal traces.
3. Functions of three variables can be described by level surfaces, and are generally more difficult to visualize than functions of two variables.

QUIZ QUESTIONS

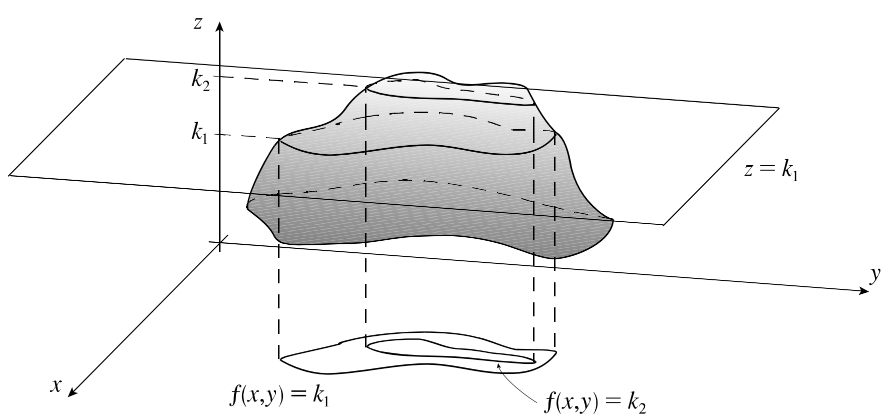
- **Text Question:** Why is the domain of the function in Example 4, $g(x, y) = \sqrt{9 - x^2 - y^2}$, shaped like a disk?

  Answer: Because it is the region $9 - x^2 - y^2 \geq 0$ or $x^2 + y^2 \leq 9$
- **Drill Question:** If f is a function of two variables and $f(3, -4) = -1$, give the coordinates of a point on the graph of f.

  Answer: $(3, -4, -1)$

MATERIALS FOR LECTURE

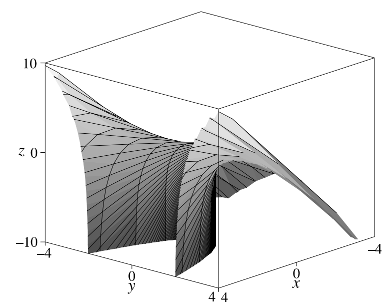
- One way to describe functions of two variables $f(x, y)$ is to have the student think of the contour curve $f(x, y) = t$ as existing at time t. One can invoke the image of a moving plane slicing a three-dimensional surface, perhaps displaying a picture like the following:



  With this way of looking at things, a sphere is a point that becomes a circle, grows, and then shrinks back to a point. This approach then makes it easier to describe functions of three variables. A function of three variables can be thought of a level surface that changes with time. Example 15 can be revisited in this context. $f(x, y, z) = x^2 + y^2 + z^2$ can be pictured as a point at time $t = 0$, a sphere of radius 1 at time $t = 1$, a sphere of radius $\sqrt{2}$ at $t = 2$, and so on. In other words, $f(x, y, z) = x^2 + y^2 + z^2$ can be pictured as a growing sphere, and the "level surfaces" of the function as snapshots of the process. Another good function to describe with this method is $f(x, y, z) = x^2 + y^2 - z$.

- An alternate way to approach the subject is to think of one dimension as "color". A surface such as $f(x, y) = \sin x + \cos y$ could be then drawn on a sheet of graph paper, with red representing the contour curve $f(x, y) = -2$, violet representing the contour curve $f(x, y) = 2$, and any number in between represented by the appropriate color. Some software packages represent functions of three variables using a method similar to this one.

- Revisit the function $f(x, y) = x \ln(y^2 - x)$. Sketch the domain, as done in Figure 3 of the text. Then go on to sketch the set of points (x, y) where $f(x, y) = 0$, $f(x, y) > 0$, $f(x, y) < 0$.

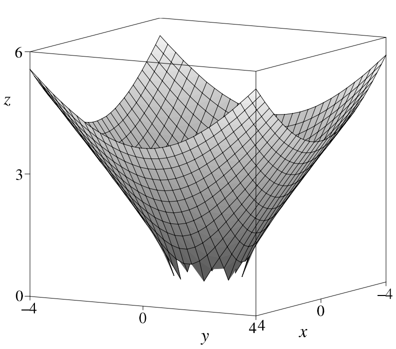


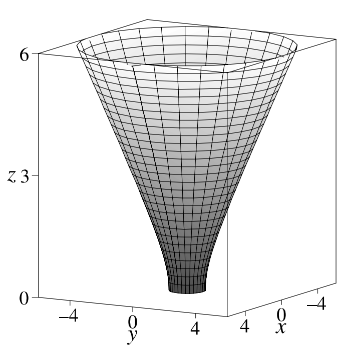
- Pass around some interesting solid figures, and have the students attempt to sketch the appropriate contour lines for the solids.

WORKSHOP/DISCUSSION

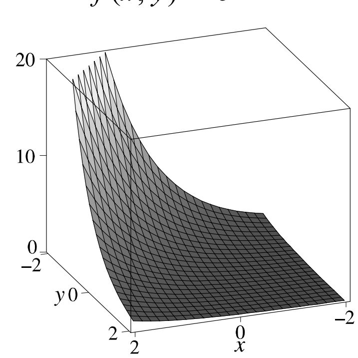
- Use mathematical reasoning to describe the domains and the graphs of the following functions, perhaps later putting up transparencies of the solutions to verify the reasoning:

$f(x, y) = \sqrt{x^2 + y^2 - 1}$





$f(x, y) = e^{x-y}$



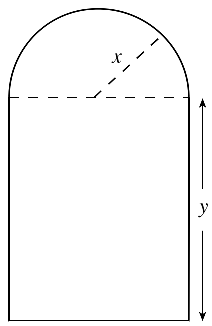
- Discuss the level curves for $f(x, y) = e^{x-y}$. Point out that they reduce to the simple equations $y = x - \ln k$ for $z = k > 0$.

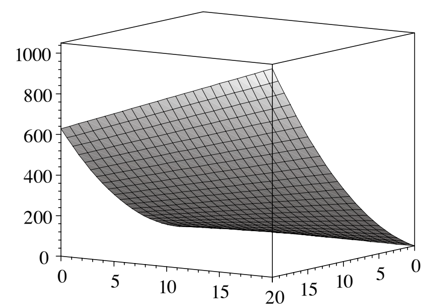
- Domain calculations often involve solving inequalities (sometimes nontrivial ones), and thus are usually not so simple for the students. Go over some examples, such as calculating domains for the following functions:

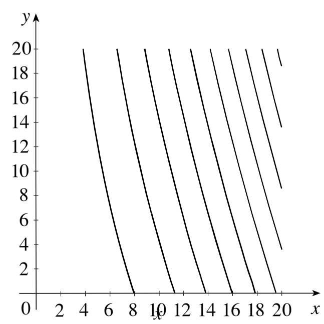
$$f(x,y) = \frac{\sqrt{x^2+y^3}}{x^2+3x-8} \qquad f(x,y) = -2\cos 2x + y$$

$$f(x,y) = \sin\left(\sqrt{1-\left(x^2+y^2\right)}\right) \qquad f(x,y) = \exp\left(\frac{x+y}{xy}\right)$$

- Let A be the area of the Norman window shown below. Lead the students to see that A can be expressed as a function of two variables x and y. Have them figure out the domain of A and use level curves to determine what the graph of A looks like.







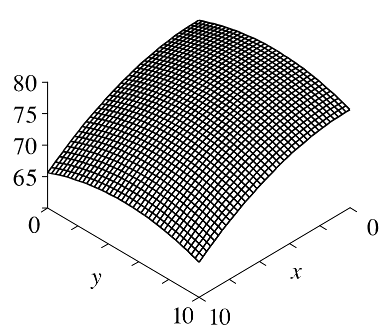
- One mildly interesting function of several variables is the *Evan function* $f(a,b) = a\left(10^{\lfloor \log_{10} b \rfloor + 1}\right) + b$. After finding this function's domain and range, have the students explore what the function does when a and b are positive integers. (It concatenates them.)

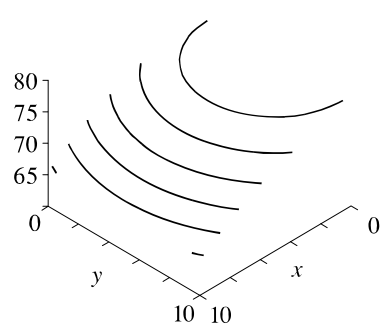
GROUP WORK 1: Staying Cool

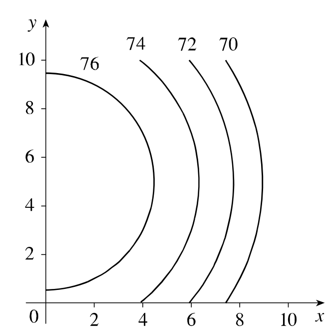
The students may try to draw three-dimensional graphs here, but a contour map should also be given full credit.

Answers:

1. (a)

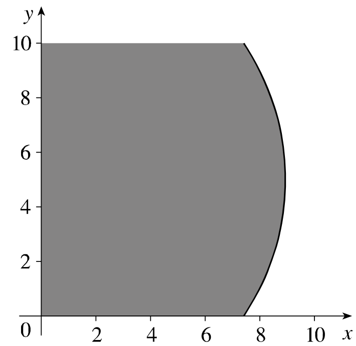






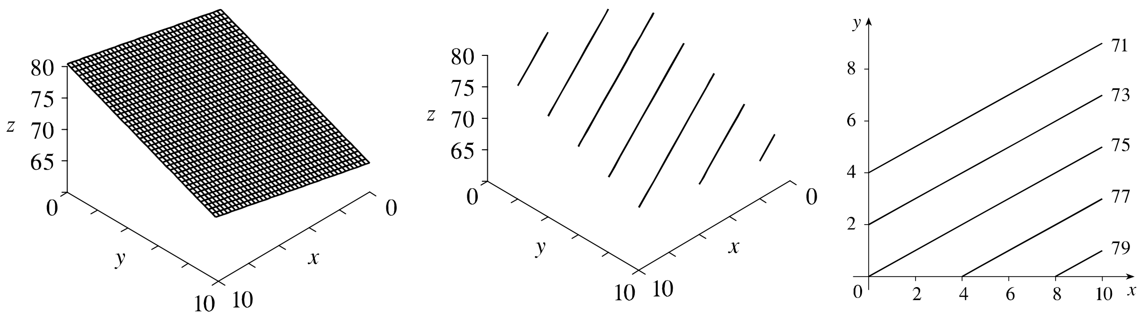
(b) The heat vent is at $(0, 5)$.

(c)



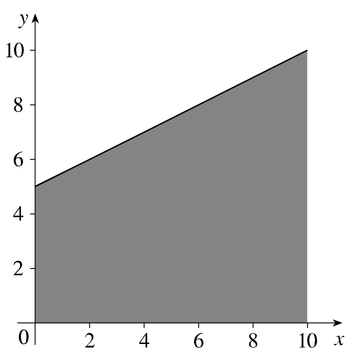
$$78 - \tfrac{1}{10}\left(x^2 + (y-5)^2\right) \le 70$$

2. (a)



(b) The heat vent is at $(10, 0)$.

(c)



$$\tfrac{1}{2}x - y + 75 = 70$$

GROUP WORK 2: The Matching Game

Don't let the students give up too easily. This activity is deliberately given early in the chapter, as an introduction to three dimensional surfaces. After the students are done, have each group present their reasoning. Some of the correct methods they come up with will surprise you.

Answers: 1. III **2.** VI **3.** V **4.** I **5.** IV **6.** II

GROUP WORK 3: Dali's Target

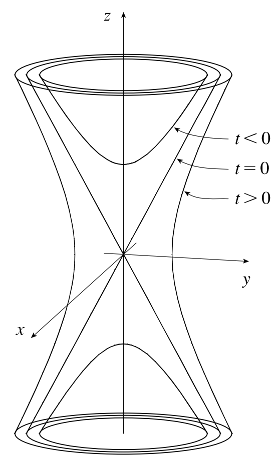
This activity is designed for students who are having some difficulty with the concept of level curves.

Answers: 1. $3 \le y \le 6.5$, $7.5 \le y \le 10$ **2.** 15 **3.** 20 **4.** 2 **5.** 3

GROUP WORK 4: Level Surfaces

Part 2 of Problem 4 requires that the students have some familiarity with conic sections, or have some type of graphing software.

Answer:



GROUP WORK 5: The MmR Project

This exercise involves looking at some interesting functions given a particular domain in the xy-plane or in space. Each group should get the same worksheet, but a different domain. (There is a blank space on the sheet in which to write the assigned domain.) Possible domains to give the students are:

$$\text{(Two-dimensional)} \quad 0 \le x, y \le 1, z = 0$$
$$\text{(Two-dimensional)} \quad x^2 + y^2 \le 1, z = 0$$
$$0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$$
$$x^2 + y^2 \le 1, 0 \le z \le 1$$
$$x^2 + y^2 + z^2 = 1$$
$$x^2 + y^2 = 1, |z| \le 1$$

If a group finishes early, they could be given another domain to do, or instructed to prepare a presentation about their solution to give to the class. Ideally, each group should solve the problem themselves for at least one domain, and see a discussion of at least two other domains.

It may be most appropriate to go through the domain $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$ with the students as an example.

If the students are able to do the last problem well, make sure to point out that what they are really doing is trying to comprehend a four-dimensional object, with the fourth dimension being time.

EXTENDED GROUP PROJECT: Applied Contour Maps

Give the students a list of today's temperature in various cities, along with a map of the country. The temperatures are available in many newspapers. Have the students draw a contour map showing curves of constant temperature. Then give them a copy of the temperature contour map from a newspaper to compare with their map to see how they did. Discuss what a three-dimensional representation of today's weather would be like. You can have the students cut the contours out of corrugated cardboard and make actual three-dimensional weather maps.

HOMEWORK PROBLEMS

Core Exercises: 2, 5, 6, 13, 16, 24, 30, 34, 41, 55

Sample Assignment: 2, 4, 5, 6, 8, 13, 16, 17, 20, 24, 25, 28, 30, 32, 34, 37, 41, 44, 49, 54, 55, 62, 67

Exercise	D	A	N	G
2	×		×	
4	×	×	×	
5	×		×	
6		×		
8		×		×
13		×		×
16		×		×
17		×		×
20		×		×
24				×
25				×
28				×
30	×			×
32	×			×
34				×
37				×
41				×
44				×
49	×			×
54				×
55				×
62	×			
67				×

GROUP WORK 1, SECTION 15.1

Staying Cool

Let $T(x, y)$ be the temperature in a 10 ft $\times$ 10 ft room on a winter night, where one corner of the room is at (0, 0) and the opposite corner is at (10, 10). For each of the following functions T,

(a) Draw a graph of the temperature function.

(b) Describe the likely floor locations of the heat vents.

(c) Suppose you like to sleep with a temperature of 70° or less. Where would you put the bed?

1. $T(x, y) = 78 - \frac{1}{10}\left[x^2 + (y-5)^2\right]$

2. $T(x, y) = \frac{1}{2}x - y + 75$

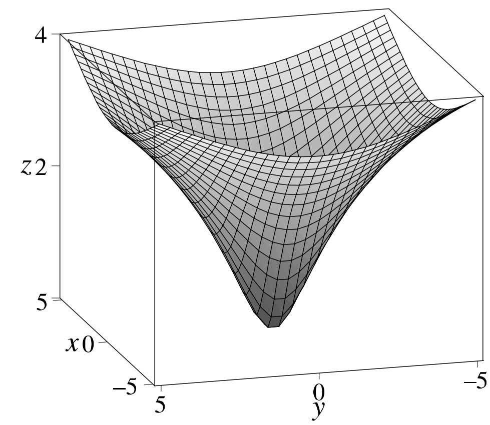
GROUP WORK 2, SECTION 15.1

The Matching Game

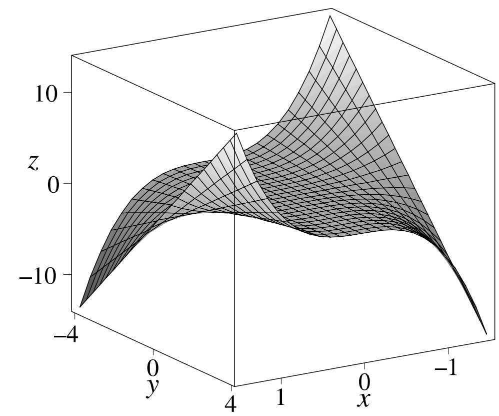
Match each function with its graph. Give reasons for your choices.

1. $f(x, y) = \dfrac{1}{x+1} + \sin y$ **2.** $f(x, y) = \sqrt{4 - x^2 - y^2}$ **3.** $f(x, y) = \cos\left(x + y^2\right)$

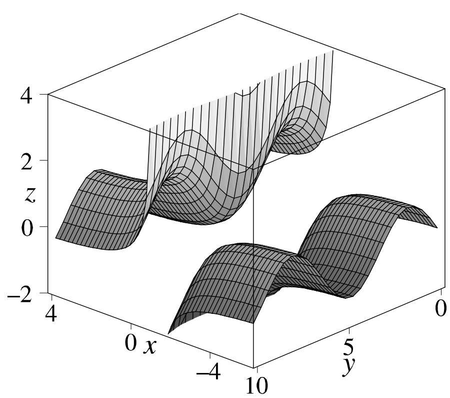
4. $f(x, y) = \ln\left(x^2 + y^2 + 1\right)$ **5.** $f(x, y) = x^2\sqrt{y}$ **6.** $f(x, y) = x^3 y$



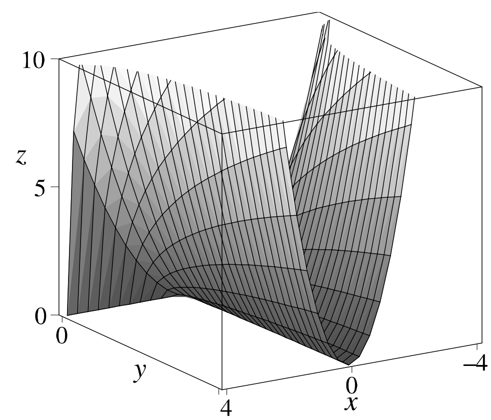
I



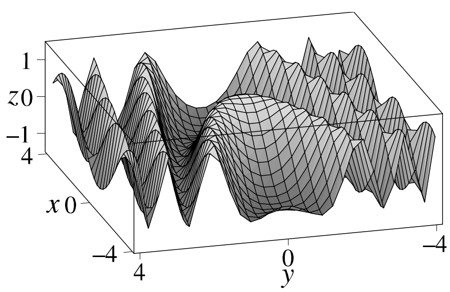
II



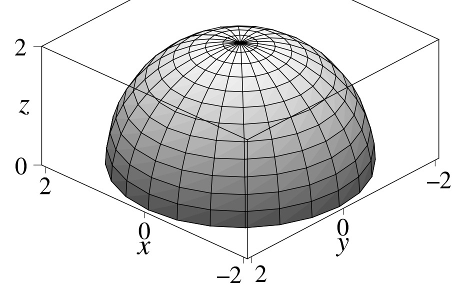
III



IV



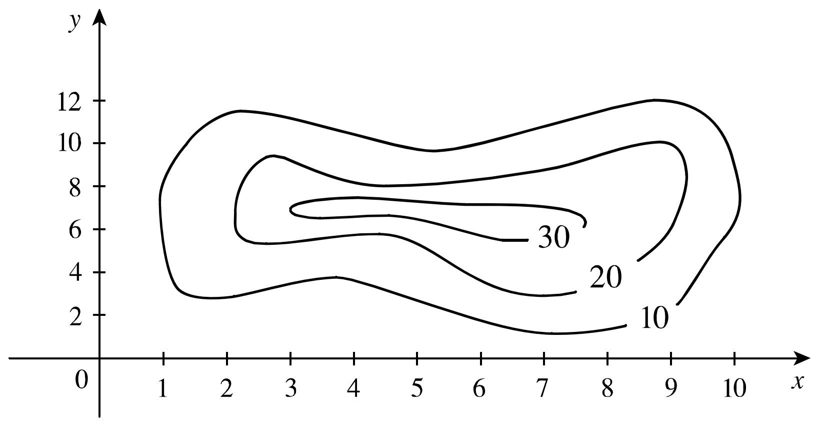
V



VI

GROUP WORK 3, SECTION 15.1
Dali's Target

Consider the following contour map of a continuous function $f(x, y)$:



1. For approximately what values of y is it true that $10 \leq f(5, y) \leq 30$?

2. What can you estimate $f(2, 4)$ to be, and why?

3. Do we have any good estimates for $f(5, 8)$? Explain.

4. How many values y satisfy $f(7, y) = 20$?

5. How many values of x satisfy $f(x, 8) = 20$?

GROUP WORK 4, SECTION 15.1

Level Surfaces

It can be difficult to visualize functions of three variables. One way to do it is by thinking of each level surface as representing a different point in time. As we let t vary in the equation for the level surface $f(x, y, z) = t$ we can think of the function $f(x, y, z)$ as a surface whose shape and size varies as time changes.

Consider the function $f(x, y, z) = x^2 + y^2 - z^2$.

1. What is the level surface $f(x, y, z) = 0$?

2. What is the level surface $f(x, y, z) = 1$?

3. For $t > 0$, what do the level surfaces $f(x, y, z) = t$ look like?

4. What is the level surface $f(x, y, z) = -1$?

5. Describe all the level surfaces $f(x, y, z) = t$.

GROUP WORK 5, SECTION 15.1

The MmR Project

Consider the region ______________________________ .

1. Sketch or describe this region.

We are now going to describe some functions of three variables for which the region in Part 1 is the domain. In other words, every point in your domain will have a function value for the functions below. The functions are:

$M(x, y, z) = \max(x, y, z)$ $\qquad m(x, y, z) = \min(x, y, z)$ $\qquad R(x, y, z) = x + y + z$

2. Evaluate M, m, and R at several different points in your domain. The first line in the following table is an example for you to look at.

Point	$M(x, y, z)$	$m(x, y, z)$	$R(x, y, z)$
$(1, 2, 3)$	3	1	6
(, ,)			
(, ,)			
(, ,)			

3. Find the maximum values of M, m, and R on your domain.

4. Sketch the level surfaces $M = \frac{1}{2}$, $R = \frac{1}{2}$, $R = 0$, and $m = \frac{1}{2}$ for your domain.

5. For an extra challenge, try to describe the level surfaces $M = t$, $R = t$, and $m = t$, for $0 \leq t \leq 2$. If we let t stand for time, and make a movie of the level surface changing as t goes from 0 to 2, what would the movie look like?

15.2 LIMITS AND CONTINUITY

SUGGESTED TIME AND EMPHASIS

1 class Recommended material. This material can be covered from a variety of perspectives, and at a variety of depths. (For example, the nonexistence of certain limits can be de-emphasized.) The instructor should feel especially free to pick and choose from the suggestions below.

POINTS TO STRESS

1. While the definitions of limits and continuity for multivariable functions are nearly identical to those of their single variable counterparts, very different behavior can take place in the multivariable case.
2. The idea of points being "close" in $\mathbb{R}^2$ and $\mathbb{R}^3$.

QUIZ QUESTIONS

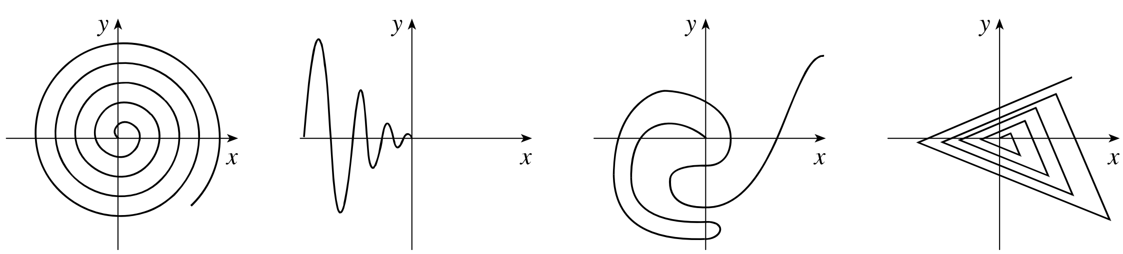
- **Text Question:** When talking about limits for functions of several variables, why isn't it sufficient to say, "$\lim_{(x,y)\to(0,0)} f(x, y) = L$ if $f(x, y)$ gets close to L as we approach $(0, 0)$ along the x-axis ($y = 0$) and along the y-axis ($x = 0$)"?

  Answer: Path independence is important, and only two paths are discussed above.
- **Drill Question:** Show that $\lim_{(x,y)\to(0,0)} \left(x^2 + y^2\right) = 0$.

  Answer: Any answer that correctly addresses path independence should be given credit.

MATERIALS FOR LECTURE

- Stress that $f(x, y) \to L$ as $(x, y) \to (a, b)$ means that $f(x, y)$ gets as close to L as we like as (x, y) gets close to (a, b) in distance, that is, *regardless of path*. Give examples of exotic paths to $(0, 0)$ such as the following:



- This is a rich example of a limit that exists:

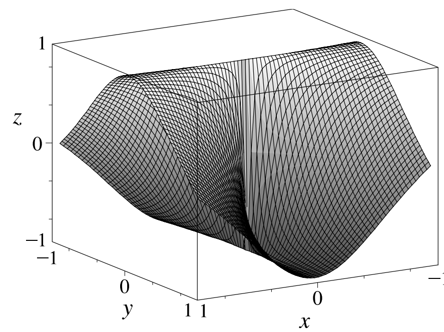
$$\lim_{(x,y)\to(0,0)} \frac{\sin\left(x^2 + y^2\right)}{x^2 + y^2} = 1$$

  Since $\frac{\sin\left(x^2 + y^2\right)}{x^2 + y^2}$ is constant on circles centered at the origin, we want to look at the distance w between (x, y) and $(0, 0)$: $w = \sqrt{x^2 + y^2}$. Computationally, it is best to look at what happens when $w^2 \to 0$. In this case, $\frac{\sin\left(x^2 + y^2\right)}{x^2 + y^2} = \frac{\sin w^2}{w^2}$, and single-variable calculus gives us that $\lim_{w\to 0} \frac{\sin w^2}{w^2} = 1$. Stress that in general it does *not* suffice to just let $x = 0$ or $y = 0$ and then compute the limit.

- This is a good example of a limit that does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

The text shows this fact in an interesting way: If we let $x = 0$, then we get $\lim_{y\to 0} \frac{-y^2}{y^2} = -1$, but if we let $y = 0$ then we get $\lim_{x\to 0} \frac{x^2}{x^2} = 1$. So if we approach the origin by one radial path, we get a different limit than we do if we go by a different radial path. In fact, assume we go to the origin by a straight line $y = mx$. Then $\frac{x^2 - y^2}{x^2 + y^2} = \frac{x^2\left(1 - m^2\right)}{x^2\left(1 + m^2\right)}$ and $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \frac{1 - m^2}{1 + m^2}$. So this limit can take any value from -1 to 1 if we approach the origin by a straight line. For example, if we use the line $y = \frac{1}{\sqrt{3}}x$, we get $\frac{1}{2}$ as a limit.

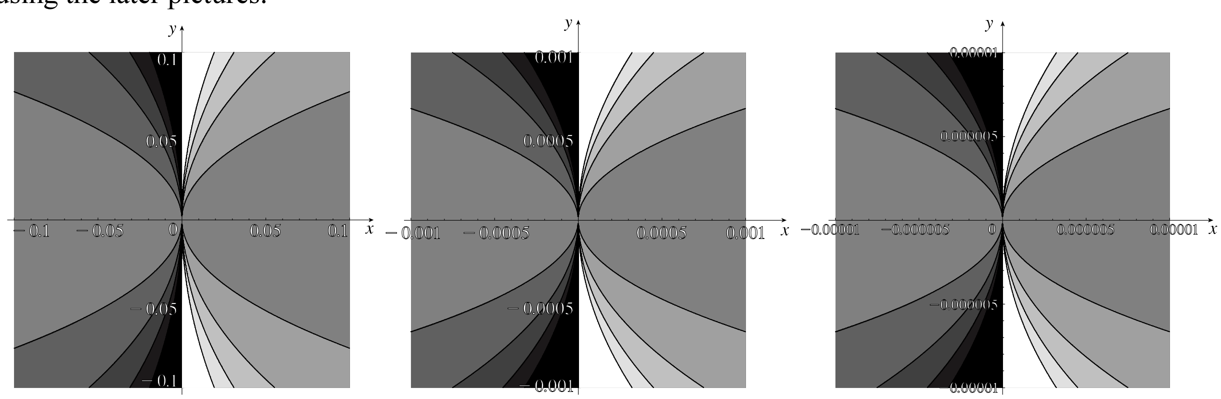


- Note that just because a function has a limit at a point doesn't imply that it is continuous there. For example,

$$f(x, y) = \begin{cases} \dfrac{\sin\left(x^2 + y^2\right)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

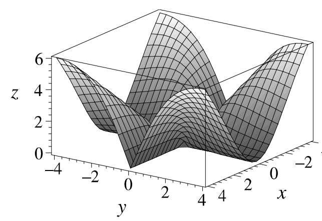
is discontinuous at the origin even though $\lim_{(x,y)\to(0,0)} f(x, y)$ exists.

- Expand the explanation of Example 3 using a visual approach, perhaps using figures like the following or using algebra to compute the limit along the general parabola $x = my^2$. The figures show progressively smaller viewing rectangles centered at the origin. The black regions correspond to larger negative values of $f(x, y) = \frac{xy^2}{x^2 + y^4}$, and the white regions correspond to larger positive values. Notice that when travelling along any straight line $y = mx$, the color of the points on the path eventually becomes gray [at points where $f(x, y) = 0$] as the origin is approached. This effect is best observed (even if m is large) using the later pictures.

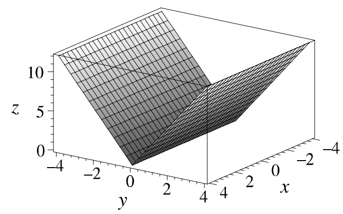


However, when approaching the origin on a parabolic path, $x = my^2$, the color of the points on the path always stays the same! This phenomenon is best illustrated by the earlier pictures. Therefore, this set of plots illustrates how the limit as $(x, y) \to (0, 0)$ of $f(x, y) = \dfrac{xy^2}{x^2 + y^4}$ does not exist, although one would erroneously believe it to be 0 if one looked only at the "obvious" linear paths.

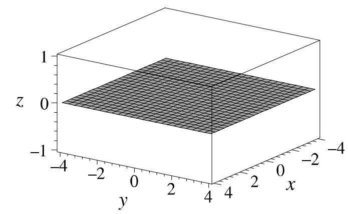
- Discuss the squeeze principle by looking at the three graphs below, and computing $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y)$ by squeezing f between g and h.



$f(x, y) = \dfrac{3x^2\,|y|}{x^2 + y^2}$



$g(x, y) = 3\,|y|$



$h(x, y) = 0$

WORKSHOP/DISCUSSION

- Introduce the use of polar coordinates by trying to compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{\sqrt{x^2 + y^2}}$. Point out how intractable the problem looks at first glance. But notice that in polar coordinates, the statement "$(x, y) \to (0, 0)$" translates to the much simpler "$r \to 0$", so the limit can be rewritten as $\displaystyle\lim_{r\to 0} \frac{3\,(r\cos\theta)^2\, r\sin\theta}{r}$, which simplifies to $\displaystyle\lim_{r\to 0} 3r^2\cos^2\theta\sin\theta = 0$.

- Check that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x + y - \sin(x + y)}{(x + y)^3} = \frac{1}{6}$, $x \neq -y$, by noticing that this function is constant on $x + y = k$. This suggests using the substitution $u = x + y$ and applying the single-variable version of l'Hospital's Rule. This is also a good limit to first investigate numerically, plugging in small values of x and y.

GROUP WORK 1: Even Mathematicians Have Their Limits

When a group is finished doing the problems as stated (plugging in different values of x and y) have them attempt to prove their results mathematically, establishing path-independence, either by changing to polar coordinates, or by using a substitution.

Answers: 1. 5 **2.** ∞ **3.** $-\infty$ **4.** 2

GROUP WORK 2: There Is No One True Path

The students are asked to analyze $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^4+y^8}$. This activity is not intended as an exercise in algebra; the students should make free use of some form of technology.

Answers:

1. This corresponds to a path going along the y-axis. The limit is 0.

2. 0

3. 0

4. This is the punchline. The limit is not zero! For example, following the path $y = \sqrt{x}$ gives a limit of 1. So the limit does not exist.

HOMEWORK PROBLEMS

Core Exercises: 2, 4, 8, 16, 20, 29, 36, 39

Sample Assignment: 2, 4, 7, 8, 9, 16, 20, 21, 23, 28, 29, 31, 36, 39, 42

Exercise	D	A	N	G
2	×			
4	×	×	×	
7		×		
8		×		
9		×		
16		×		
20		×		
21		×		
23	×			×
28	×			×
29		×		
31		×		
36		×		
39		×		
42		×		×

GROUP WORK 1, SECTION 15.2

Even Mathematicians Have Their Limits

Try to estimate the following limits by plugging small values of x and y into the appropriate function. Remember that path independence is important: Try some values where $x = y$ and some where $x \neq y$.

1. $\displaystyle\lim_{(x,y)\to(0,0)} 5$

2. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{xy}}{x^2 + y^2}$

3. $\displaystyle\lim_{(x,y)\to(0,0)} \ln\left(x^2 + y^2\right)$

4. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin 2(x+y)}{x+y}, y \neq -x$

GROUP WORK 2, SECTION 15.2
There Is No One True Path

In this activity, we are going to investigate

$$\lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^4+y^8}$$

1. Set $x = 0$ and let $y \to 0$. To what path does this process correspond? What is the limit along this path?

2. What limit do you get when you approach the origin along an arbitrary straight line ($y = mx$)?

3. What limit do you get when you approach the origin along a parabola such as $y = x^2$?

4. What is $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^4+y^8}$?

15.3 PARTIAL DERIVATIVES

TRANSPARENCIES AVAILABLE

#41 (Figures 2–5), #42 (Exercise 7)

SUGGESTED TIME AND EMPHASIS

1 class Essential material

POINTS TO STRESS

1. The meaning of f_x and f_y, both analytically and geometrically.
2. The various notations for f_x and f_y. [Be sure to point out that in the notation $f_x(x, y)$, x is playing two different roles: $f_x(x, y)$ can be written $f_1(x, y)$, where the 1 indicates that the derivative is taken with respect to the first variable.]
3. Higher-order partial derivatives.

QUIZ QUESTIONS

- **Text Question:** Suppose that $f(x, y)$ is continuous everywhere. Assume that $f_x(1, 1) = 2$, $f_y(1, 1) = -2$, and $f_{xy}(1, 1) = 3$. Is it possible to compute $f_{yx}(1, 1)$ from this information alone? If so, what is it? If not, why not?
  Answer: 3
- **Drill Question:** Find a function $f(x, y)$ for which $\dfrac{\partial f}{\partial x} = x + y$ and $\dfrac{\partial f}{\partial y} = x$.
  Answer: $f(x, y) = xy + \frac{1}{2}x^2 + K$

MATERIALS FOR LECTURE

- The idea of partial derivatives being continuous is going to be very important in this chapter, so make sure to stress both the hypothesis and the conclusion of Clairaut's Theorem.
- Provide an alternate geometric interpretation for the partial derivative in terms of vector functions. The graph C_1 of the function $g(x) = f(x, b)$ is the curve traced out by the *vector* function $\mathbf{g}(x) = \langle x, b, f(x, b)\rangle$ whose *vector* derivative $\mathbf{g}'(a) = \langle 1, 0, f_x(a, b)\rangle$ is determined by $f_x(a, b)$. [That is, its "slope" is $f_x(a, b)$.] Similarly, the graph C_2 of $h(y) = f(a, y)$ is the curve traced out by the vector function $\mathbf{h}(y) = \langle a, y, f(a, x)\rangle$ whose vector derivative $\mathbf{h}'(b) = \left\langle 0, 1, f_y(a, b)\right\rangle$ is determined by $f_y(a, b)$.
  This definition can be explored further, if we want to foreshadow the next section. Describe the idea of a tangent plane to, say $f(x, y) = e^{xy}$ at the point $\left(1, 2, e^2\right)$. We can set up the tangent plane determined by the partials f_x and f_y by using the vector functions $\mathbf{g}(x) = \langle x, b, f(x, b)\rangle$, $\mathbf{h}(y) = \langle a, y, f(a, y)\rangle$ and the vector derivatives $\mathbf{a} = \langle 1, 0, f_x(a, b)\rangle$, $\mathbf{b} = \left\langle 0, 1, f_y(a, b)\right\rangle$. Form the plane through $(a, b, f(a, b))$ with normal vector $\mathbf{N} = \mathbf{a} \times \mathbf{b}$. Find the plane tangent to $f(x, y) = e^{xy}$ at the point $\left(1, 2, e^2\right)$.

WORKSHOP/DISCUSSION

We believe that it is crucial that the students do not leave their workshop/discussion session without knowing how to compute partial derivatives. There should be some opportunity for the students to practice in class, even by just trying one or two easy problems, so they can get instant feedback.

- Compute $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$ for $f(x,y) = \sin(\pi x e^{xy})$.
- Let $f(x,y,z) = xy^4z^3$ or some other easy-to-differentiate function. Verify that $f_{xyz} = f_{xzy} = \cdots = f_{zyx}$. Perhaps then show that $f_{xzz} = f_{zxz}$.
- Demonstrate that the functions $f(x,y) = 5xy$, $f(x,y) = e^x \sin y$, and $f(x,y) = \arctan(y/x)$ all solve the Laplace equation $f_{xx} + f_{yy} = 0$.

GROUP WORK 1: Clarifying Clairaut's Theorem

Problem 3 foreshadows the process of solving exact differential equations by finding $f(x,y)$ given that $f_{xy} = f_{yx}$. Students should be led carefully through this component at the end, to make sure they will be able to eventually find gradient functions.

Answers:

1. By inspection, $f_{xxx} = 0$.

2. There is no similar trivial argument to be made in the case of taking the z-derivative thrice.

3. (a) $a = \frac{1}{2}, b = 1$ (b) $\frac{3}{2}x^2 + \frac{1}{2}xy^2$ (c) $\frac{3}{2}x^2 + \frac{1}{2}xy^2 + \frac{1}{2}y^2$

(d) $3x^2 + \frac{1}{2}y^2$. $f(x,y) = \frac{3}{2}x^2 + \frac{1}{2}xy^2 + \frac{1}{2}y^2 + K$

GROUP WORK 2: Back to the Park

Answers:

1. $70, \approx 98$ **2.** $0.1, 0.12$

3. Steepness in the positive x- and positive y-directions

4. Perpendicular to the contour line. Answers such as "$\langle 1, 1\rangle$" and "45° to the horizontal" are correct.

5. $70 + 20(0.1) + 10(0.12) = 73.2$

GROUP WORK 3: Mixed Partials

This activity will give the students a solid understanding of the geometry of mixed partial derivatives. Strive to get them to articulate their reasons for their answers; the very process of trying to put into words the reason that $f_{xy} < 0$ should clear up sloppy thinking.

Answer: $f_x > 0$, $f_y < 0$, $f_{xx} > 0$, $f_{yy} > 0$, $f_{xy} < 0$, $f_{yx} < 0$

HOMEWORK PROBLEMS

Core Exercises: 3, 7, 9, 11, 20, 25, 39, 46, 62, 69, 81

Sample Assignment: 2, 3, 5, 7, 9, 10, 11, 16, 20, 25, 31, 36, 39, 43, 46, 53, 59, 62, 67, 69, 72, 77, 81, 84, 92

Exercise	D	A	N	G
2	×		×	
3	×		×	
5				×
7				×
9	×			×
10				×
11		×		×
16		×		
20		×		
25		×		
31		×		
36		×		
39		×		
43		×		
46		×		
53		×		
59		×		
62		×		
67		×		
69	×		×	
72		×		
77		×		
81	×	×		
84	×	×		
92	×			

GROUP WORK 1, SECTION 15.3
Clarifying Clairaut's Theorem

Consider $f(x, y, z) = x^2 \cos(y^3 + z^2)$.

1. Why do we know that $f_{zyyxxx} = 0$ without doing any computation?

2. Do we also know, without doing any computation, that $f_{xyzzz} = 0$? Why or why not?

3. Suppose that $f_x = 3x + ay^2$, $f_y = bxy + 2y$, $f_y(1, 1) = 3$, and f has continuous mixed second partial derivatives f_{xy} and f_{yx}.

(a) Find values for a and b and thus equations for f_x and f_y. *Hint:* What does Clairaut's Theorem say about the mixed partial derivatives of a function? When does the theorem apply?

(b) Can you find a function $F(x, y)$ such that $\dfrac{\partial F}{\partial x} = f_x$ in part (a)?

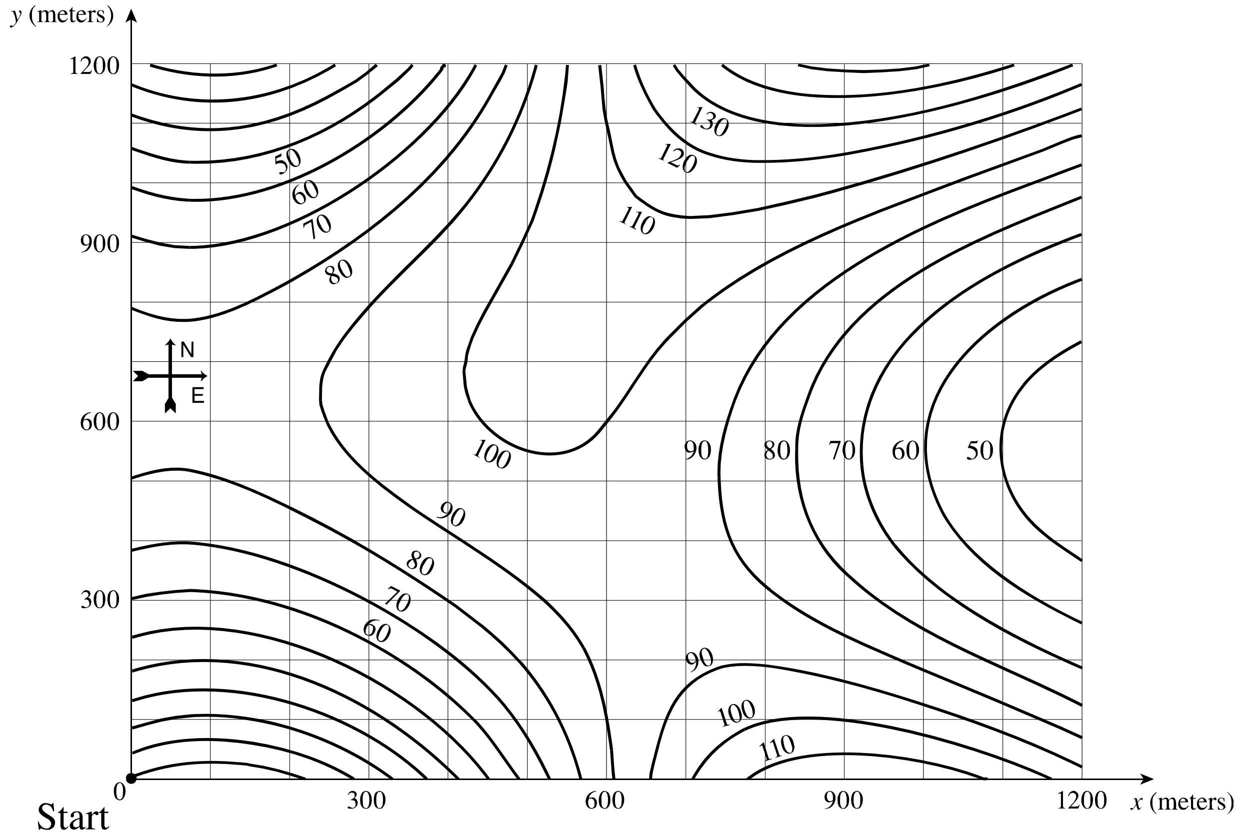
(c) Can you find a function $G(x, y) = F(x, y) + k(y)$ such that $\dfrac{\partial G}{\partial y} = f_y$ in part (a)? What is $k(y)$?

(d) What is $\dfrac{\partial G}{\partial x}$? Can you now find $f(x, y)$?

GROUP WORK 2, SECTION 15.3

Back to the Park

The following is a map with curves of the same elevation of a region in Orangerock National Park:



We define the altitude function, $A(x, y)$, as the altitude at a point x meters east and y meters north of the origin ("Start").

1. Estimate $A(300, 300)$ and $A(500, 500)$.

2. Estimate $A_x(300, 300)$ and $A_y(300, 300)$.

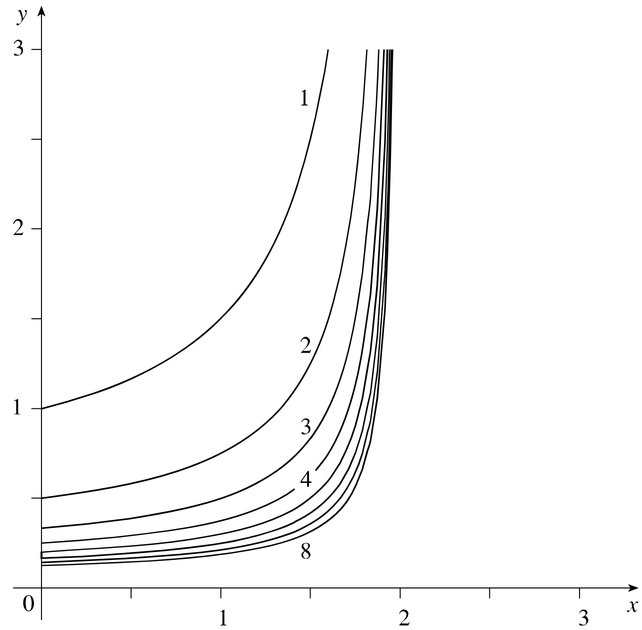
3. What do A_x and A_y represent in physical terms?

4. In which direction does the altitude increase most rapidly at the point (300, 300)?

5. Use your estimates of A_x (300, 300) and A_y (300, 300) to approximate the altitude at (320, 310).

GROUP WORK 3, SECTION 15.3
Mixed Partials

The level curves of a function $z = f(x, y)$ are given below.



Use the level curves of the function to decide the signs (positive, negative, or zero) of the derivatives f_x, f_y, f_{xx}, f_{yy}, f_{xy},, and f_{yx} of the function at the point $\left(\frac{3}{2}, \frac{1}{2}\right)$.

15.4 TANGENT PLANES AND LINEAR APPROXIMATIONS

TRANSPARENCIES AVAILABLE

#43 (Figure 2), #44 (Figure 7)

SUGGESTED TIME AND EMPHASIS

1–$1\frac{1}{2}$ classes Recommended material

POINTS TO STRESS

1. The tangent plane and its analogy with the tangent line.
2. Approximation along the tangent plane and its analogy with approximation along the tangent line.
3. The meaning of differentiability in $\mathbb{R}^2$ and $\mathbb{R}^3$.
4. The difference between f being differentiable and the existence of f_x and f_y.

QUIZ QUESTIONS

- **Text Question:** Is it possible for a function f to be differentiable at (a, b) even though f_x and f_y do not exist at (a, b)?
  Answer: No
- **Drill Question:** Is it possible for a function f to be non-differentiable at (a, b) even though f_x and f_y exist at (a, b)?
  Answer: Yes

MATERIALS FOR LECTURE

- Discuss differentiability using both the definition and this intuitive description: f is differentiable at (a, b) if both $f_x(a, b)$ and $f_y(a, b)$ exist and the linearization of f at (a, b) closely approximates $f(x, y)$ when (x, y) is sufficiently close to (a, b).
- One good example of non-differentiability is $f(x, y) = \begin{cases} \dfrac{(x-y)^2}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$. Here $f(x, 0) \equiv 1$ and $f(0, y) \equiv 1$, so f_x and f_y are both zero, but the tangent plane fails to be a good approximation, no matter how close to the origin we look. For example, $f(x, x) \equiv 0$ for all $x \neq 0$. Another good example is $g(x, y) = \sqrt{x^2 + y^2}$, which is continuous but not differentiable.
- Note that if the partial derivatives of a function exist and are continuous, then the tangent plane exists.

WORKSHOP/DISCUSSION

- Find an equation for the tangent plane to the top half of the unit sphere $x^2 + y^2 + z^2 = 1$ at the point $(0, 0, 1)$ and then at $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ using both algebraic and geometric reasoning.
- Compute some approximations to values of differentiable functions. For example, if $f(x, y) = \sin\left(\pi\left(x^2 + xy\right)\right)$, then $f\left(\frac{1}{2}, 0\right) = \frac{1}{\sqrt{2}}$. Show the students how to use this fact and the partial derivatives of f to estimate $f(0.55, -0.01)$.
- Given $f(x, y) = x^2y^3$, find the equation for the tangent plane at $(1, 1, 1)$, using the formula $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ [Answer: $z = 2(x-1) + 3(y-1) + 1$.] Then

have the students find the tangent plane at the point $(3, 1, 9)$. [Answer: $z = 6(x-3) + (y-1) + 9$.] This example can be extended by asking how well the tangent plane approximates the function at the given point, perhaps by comparing $f(1.1, 1.1) = 1.61$ to the approximating function at $(1.1, 1.1)$, which has the value 1.5, and then comparing $f(1.01, 1.01) \approx 1.051$ to the approximation at $(1.01, 1, 01)$, which has the value 1.05.

GROUP WORK 1: Trying it All Out

Problem 1 of this exercise requires the student to recognize where a complicated function is continuous. Problem 2 can be done without a picture, but students should be encouraged to draw a picture to verify their conclusions.

Answers:

1. (a) All points (b) All points where $x \neq y$ (c) All points where $x \neq y$

2. (a) $z - 2 = -\frac{3}{2}\left(x - \frac{1}{3}\right) - (y-2)$ (b) $(0, 0, \pm 3)$ (c) $(\pm 1, 0, 0)$ and $(0, \pm 3, 0)$

GROUP WORK 2: Voluminous Approximations

Problem 2 may seem like "alphabet soup" to some of the students. It may be advisable to start the activity by describing the solid in Problem 1 to the class, and then showing how 3 and 2 can be replaced by parameters, making the volume a function of two variables r and s.

Answers:

1. $\frac{28}{3}\pi$ **2.** $\frac{\pi}{3}s^2(r+2s)$

3. $dV = \frac{\pi}{3}s^2 dr + \frac{2\pi}{3}s(r+3s)\,ds$, so the maximum possible error is

$$dV = \frac{\pi}{3}(4)\left(\frac{1}{2}\right) + \frac{2\pi}{3}(2)(3+6)\left(\frac{1}{2}\right) = \frac{20}{3}\pi \approx 20.9$$

HOMEWORK PROBLEMS

Core Exercises: 1, 4, 11, 16, 20, 22, 25, 35, 41

Sample Assignment: 1, 4, 5, 7, 11, 14, 16, 17, 20, 22, 24, 25, 26, 30, 33, 35, 40, 41

Exercise	D	A	N	G
1		×		
4		×		
5		×		
7				×
11	×	×		
14	×	×		
16	×	×		
17		×		
20		×		×

Exercise	D	A	N	G
22	×		×	
24	×	×	×	
25		×		
26		×		
30		×		
33	×	×		
35	×	×		
40	×	×		
41	×	×		

GROUP WORK 1, SECTION 15.4

Trying it All Out

1. Determine for what points the following functions are differentiable, and describe (qualitatively) why your answer is correct.

(a) $f(x, y) = e^{xy} \cos(\pi(xy + 1))$

(b) $g(x, y) = \dfrac{x^4 - y^4}{x + y}$

(c) $h(x, y) = x - 2y \ln|x + y|$

2. Consider the surface $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{9} = 1$.

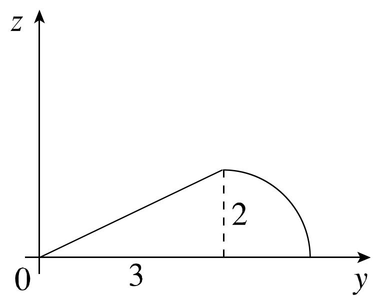
(a) Find the equation of the tangent plane to this surface at the point $\left(\frac{1}{3}, 2, 2\right)$.

(b) Find a point at which the tangent plane to this surface is horizontal. Are there any other such points?

(c) Find a point at which the tangent plane to this surface is vertical. Are there any other such points?

GROUP WORK 2, SECTION 15.4
Voluminous Approximations

Consider the solid obtained when rotating the following region about the x-axis.



1. Compute the volume of this solid.

2. Find the volume $V(r, s)$ of a similar solid created by rotating a region with dimensions r and s instead of 3 and 2.

3. Oh, we forgot to tell you that in Problem 1, the "2" and the "3" were really just rounded-off numbers. The actual quantities can be off by up to 0.5 in either direction. Use linear approximation to estimate the maximum possible error in your answer to Problem 1.

SUGGESTED TIME AND EMPHASIS

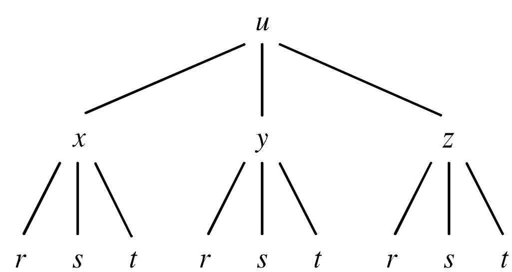
1 class Essential material

POINTS TO STRESS

1. The extension of the Chain Rule for functions of several variables.
2. Tree diagrams.
3. Implicit differentiation.

QUIZ QUESTIONS

- **Text Question:** What was the following figure illustrating in the text? Specifically, how was it used?



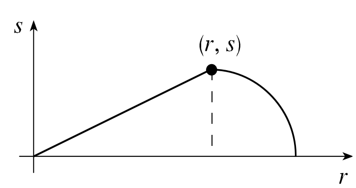
  Answer: It was used to find a Chain Rule formula for $\dfrac{\partial u}{\partial r}$, $\dfrac{\partial u}{\partial s}$, or $\dfrac{\partial u}{\partial t}$.

- **Drill Question:** Suppose that $f(x, y) = -x + y^2$, $x = u^2 + v^3$, and $y = 2u - 3v$. Compute $\partial f/\partial u$ when $u = 1$ and $v = -1$.
  Answer: 18

MATERIALS FOR LECTURE

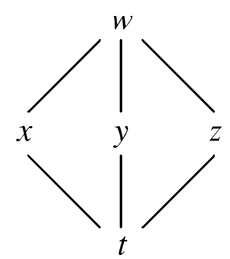
- Review the single-variable Chain Rule $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt}$ using the same language and symbols that will be used in presenting the multivariable Chain Rule. Then develop the formulas and derivatives for chain rules involving two and three independent variables.

- If Group Work 2: Voluminous Approximations was covered in the previous section, extend it as follows: Suppose that $r = t + \sin t$ and $s = e^t - \cos t$ vary with time. Compute dV/dt.

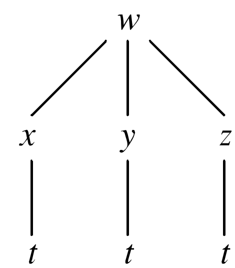


- Set up tree diagrams in two ways for the set of functions

$$w = f(x, y, z) \qquad x = g(t) \qquad y = h(t) \qquad z = k(t)$$



or



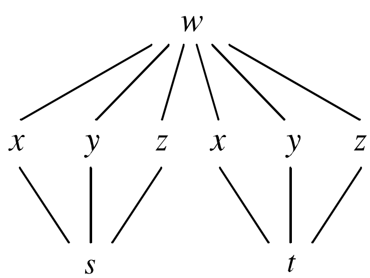
Then write out the Chain Rule for this case:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

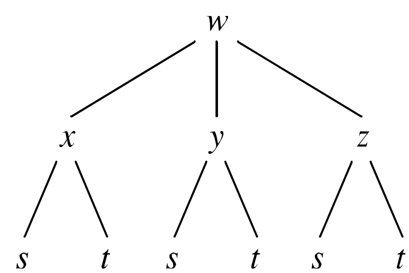
Extend the demonstration by considering

$$w = f(x, y, z) \qquad x = g(s, t) \qquad y = h(s, t) \qquad z = k(s, t)$$

Set out the relevant tree diagrams as shown below and write out the Chain Rule for $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$.



or



- State the Implicit Function Theorem. Illustrate it for the "fat circle" $x^4 + y^4 = 1$, and show why it fails when $\frac{\partial F}{\partial y} = 0$ [that is, at the points $(-1, 0)$ and $(1, 0)$.]

WORKSHOP/DISCUSSION

- Give an example of implicit differentiation on the ellipsoid $x^2 + \frac{y^2}{2} + \frac{z^2}{3} = 1$. Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, both in general and at the point $\left(0, 0, \sqrt{3}\right)$.

- Consider a cylindrical can of radius r and height h. Let V and S be the volume and surface area of the can. Find $\frac{\partial V}{\partial r}$ and $\frac{\partial S}{\partial h}$ and discuss what these quantities mean in practical terms. Then find $\frac{\partial V}{\partial h}$ when $r = 5$.

- Consider the function $f(x, y) = x^2 + y^2$ where $x = 3t$ and $y = e^{2t}$. Compute $\frac{df}{dt}$, first by using the Chain Rule, and then by actually performing the substitution to get $f(t) = 9t^2 + e^{4t}$ and taking the derivative. Show how this process is more complicated when using functions of two variables by discussing the function $g(x, y) = x^2 + xy + y^2$ with $x = 3(t + s)$ and $y = e^{2st}$.

GROUP WORK 1: The Mutual Fund

Problem 3 can be done using the Chain Rule or directly.

Answers:

1. $q = 30.048$, $r = 15.052$, $s = 21.3$, $v = 22.06$

2. $$S = \frac{30{,}000a + 20{,}000b + 28{,}000j + 22{,}000l}{500{,}000}$$
$$= \frac{30{,}000\left(\frac{w^2}{10} - 2p + 30.048\right) + 20{,}000\left(\frac{w}{100} - p + 15.052\right) + 28{,}000\,(2w + 21.3) + 22{,}000\,(-p + 22.06)}{500{,}000}$$

3. $$\frac{\partial S}{\partial w} = \frac{\partial S}{\partial a}\frac{\partial a}{\partial w} + \frac{\partial S}{\partial b}\frac{\partial b}{\partial w} + \frac{\partial S}{\partial j}\frac{\partial j}{\partial w} + \frac{\partial S}{\partial l}\frac{\partial l}{\partial w}$$
$$= \frac{1}{500{,}000}\left[30{,}000\,(0.17) + 20{,}000\,(0.01) + 28{,}000\,(2) + 22{,}000\,(0)\right] = 0.1226$$

4. $$\frac{\partial S}{\partial p} = \frac{\partial S}{\partial a}\frac{\partial a}{\partial p} + \frac{\partial S}{\partial b}\frac{\partial b}{\partial p} + \frac{\partial S}{\partial j}\frac{\partial j}{\partial p} + \frac{\partial S}{\partial l}\frac{\partial l}{\partial p}$$
$$= \frac{1}{500{,}000}\left[30{,}000\,(-2) + 20{,}000\,(-1) + 28{,}000\,(0) + 22{,}000\,(-1)\right] = -0.204$$

GROUP WORK 2: Chemistry 101

In case the students ask, there is no such thing as a "millikent"!

Answers:

1. $\dfrac{\partial P}{\partial t} = \left(\dfrac{500R}{V}\right)\dfrac{1}{(t+1)^2} - \dfrac{60RTs}{V^2}\left(e^{-3ts}\right)$

2. $\dfrac{\partial P}{\partial s} = -\dfrac{60RTt}{V^2}\left(e^{-3ts}\right)$

3. $P = 64$ atmospheres

4. 66.294, 0

HOMEWORK PROBLEMS

Core Exercises: 1, 5, 11, 15, 17, 23, 30, 35, 42

Sample Assignment: 1, 2, 5, 9, 11, 14, 15, 17, 18, 21, 23, 28, 30, 34, 35, 37, 39, 42, 45, 53

Exercise	D	A	N	G
1		×		
2		×		
5		×		
9		×		
11		×		
14		×		
15		×	×	
17		×		
18		×		
21		×		

Exercise	D	A	N	G
23		×		
28		×		
30		×		
34		×		
35	×	×		
37	×	×		×
39	×	×		
42	×	×		
45		×		
53		×		

GROUP WORK 1, SECTION 15.5

The Mutual Fund

One of the hottest investments on Wall Street today is the Share-All Mutual Fund. The Share-All Fund has issued 500,000 shares for eager investors to buy. Each share, therefore, represents $\frac{1}{500,000}$ of the fund's total net asset value. The fund owns 100,000 shares of stock in four companies, as described below, on a given day.

Company Name	Current price/share	Number of Shares Owned by Share-All	Total $ value
Allied Oil	30	30,000	900,000
Beck Keyboard Manufacturing	15	20,000	300,000
Jasmine Tea	23	28,000	644,000
Lapland Importing-Exporting	22	22,000	484,000
Total asset value			2,328,000

So, on this day, the total asset value is $2,328,000, and the price of one share of Share-All is $\frac{2,328,000}{500,000} = \$4.656$.

Many factors affect the price of a stock. For example, the worldwide exchange rate* w affects the Lapland Importing-Exporting company much more than it does the primarily domestic Beck Keyboard Manufacturing company. Similarly, the United States prime lending rate p affects the highly indebted Allied Oil company more than it affects the relatively debt-free Jasmine Tea company. Thus, we can develop models for the price of these stocks as functions of the world-wide exchange rate, the prime lending rate, and other economic factors q, r, s, and v, which are independent of these variables.

Let w be the world-wide exchange rate, and p be the U.S. prime lending rate. Let $a(w, p, q)$, $b(w, p, r)$, $j(w, p, s)$, and $l(w, p, v)$ be the current price per share of Allied, Beck, Jasmine and Lapland respectively. We have

$$\begin{aligned} a(w,p,q) &= 0.1w^2 - 2p + q \\ b(w,p,r) &= 0.01w - p + r \\ j(w,p,s) &= 2w + s \\ l(w,p,v) &= -p + v \end{aligned}$$

where q, r, s and v are composite variables representing the myriad other factors that affect the price of these stocks, and are independent of w and p. (*Note:* If we knew q, r, s, and v exactly, then we would be able to predict the price of the stock with unrealistic accuracy.)

* The worldwide exchange rate measures the value of the dollar measured versus a weighted average of other relevant currencies. In other words, it is a statistic which we made up, but which could probably fool some people.

1. If the current worldwide exchange rate is 0.85, and the current prime lending rate is 0.06, what are the current values of q, r, s, and v?

2. Assume that the values of q, r, s, and v are fixed at the quantities computed in Problem 1. Let S be the price of a share of Share-All. Write S as a function of w and p.

3. If S is the current price of a share of Share-All, what is $\dfrac{\partial S}{\partial w}$?

4. What is $\dfrac{\partial S}{\partial p}$?

GROUP WORK 2, SECTION 15.5
Chemistry 101

Given n moles of gas, the relationship between pressure P, temperature T, and volume V can be approximated by the formula

$$PV = nRT$$

where P is in atmospheres, V is in Liters, T is in degrees Kelvin (degrees Kelvin = degrees Celsius $+273.15$), and R is the ideal gas constant $[0.08206\ \mathrm{L} \cdot \mathrm{atm}/\ (\mathrm{mol} \cdot \mathrm{K})]$

Assume we have 10 moles of gas in a balloon-type bladder. Initially we have a volume of 1 liter at "STP" ($T = 273.15$, $P = 1$). As time goes on, the gas is heated. The following expresses the temperature T of the gas as a function of the time elapsed t since the beginning of the experiment:

$$T = 323.15 - \frac{50}{t+1}$$

The bladder begins to expand over time as a function also of the strength s of its material, with the following formula describing how the volume V of the gas which can occupy the bladder changes as a function of the number of minutes t and the material strength s (where s is measured in millikents.)

$$V = 2\left(2 - e^{-3ts}\right)$$

1. Describe how the pressure of the gas in the bladder changes as a function of time.

2. Describe how the pressure of the gas in the bladder changes as a function of the strength of the bladder.

3. If the experiment takes place in a one-millikent bladder, what is the pressure of the gas in the box after 4 minutes?

4. If the experiment is allowed to run for a very long time, what value will P approach? What value will $\dfrac{dP}{dt}$ approach?

15.6 DIRECTIONAL DERIVATIVES AND THE GRADIENT VECTOR

TRANSPARENCY AVAILABLE

#45 (Figures 3 and 5)

SUGGESTED TIME AND EMPHASIS

$1\text{–}1\frac{1}{2}$ classes   Essential material

POINTS TO STRESS

1. The computation and geometric meaning of a directional derivative.

2. The computation of a gradient vector.

3. The geometric meanings of a gradient vector: A normal vector to a surface, the direction of greatest change, a perpendicular vector to contour curves and surfaces, a vector with length equal to the maximum value of the directional derivative.

4. The relationships between tangent planes, gradient vectors, and directional derivatives.

QUIZ QUESTIONS

- **Text Question:** The text shows that $D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$, where $\mathbf{u}$ is a unit vector Why does this express the directional derivative in the direction of $\mathbf{u}$ as the scalar projection of the gradient vector onto $\mathbf{u}$?
  Answer: This follows directly from the scalar projection formula.
- **Drill Question:** If $f(x, y) = xy^2 + x$, what is $\nabla f(x, y)$?
  Answer: $\langle y^2 + 1, 2xy \rangle$

MATERIALS FOR LECTURE

- Show how the gradient, while relatively simple to compute, yields a treasure trove of useful information.
- Define S to be a level surface of the function $f(x, y, z)$. Explain why $\nabla f(x_0, y_0, z_0)$ is orthogonal to the surface at any point $P(x_0, y_0, z_0)$ on S. Define the tangent plane to S at P to be the plane with normal vector $\nabla f(x_0, y_0, z_0)$.
- Review that the direction of any vector $\mathbf{v} \neq \mathbf{0}$ is determined by the unit vector $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \langle \cos\theta, \sin\theta \rangle$ where θ is the angle that $\mathbf{v}$ makes with the positive x-axis. Using this interpretation, the directional derivative formula can be rewritten as

$$D_u f = \frac{\partial f}{\partial x} \cos\theta + \frac{\partial f}{\partial y} \sin\theta$$

- Let $f(x, y) = x^2y^2$. Compute $D_{\mathbf{u}} f(x, y)$ for unit vectors $\mathbf{u}$ making angles of 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$ with the positive x-axis, and fill in the following table. Point out that the coefficient of $2xy^2$ decreases from 1 to 0 while the coefficient of $2x^2y$ increases from 0 to 1. Have the students reason intuitively why this should be the case, just using the concept of "directional derivative". Have them figure out (intuitively) $D_{\mathbf{u}} f(x, y)$

for angles π and $\frac{3\pi}{2}$.

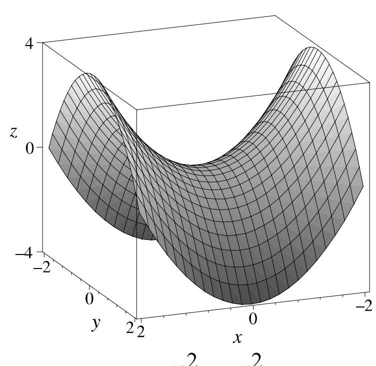
Angle	$D_u f(x, y)$
0	$(1)\,2xy^2 + (0)\,2x^2y$
$\frac{\pi}{6}$	$(0.866\ldots)\,2xy^2 + (0.5)\,2x^2y$
$\frac{\pi}{4}$	$(0.707\ldots)\,2xy^2 + (0.707\ldots)\,2x^2y$
$\frac{\pi}{3}$	$(0.5)\,2xy^2 + (0.866...)\,2x^2y$
$\frac{\pi}{2}$	$(0)\,2xy^2 + (1)\,2x^2y$

Note: At a given point (x, y), one cannot maximize D_u merely by looking at the chart.

- Show how Equation 2 in the Section 15.4 is really just a special case of Equation 19 in this section.

WORKSHOP/DISCUSSION

- Consider the surface $f(x, y) = xy$ at the point $(0, 0)$. Note that although the maximum rate of change is zero at that point, it is not the case that the function is identically zero near the origin. Thus, if $\nabla f(a, b) = \mathbf{0}$, we cannot talk about the direction of maximal change at (a, b).
- Have the students practice finding the directional derivatives of $f = x^2y$ and $f = e^{xy}$ in the directions $-\mathbf{i}$, $\mathbf{i} + \mathbf{j}$, $-\mathbf{i} - \mathbf{j}$, and $\mathbf{i} - \mathbf{j}$. Also have them find the directional derivatives of $f(x, y, z) = z^2e^{xy^2}$ in the directions of $\langle -1, -1, -1 \rangle$ and $\langle 0, 0, -1 \rangle$.
- Show that the gradient vector $\nabla f(x_0, y_0)$ is normal to the line tangent to the level curve $k = f(x, y)$ at the point (x_0, y_0). Look at the example $f(x, y) = 5x^4 + 4xy + 3y^2$ and show that $\nabla f(-1, -1) = \langle -24, -10 \rangle$. Conclude that we now know that f is decreasing in both the x- and y-directions and that the direction of maximal increase is $\langle -24, -10 \rangle$. Ask the students to resolve these seemingly contradictory observations: That the gradient is supposed to point in the direction of maximal *increase*, yet the components $f_x(-1, -1) = -24$ and $f_y(-1, -1) = -10$ of the gradient are pointing in the direction of decreasing x and y. (Although no real paradox exists, students are often confused by this type of situation.)
- Analyze Figure 13. Ask why the gradients near the y-axis point toward the vicinity of the origin and downhill (have negative z-coordinate) while those near the x-axis are pointing uphill, as the text claims. Show how the shape of $z = x^2 - y^2$ reflects this behavior.



$z = x^2 - y^2$

GROUP WORK 1: Two Ways

This exercise ties together many of the key concepts from this section. Therefore, closure is particularly important here.

Answers:

1. 5 **2.** $7\sqrt{2}$ **3.** $6\cos\theta + 8\sin\theta$

4. 10 (They should find this by optimizing the single-variable function $f(\theta) = 6\cos\theta + 8\sin\theta$.)

5. $\theta = 0.9273$ radians **6.** $\langle 0.6, 0.8\rangle$ **7.** 10, $\langle 0.6, 0.8\rangle$

GROUP WORK 2: Computation Practice

It is a good idea to give the students a chance for guided practice using the types of computations that will be required on the homework. We recommend having them do either Problem 1 or Problem 2 in groups, and then handing the remaining problem out as a worksheet.

Answers:

1. (a) $e - 2$ (b) e (c) $\sqrt{2}(e-1)$ (d) $-\dfrac{\ln 2}{\sqrt{3}}$ (e) $-\frac{\ln 2}{\sqrt{6}}$

2. (a) Maximum: $\langle e-2, e\rangle$, minimum: $\langle 2-e, -e\rangle$ (b) Maximum: $\langle 0, 0, \ln 2\rangle$, minimum: $\langle 0, 0, -\ln 2\rangle$

GROUP WORK 3: Bowling Balls and Russian Weebles

This exercise may seem trivial, but it is a good setup for discussions of Lagrange multipliers. If the students do this exercise, ask them to remember the result, and make sure to remind them of the bowling balls and Russian weebles when discussing Lagrange multipliers.

Answers:

1. Parallel, opposite directions. Same tangent plane. No difference.

2. Parallel, same direction. Same tangent plane **3.** Parallel, same tangent plane

HOMEWORK PROBLEMS

Core Exercises: 1, 6, 8, 11, 23, 30, 36

Sample Assignment: 1, 5, 6, 8, 11, 12, 15, 19, 23, 25, 28, 30, 34, 36, 38, 40, 43, 47, 52, 62

Exercise	D	A	N	G
1		×		×
5		×		
6		×		
8		×		
11		×		
12		×		
15		×		
19		×		
23		×		
25		×		

Exercise	D	A	N	G
28		×		
30	×	×		
34	×	×		
36				×
38	×			×
40		×		
43		×		
47		×		×
52		×		
62	×	×		×

GROUP WORK 1, SECTION 15.6
Two Ways

Consider the function $f(x, y) = x^2 + 4xy^2$.

1. What is $f(1, 1)$?

2. What is the directional derivative $D_u(1, 1)$ if $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$?

3. What is the directional derivative $D_u(1, 1)$ if $\mathbf{u}$ is the unit vector that makes an angle $\boldsymbol{\theta}$ with the positive x-axis?

4. In Problem 3, you expressed $D_u(1, 1)$ as a function of the angle $\boldsymbol{\theta}$. Let's say we want to find the maximum value of the directional derivative. This is now a single-variable calculus problem! Use your single-variable calculus techniques, coupled with your answer to Problem 3, to find the find the maximum value of the directional derivative.

5. What is the angle $\boldsymbol{\theta}$ for which f increases the fastest? (You should be able to use your computations for Problem 4 to answer this one quickly.)

6. What is the unit vector that makes that angle $\boldsymbol{\theta}$ with the positive x-axis?

7. Now compute $|\nabla f|$ and $\dfrac{\nabla f}{|\nabla f|}$, but before you do so, discuss with your group members what the answers should be. You should be able to anticipate the correct answers.

GROUP WORK 2, SECTION 15.6
Computation Practice

1. Find the directional derivative of the function at the given point in the direction of the given vector $\mathbf{v}$.

(a) $f(x, y) = e^{xy} - x^2$, $(1, 1)$, $\mathbf{v} = \langle 1, 0 \rangle$

(b) $f(x, y) = e^{xy} - x^2$, $(1, 1)$, $\mathbf{v} = \langle 0, 1 \rangle$

(c) $f(x, y) = e^{xy} - x^2$, $(1, 1)$, $\mathbf{v} = \langle 1, 1 \rangle$

(d) $f(x, y, z) = z \ln\left(x^2 + y^2\right)$, $(-1, 1, 0)$, $\mathbf{v} = \langle 1, 1, -1 \rangle$

(e) $f(x, y, z) = z \ln\left(x^2 + y^2\right)$, $(-1, 1, 0)$, $\mathbf{v} = \langle 2, 1, 1 \rangle$

2. Find the maximum and minimum rates of change of f at the given point and the directions in which they occur.

(a) $f(x, y) = e^{xy} - x^2$, $(1, 1)$

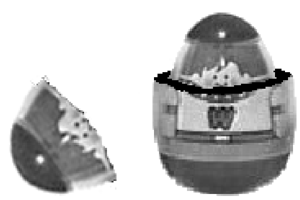
(b) $f(x, y, z) = z \ln\left(x^2 + y^2\right)$, $(-1, 1, 0)$

GROUP WORK 3, SECTION 15.6
Bowling Balls and Russian Weebles

1. Assume that there are two bowling balls in a ball-return machine. They are touching each other. At the point at which they touch, what can you say about their respective normal vectors? What about their tangent planes? What would happen if the bowling balls were of different sizes?

2. A *weeble* is a doll that is roughly egg-shaped. It is an ideal toy for little children, because weebles wobble but they don't fall down.

A *Russian weeble* is a hollow weeble, with one or more weebles inside it. Picture two nested hollow eggs as shown.



At the point at which two nested Russian weebles touch each other, what can you say about their respective normal vectors? What about their tangent planes?

3. Now picture your two favorite differentiable surfaces that touch at exactly one point. What can you say about their normal vectors at the point where they touch? What about their tangent planes?

TRANSPARENCY AVAILABLE

#46 (Figures 7–9)

SUGGESTED TIME AND EMPHASIS

1 class   Essential material

POINTS TO STRESS

1. The contrast between optimization problems in single-variable calculus (relatively few cases) and in multivariable calculus (many possible solutions)
2. Critical points and local maxima and minima
3. The Second Derivative Test.
4. Absolute maxima and minima

QUIZ QUESTIONS

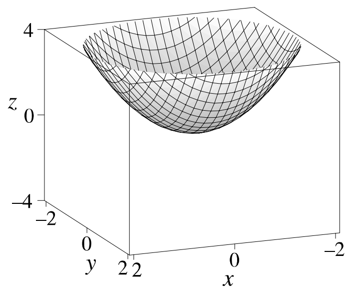
- **Text Question:** Can a differentiable function f have a local maximum at a point (a, b) with $f_x(a, b) = 3$?
  Answer: No
- **Drill Question:** Can you give an example of a function f with the property that $f_x(a, b) = 0$, $f_y(a, b) = 0$, and f does *not* have a local maximum or minimum at (a, b)?
  Answer: $f(x, y) = xy$ at $(0, 0)$.

MATERIALS FOR LECTURE

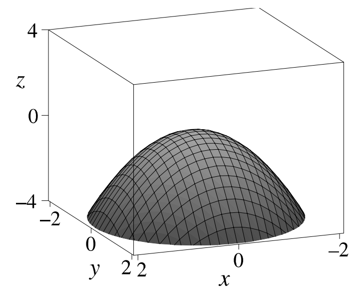
A good way to introduce this topic may be to have the students do Group Work 1: Foreshadowing Critical Points and Extrema.

- Stress geometric interpretations: If f is differentiable at a local maximum or minimum, then the tangent plane must be horizontal. Note that there are critical points at which there is no local maximum or minimum. For example, examine the saddle points at the origin for $f(x, y) = xy$ and $g(x, y) = x^2 - y^2$.

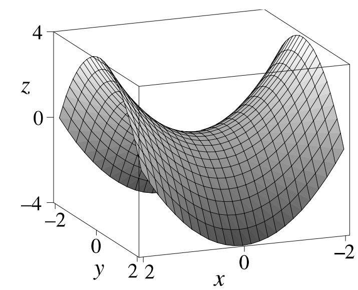
- Illustrate the idea behind the Second Derivatives Test using $f(x, y) = \frac{1}{2}\left(ax^2 + by^2\right)$. Note that $D = ab$ means that
  - $D > 0$, $a > 0$ gives $b > 0$ and hence $f(x, y)$ has a local minimum at $(0, 0)$ [See Picture (a)]
  - $D > 0$, $a < 0$ gives $b < 0$ and hence $f(x, y)$ has a local maximum at $(0, 0)$ [See Picture (b)]
  - $D < 0$, $a > 0$ gives $b < 0$ and hence $f(x, y)$ has a saddle point at $(0, 0)$ [See Picture (c)]



(a) Minimum



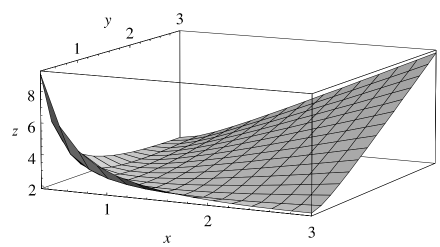
(b) Maximum



(c) Saddle Point

If there is time, discuss the ideas of the proof given in the text.

- Describe some of the ways saddle points can occur for functions of two variables. (For example, see Figures 3, 4, 7, and 8.) Contrast with the single-variable case, where there are fewer possibilities.
- Discuss local and absolute maxima and minima for $f(x, y) = xy + 1/(xy)$.



WORKSHOP/DISCUSSION

- Use $f(x, y) = x^4 + y^4$, $g(x, y) = x^4 - y^4$, $h(x, y) = -\left(x^4 + y^4\right)$ to show that no information is given about local extrema when $D = 0$.
- Consider the problem of finding the distance between a point and a plane. Contrast this chapter's approach to the method used in Example 8 in Section 13.5.
- Pose the problem of finding the maximum of $f(x, y) = ax + by + c$ on the set of points $x^2 + y^2 \leq 4$. Note that the gradient of f is never zero, so the maximum and minimum values must occur on the boundary. One way to find these maximum and minimum values is by parametrizing the boundary $x^2 + y^2 = 4$ by $\mathbf{r}(\theta) = \langle 2\cos\theta, 2\sin\theta \rangle$, where θ is the angle made by the position vector with the x-axis, and then optimizing the function $g(\theta) = f(\mathbf{r}(\theta))$.
- Illustrate that it is often easier to optimize $f^n(x, y)$ instead of $f(x, y)$ for a function f that is always positive. Point out that f^n has the same maxima as f for any n. One good example to use is $f(x, y) = \left[(x-1)^2 + (y+1)^2 + 1\right]^{1/3}$. Another example is the problem of finding the point on the surface $z^2 = xy + x^2 + 1$ that is closest to the origin. [Answer: $\left(\frac{1}{7}, -\frac{4}{7}, \frac{3\sqrt{2}}{7}\right)$]

GROUP WORK 1: Foreshadowing Critical Points and Extrema

This group work is best done just before this section is covered. First present the single-variable definitions of local and global maximum and minimum. (This was done in single-variable calculus, but the students have probably forgotten the technical definitions by this point.) Then put the students into groups and ask them to come up with good multivariable definitions of the same concepts. They should present their definitions and discuss them. At the end of the activity, look up the definition presented in the text, and compare it with the student definitions.

If there is time, do a similar activity for the various types of critical points. Graph $y = x^2$, $y = -x^2$, $y = x^3$, $y = -x^3$, $y = |x|$, and $y = -|x|$ on the board to show different types of critical values at $x = 0$. Then have the students try to come up with the variety of types that can occur for functions of two variables.

GROUP WORK 2: The Squares Conjecture

Note that calculus is not needed to solve this problem; students should be able to get the answer intuitively. They can be asked to use the techniques of this section to verify that their intuition was true.

Answer: $x = y = z = \sqrt[3]{100}$ (other answers are possible)

GROUP WORK 3: Strange Critical Points

In this case, f_x and f_y do not exist at the critical point $(1, -1)$ and so the students cannot use the Second Derivative Test. Acceptable answers include graphing the surface or recognizing that it is an elliptic cone.

Answers: 1. $(1, -1)$ **2.** The absolute minimum is 2, and it occurs at $(1, -1)$.

EXTENDED LABORATORY PROJECT: The Genetic Algorithm

The use of "genetic algorithms" for finding maxima and minima for functions of several variables has become popular in recent years. Usually this technique is used to optimize functions of hundreds of variables, but we'll look at the simpler case of functions of two variables.

Although we don't intend to give a complete description of how genetic algorithms work, an outline is as follows:
Suppose you want to maximize a function of several variables. Start by selecting several arbitrary points (at random or otherwise) from your domain. Select two points among these which give the two largest values of your function. Now choose several more arbitrary points close to these selected points. Continue to repeat this process until you have what seems to be a maximum value.

We will study this process for the complicated function $100e^{-(|x|+1)(|y+1|+1)}\dfrac{\sin(y\sin x)}{1+x^2y^2}$. Let D be the square $[-3,3]\times[-3,3]$.

(i) Use your computer program to select 5 points at random in this square and then evaluate the function at these 5 points.

(ii) Select the points which give the two largest values for $f(x,y)$ and then select 4 points at random close to each of these points. Again, selecting the points at random near these points isn't so trivial. Evaluate the function at the 10 points you now have. Select the two points among these which give the largest value for $f(x,y)$. Repeat (b) until it appears that you have a maximum.

(iii) Is the value you found in (ii) likely to be an absolute maximum?

HOMEWORK PROBLEMS

Core Exercises: 3, 6, 13, 21, 39, 44, 50

Sample Assignment: 1, 3, 6, 8, 13, 18, 19, 21, 23, 26, 31, 35, 39, 41, 44, 48, 50, 53

Exercise	D	A	N	G
1	×			
3		×		×
6		×		×
8		×		×
13		×		×
18		×		×
19		×		
21		×		×
23		×		×
26		×		×
31		×		
35		×		
39		×		
41		×		
44		×		
48		×		
50	×	×		
53		×		

GROUP WORK 2, SECTION 15.7
The Squares Conjecture

You are given a government grant to prove or disprove the Squares Conjecture:

> There exist three positive numbers, r, s, t whose product is 100, yet have the property that the sum of their squares is less than 65.

Either find three such numbers, or show that none exist.

GROUP WORK 3, SECTION 15.7

Strange Critical Points

Let $f(x, y) = 2 + \sqrt{3(x-1)^2 + 4(y+1)^2}$.

1. Find the critical points of f.

2. Find the local and absolute minimum values of f. Where do these values occur?

APPLIED PROJECT Designing a Dumpster

This project requires the students to solve an extended real-world problem that involves them going out and measuring a nearby dumpster. They will have to make approximations, and figure out how best to get an accurate answer. A good sample answer is given in the *Complete Solutions Manual*.

The project can be made more applied if the students are instructed to actually research the costs involved, instead of using the ones provided in the problem statement.

DISCOVERY PROJECT Quadratic Approximations and Critical Points

Problems 1–3 serve as a good introduction to Taylor's Theorem for two variables, and to quadratic polynomial approximation in two variables. Problems 4 and 5 justify the Second Derivative Test, the proof of which is given in Appendix F.

15.8 LAGRANGE MULTIPLIERS

TRANSPARENCY AVAILABLE

#47 (Figures 2 and 3)

SUGGESTED TIME AND EMPHASIS

1 class Essential material

POINTS TO STRESS

1. The geometric justification for the method of Lagrange multipliers

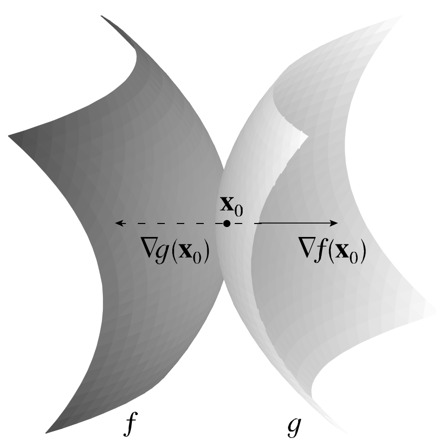
2. How to apply the method of Lagrange multipliers, including the extension of the method for two-constraint problems

QUIZ QUESTIONS

- **Text Question:** How does the equation $\nabla f(x, y) = \lambda \nabla g(x, y)$, subject to the constraint $g(x, y) = k$, lead to three equations with three unknowns? What are the unknowns?
  Answer: The constraint gives one equation, and the gradient (consisting of two dimensions) gives the other two. x, y, and λ are the unknowns.

MATERIALS FOR LECTURE

- Draw a picture like the one below illustrating that if two surfaces are tangent, they have parallel normals at the point of tangency.



  If Group Work 3: Bowling Balls and Russian Weebles was covered in Section 15.6, this is a good time to remind them of the lesson of the Russian weebles.

- Make sure that students understand the actual "nuts and bolts" of the one-constraint method.

- Give an example to show that, with functions of two variables, there are often alternate methods other than Lagrange multipliers to solve optimization problems. Perhaps redo Example 2, substituting $x^2 = 1 - y^2$ into $f(x, y) = x^2 + 2y^2$ to get the single-variable problem $g(y) = 1 + y^2$, minimize to get $y = 0$ (with $x = \pm 1$), and then get $h(x) = 2 - x^2$ with maximum $x = 0$, $y = \pm 1$. Perhaps also note that, for the two-variable case, $\nabla f = \lambda \nabla g$ implies that $\nabla f \times \nabla g = \mathbf{0}$. This condition can sometimes be used to replace Lagrange multipliers.

WORKSHOP/DISCUSSION

- Find the volume of the largest rectangular solid that can be inscribed in a sphere, that is, maximize $V(x, y, z) = (2x)(2y)(2z)$ given that $x^2 + y^2 + z^2 = a^2$.
- Discuss the geometric solution to Example 4. What basic geometric principle is being used?

GROUP WORK 1: The Inscribed Rectangle Race

Divide the class in half. Write the following problem on the board: "What is the area of the largest rectangle that can be inscribed in a circle of radius 4?" Have one half of the class try to solve this problem using Lagrange multipliers, and the other half try to use single-variable calculus. See which side finishes first, and which side found the problem more difficult. At the end, the students should see both methods presented.

If a group finishes early, or after all groups have presented, have the students further practice the two techniques by maximizing xy^2 on the ellipse $\frac{1}{9}x^2 + \frac{1}{4}y^2 = 1$.

Answers: 4, $\frac{8\sqrt{3}}{3}$

GROUP WORK 2: Biggest and Smallest on Closed and Bounded Sets

This activity involves finding the absolute maximum and minimum values of a function of several variables on a closed and bounded set. Review the necessary steps outlined in Section 15.7.

Answers: 2, -1

GROUP WORK 3: The Heated Cannonball

This problem appears to be quite difficult at first reading, but letting x, y, and z be the angles (in radians) and using Lagrange multipliers leads to a very easy solution.

Answers:

1. The minimum is -60° at the points $(\pm 1, 0, 0)$. The maximum is 60° wherever $y^2 + z^2 = 1$.
2. It is a circular frame in the yz-plane.

GROUP WORK 4: The Sum of the Sines

Answers: 1. Equilateral **2.** $\frac{3\sqrt{3}}{2}$ **3.** 45°-45°-90°, $1+\sqrt{2}$

HOMEWORK PROBLEMS

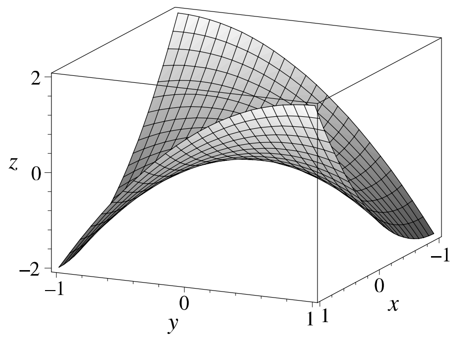
Core Exercises: 1, 3, 10, 18, 20, 25, 36

Sample Assignment: 1, 3, 6, 10, 11, 18, 20, 25, 27, 32, 36, 38, 43

Exercise	D	A	N	G
1	×			×
3		×		
6		×		
10		×		
11		×		
18		×		
20	×	×		×
25		×		
27		×		
32		×		
36		×		
38		×		
43		×		×

GROUP WORK 2, SECTION 15.8
Biggest and Smallest on Closed and Bounded Sets

Let $f(x, y) = x^2 - y^2 + 2xy$.



What are the absolute maximum and absolute minimum values of this function on the unit square $\{0 \le x \le 1, 0 \le y \le 1\}$?

GROUP WORK 3, SECTION 15.8
The Heated Cannonball

One of the wonderful things about the British army in the eighteenth century was that they were very polite. For example, during the Revolutionary War, during the battle of Valley Forge, it was standard practice for them to gently warm their cannonballs before firing them at the colonists. Suppose that a particular cannonball with radius 1 foot has a temperature distribution $T(x, y, z) = 60\left(y^2 + z^2 - x^2\right)$ (where the center of the cannonball is at the origin).

1. What are the maximum and minimum temperatures in the cannonball, and where do they occur?

2. What is the shape of the wire frame used to apply the heat to the surface of the cannonball?

GROUP WORK 4, SECTION 15.8
The Sum of the Sines

1. What is the description of the triangle for which the sum of the sines of its angles is a maximum?

2. Find the maximal value of the sums of the sines of the angles of a triangle.

3. Repeat Problems 1 and 2 if we now assume that the triangle is a right triangle.

APPLIED PROJECT Rocket Science

This is an excellent example of Lagrange multipliers presented in a realistic setting. If not assigned as a project, it can be given as a supplementary reading. The computations required for this problem are extensive. A CAS might help, but is not required.

APPLIED PROJECT Hydro-Turbine Optimization

This problem has not been simplified. The Great Northern Paper Company is a real company that has hired people to solve the same problem that the students are faced with. If this project is assigned, the students should be informed that they have the opportunity to solve a real engineering problem. You can specify that the final report be written up in a professional manner so that the students can show the report to prospective employers.

Problems marked with an asterisk (*) are particularly challenging and should be given careful consideration.

1. (a) Consider the function $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$. Find equations for the following level surfaces for f, and sketch them.

(i) $f(x, y) = \frac{1}{5}$

(ii) $f(x, y) = \frac{1}{10}$

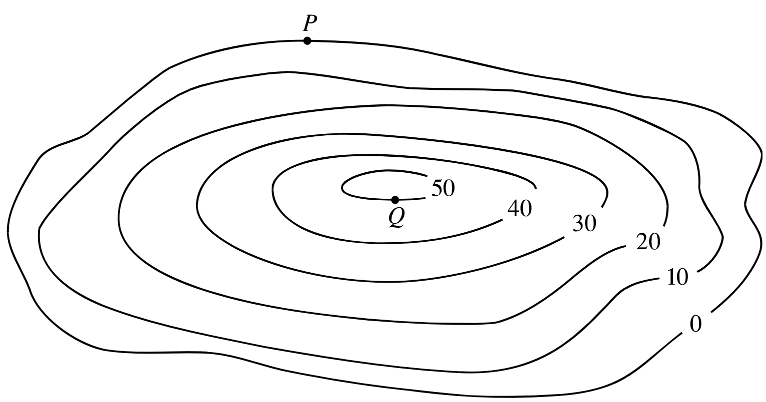
(b) Find k such that the level surface $f(x, y) = k$ consists of a single point.

(c) Why is k the global maximum of $f(x, y)$?

2. Is the function $f(x, y) = \sin^2\left(xy^2\right)$ a solution to the partial differential equation $\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y} = (2x + y)(2y)\cos\left(xy^2\right)\sqrt{f}$ when $\sin\left(xy^2\right) \geq 0$?

3. Is it possible to find a function for which it is true that, for all $x > 0$ and $y > 0$, $f_x > 0$ and $f_y < 0$, and $f(x, y) > 0$? If so, give an example. If not, why not?

4.



The above is a topographical map of a hill.

(a) Starting at P, sketch the path of steepest ascent to the peak elevation of 50 yards.

(b) Suppose it rains, and water runs down the hill starting at Q. At what point would you expect the water to reach the bottom? Justify your answer.

5. Find the absolute maximum and minimum of $f(x, y) = x^2 + xy + y^2$ on the disk $\left\{(x, y) \mid x^2 + y^2 \leq 9\right\}$.

6. Consider the ellipsoid $\dfrac{x^2}{4} + 2z^2 + \dfrac{y^2}{4} = 1$. Using geometric reasoning or otherwise, find the equation of the tangent plane at

(a) $\left(\sqrt{2}, \sqrt{2}, 0\right)$.

(b) $\left(0, 0, \frac{1}{\sqrt{2}}\right)$.

7. Describe the level surfaces $f(x, y, z) = k$ for the function $f(x, y, z) = 1 - x^2 - \dfrac{y^2}{2} - \dfrac{z^2}{3}$ and the values $k = -1$, $k = 1$, and $k = 2$.

8. Suppose that the amount of energy $F(x, y, z)$ emanating from a source at $(0, 0, 0)$ is inversely proportional to one more than the square of the distance from the origin measured only in the the xy-plane, and is directly proportional to the height above the xy-plane. Assume that all of the constants of proportionality are equal to 1.

(a) What is an equation for the energy as a function of x, y, and z?

(b) Where is there no energy at all?

(c) Sketch the level surface $F(x, y, z) = 1$.

9. Consider the function

$$f(x, y) = \frac{x + y}{|x| + |y|}$$

(a) Evaluate the following

(i) $f(1, 1)$

(ii) $f(1, -1)$

(iii) $f(-1, 1)$

(iv) $f(-1, -1)$

(b) Does this function have a limit at $(0, 0)$?

10. Consider the function

$$f(x, y) = \begin{cases} \dfrac{2x^2 + 3y^2}{x - y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

(a) Compute $f_x(0, 0)$ directly from the limit definition of a partial derivative

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

(b) Compute $f_y(0, 0)$.

11. If $f(0, 0) = 0$, $f_x(0, 0) = 0$, $f_y(0, 0) = 0$, and $f(x, y)$ is differentiable at $(0, 0)$, does this imply that $f(x, y) = 0$ for some point $(x, y) \neq (0, 0)$? Justify your result, or give a counterexample.

12. Consider the sphere $x^2 + y^2 + z^2 = 9$. Find the equation of the plane tangent to this sphere at

(a) $(3, 0, 0)$.

(b) $(2, 2, 1)$.

13. Suppose that $f(x, y) = e^{x-y}$ and $f(\ln 2, \ln 2) = 1$. Use the technique of linear approximation to estimate $f(\ln 2 + 0.1, \ln 2 + 0.04)$.

14. Let $g(u)$ be a differentiable function and let $f(x, y) = g\left(x^2 + y^2\right)$.

(a) Show that $yf_x = xf_y$.

(b) Find the direction of maximal increase of f at $(1, 1)$ in terms of g'.

15. Let f be a function of two variables with the following properties:

- $\dfrac{\partial f}{\partial x}$ is defined near $(0, 0)$, continuous at $(0, 0)$ and $\dfrac{\partial f}{\partial x}(0, 0) = 0$
- $\dfrac{\partial f}{\partial y}$ is defined near $(0, 0)$, continuous at $(0, 0)$ and $\dfrac{\partial f}{\partial y}(0, 0) = 0$
- $\dfrac{\partial^2 f}{\partial x \partial y}(0, 0) = 1$
- $\dfrac{\partial^2 f}{\partial y \partial x}(0, 0) = -1$

Answer true or false to the following, and give reasons for your answers.

(a) f is differentiable at $(0, 0)$.

(b) There is a horizontal plane that is tangent to the graph of f at $(0, 0)$.

(c) The functions $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ are both continuous at $(0, 0)$.

(d) The linear approximation to $f(x, y)$ at $(0, 0)$ is $L(x, y) = x - y$.

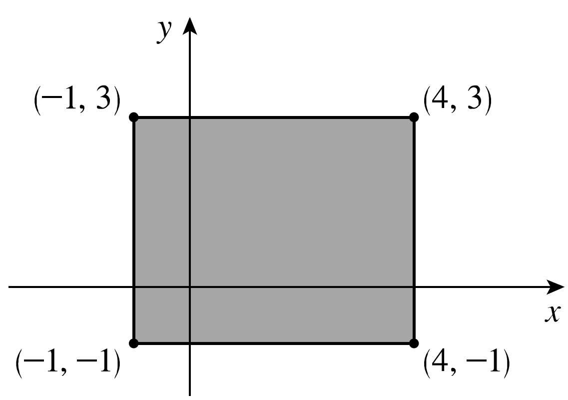
16. Suppose $\mathbf{u} = \langle 1, 0 \rangle$, $\mathbf{v} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, $D_{\mathbf{u}}(f(a, b)) = 3$ and $D_{\mathbf{v}}(f(a, b)) = \sqrt{2}$.

(a) Find $\nabla f(a, b)$.

(b) What is the maximum possible value of $D_{\mathbf{w}}(f(a, b))$ for any $\mathbf{w}$?

(c) Find a unit vector $\mathbf{w} = \langle w_1, w_2 \rangle$ such that $D_{\mathbf{w}}(f(a, b)) = 0$.

17. Let $f(x, y) = e^{-(x^2+y^2)}$. Find the maximum and minimum values of f on the rectangle shown below. Justify your answer.

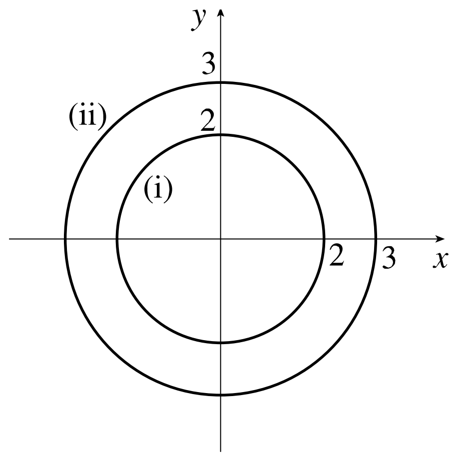


18. Which point on the surface $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1$, $x, y, z > 0$ is closest to the origin?

1. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$

(a) (i) $f(x, y) = \frac{1}{5} \Rightarrow 5 = x^2 + y^2 + 1$ $\Rightarrow x^2 + y^2 = 4$

(ii) $f(x, y) = \frac{1}{10} \Rightarrow x^2 + y^2 = 9$



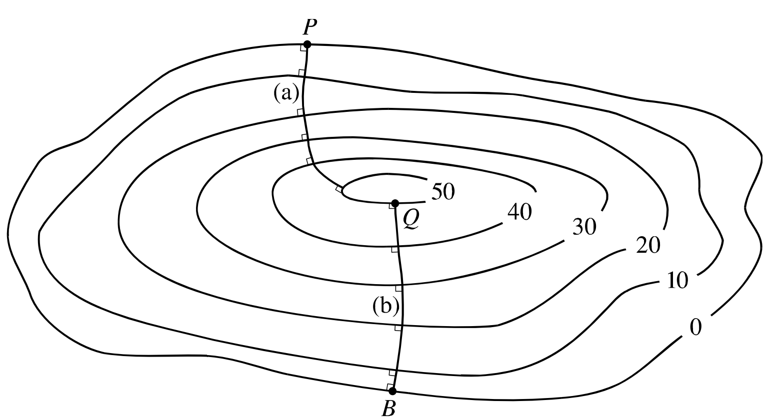
(b) $f(x, y) = 1$ consists of a single point $(0, 0)$. Otherwise, $k < 1$ always gives the circle $x^2 + y^2 = 1 - 1/k$.

(c) $\dfrac{1}{x^2 + y^2 + 1} \leq 1$ for any point (x, y), since $x^2 + y^2 + 1 \geq 1$.

2. Yes. On the left-hand side we get $(2x + y)\, 2y \cos\left(xy^2\right) \sin\left(xy^2\right)$ and on the right-hand side we get $(2x + y)\, 2y \cos\left(xy^2\right) \left|\sin xy^2\right|$, so these are equal for $\sin\left(xy^2\right) \geq 0$.

3. Yes. There are many examples of such functions. One which works for all x and y is $f(x, y) = e^x + e^{-y}$, which has $f_x = e^x$ and $f_y = -e^{-y}$. A good strategy is to write $f(x, y) = g(x) + h(y)$, where $g'(x) > 0$, $h'(y) < 0$.

4.



5. $f(x,y) = x^2 + xy + y^2$ on the disk $\{(x,y) \mid x^2 + y^2 \le 9\}$.

$\nabla f(x,y) = \langle 2x + y, 2y + x \rangle = \langle 0, 0 \rangle \quad \Leftrightarrow \quad y = -2x$ and $x = -2y \quad \Leftrightarrow \quad (x,y) = (0,0)$. So the minimum value on the interior of the disk is $f(0,0) = 0$.

Using Lagrange multipliers for the boundary, we solve $\nabla f = \lambda \nabla g$ where $g(x,y) = x^2 + y^2 = 9$. So $2x + y = \lambda 2x \quad \Rightarrow \quad \lambda = 1 + y/(2x)$ and $2y + x = \lambda 2y \quad \Rightarrow \quad \lambda = 1 + x/2y \quad \Rightarrow \quad x^2 = y^2$. But $x^2 + y^2 = 9$, so $2x^2 = 9 \quad \Rightarrow \quad x = \pm\frac{3}{\sqrt{2}}$ and $y = \pm\frac{3}{\sqrt{2}}$. Thus the maximum value on the boundary is $f\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = f\left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = \frac{27}{2}$ and the minimum value on the boundary is $f\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = f\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = \frac{9}{2}$.

The absolute minimum value is $f(0,0) = 0$ and the absolute maximum value is $f\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = f\left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = \frac{27}{2}$.

6. (a) Let $g(x,y,z) = \dfrac{x^2}{4} + \dfrac{y^2}{4} + 2z^2$, so $\nabla g = \left\langle \frac{x}{2}, \frac{y}{2}, 4z \right\rangle$ and $\nabla g\left(\sqrt{2}, \sqrt{2}, 0\right) = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle$ which is normal to the surface. So the tangent plane satisfies $\dfrac{x}{\sqrt{2}} + \dfrac{y}{\sqrt{2}} = k$ and goes through $\left(\sqrt{2}, \sqrt{2}, 0\right)$.

Thus $k = 1$ and the tangent plane is $\dfrac{x}{\sqrt{2}} + \dfrac{y}{\sqrt{2}} = 1$.

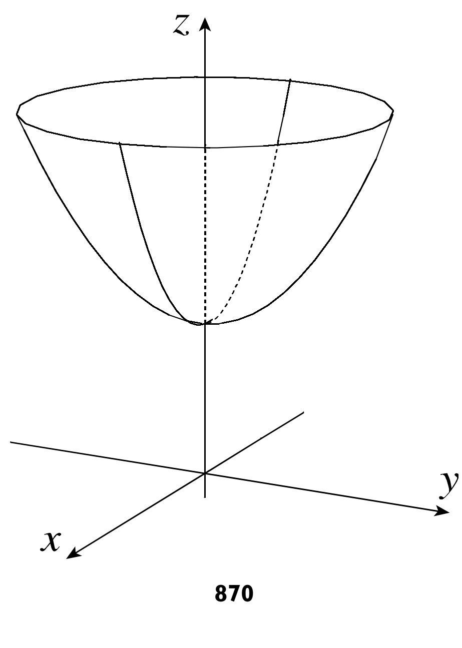
(b) Since this is a maximum value of z, the tangent plane is horizontal, that is, $z = \frac{1}{\sqrt{2}}$. Analytically, $\nabla g\left(0, 0, \frac{1}{\sqrt{2}}\right) = \left\langle 0, 0, 2\sqrt{2} \right\rangle$, so the tangent plane is $2\sqrt{2}z = 2$ or $z = \frac{1}{\sqrt{2}}$.

7. Ellipsoid for $k = -1$, single point $(0,0,0)$ for $k = 1$, no surface for $k = 2$.

8. (a) $F(x,y,z) = \dfrac{z}{1 + x^2 + y^2}$

(b) $z = 0$ is the only place where $F(x,y,z) = 0$. So there is no energy on the xy-plane.

(c) $F(x,y,z) = 1$ gives $1 = \dfrac{z}{1 + x^2 + y^2}$ or $z = 1 + x^2 + y^2$, a circular paraboloid.



9. $f(x,y) = \dfrac{x+y}{|x|+|y|}$

(a) (i) $f(1,1) = 1$

(ii) $f(1,-1) = 0$

(iii) $f(-1,1) = 0$

(iv) $f(-1,-1) = -1$

(b) No, the function does not have a limit at $(0,0)$, since if $y = -x$, then $f(x,-x) = 0$ and if $y = x$, $f(x,x) = \dfrac{x}{|x|} = \pm 1$.

10. $f(x,y) = \begin{cases} \dfrac{2x^2+3y^2}{x-y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

(a) $f_x(0,0) = \displaystyle\lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{2h^2}{h}-0}{h} = \lim_{h\to 0} \frac{2h^2}{h^2} = \lim_{h\to 0} 2 = 2$

(b) $f(0,y) = \dfrac{3y^2}{-y} = 3y = g(y)$. Then $f_y(0,0) = g'(0) = -3$.

11. A counterexample is $f(x,y) = x^2+y^2$. For this function $f_x(0,0) = f_y(0,0) = 0$; $f(0,0) = 0$ and $f(x,y) \neq 0$ for $(x,y) \neq (0,0)$.

12. $x^2+y^2+z^2 = 9$

(a) The tangent plane at $(3,0,0)$ is $x = 3$.

(b) Let $g(x,y,z) = x^2+y^2+z^2$. Then $\nabla g = \langle 2x, 2y, 2z\rangle$ and $\nabla g(2,2,1) = \langle 4,4,2\rangle$, which is normal to the surface. So the tangent plane is $4x+4y+2z = k$ and goes through $(2,2,1)$, so $k = 18$, and the tangent plane is $2x+2y+z = 9$.

13. $f(x,y) = e^{x-y}$, $f_x(x,y) = e^{x-y}$, $f_y(x,y) = -e^{x-y}$.
$L(x,y) = f(\ln 2, \ln 2) + f_x(\ln 2, \ln 2)(x-\ln 2) + f_y(\ln 2,\ln 2)(y-\ln 2)$. So the linear approximation is $f(\ln 2 + 0.1, \ln 2 + 0.04) \approx L(\ln 2 + 0.1, \ln 2 + 0.04) = 1 + 1(0.1) - 1(0.04) = 1.06$.

14. (a) $yf_x = y\left[g'\left(x^2+y^2\right)2x\right] = 2xyg'\left(x^2+y^2\right)$, $xf_y = x\left[g'\left(x^2+y^2\right)2y\right] = 2xyg'\left(x^2+y^2\right)$.

(b) The maximal increase is in the direction of $\mathbf{u} = \left\langle 2g'(2), 2g'(2)\right\rangle$, which is the same as that of $\mathbf{w} = \langle 1,1\rangle$.

15. (a) True; the partials are continuous.

(b) True (in fact the plane is $z = 0$).

(c) False; if they were continuous, then we would have $f_{xy}(0,0) = f_{yx}(0,0)$.

(d) False; the linear approximation is $L(x,y) = 0$.

16. $\mathbf{u} = \langle 1, 0 \rangle$, $\mathbf{v} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, $D_{\mathbf{u}}(f(a,b)) = 3$ and $D_{\mathbf{v}}(f(a,b)) = \sqrt{2}$

(a) $\nabla f(a,b) = \langle f_1, f_2 \rangle$ and $\langle f_1, f_2 \rangle \cdot \mathbf{u} = 3 \quad\Rightarrow\quad f_1 = 3$. $\langle f_1, f_2 \rangle \cdot \mathbf{v} = \sqrt{2} \quad\Rightarrow\quad \dfrac{f_1}{\sqrt{2}} + \dfrac{f_2}{\sqrt{2}} = \sqrt{2}$

$\Rightarrow \quad \dfrac{3}{\sqrt{2}} + \dfrac{f_2}{\sqrt{2}} = \sqrt{2} \quad\Rightarrow\quad 3 + f_2 = 2 \quad\Rightarrow\quad f_2 = -1$. So $\nabla f(a,b) = \langle 3, -1 \rangle$.

(b) $D_{\mathbf{w}}(f(a,b))$ is maximized when $\mathbf{w}$ is in the direction of $\langle 3, -1 \rangle$. So $\mathbf{w} = \left\langle \frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}} \right\rangle$ and since $\mathbf{w} = \frac{4}{\sqrt{10}}\mathbf{u} - \frac{1}{\sqrt{5}}\mathbf{v}$, $D_{\mathbf{w}}(f(a,b)) = \frac{4}{\sqrt{10}} D_{\mathbf{u}}(f(a,b)) - \frac{1}{\sqrt{5}} D_{\mathbf{v}}(f(a,b)) = \frac{4}{\sqrt{10}} \cdot 3 - \frac{1}{\sqrt{5}} \cdot \sqrt{2} = \sqrt{10}$

(c) $D_{\mathbf{w}}(f(a,b)) = 0$ if $\mathbf{w} \cdot \langle 3, -1 \rangle = 0$, so $3w_1 - w_2 = 0$ and $w_1^2 + w_2^2 = 1$ gives $w_1^2 + 9w_1^2 = 1$, $w_1 = \frac{1}{\sqrt{10}}$ and $w_2 = \frac{3}{\sqrt{10}}$, so $\mathbf{w} = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$.

17. Since f is a function which is constant on circles $x^2 + y^2 = R$ and since f is decreasing as the radius of the circle increases, then the maximum is $f(0,0) = 1$ and the minimum is $f(4,3) = e^{-25}$.

18. Let $d^2 = x^2 + y^2 + z^2$ and minimize d^2 subject to the constraint $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1$, $x, y, z > 0$. The method of Lagrange multipliers gives the point $(3, 3, 3)$.