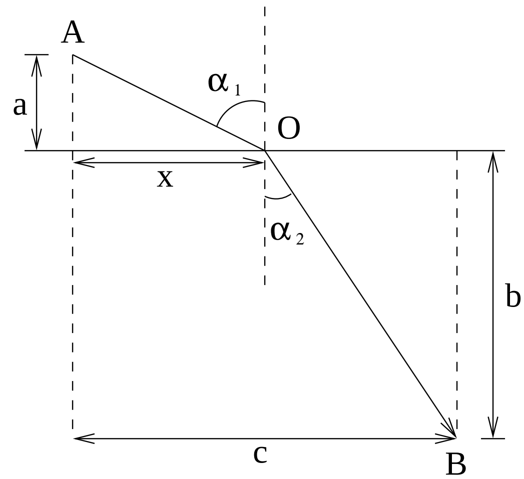
溜滑梯(最速下降曲線)

一個小圓盤, 由A點往下滑到B點, 要以什麼樣的路徑往下滑, 所花的時間最短呢? 假設A, B兩點之間無摩擦力, 如果路徑如下圖:



小球由A到O的速度是V_1, 由O到B的速度是V_2, 則從A到B所花的時間

$$T = \frac{\sqrt{a^2+x^2}}{V_1} + \frac{\sqrt{(c-x)^2+b^2}}{V_2}$$

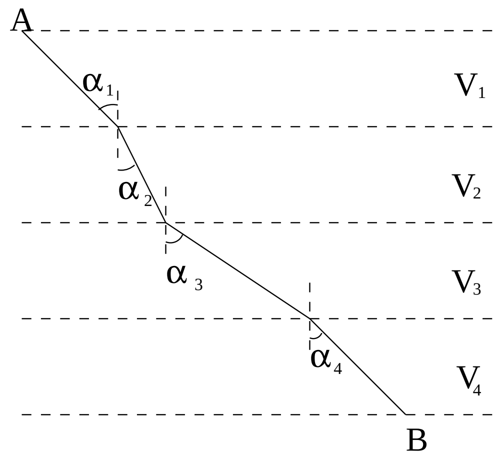
因為要求最短時間, $\therefore \frac{dT}{dx} = 0$

$\frac{dT}{dx} = \frac{x}{\sqrt{a^2+x^2} \cdot V_1} + \frac{-(c-x)}{\sqrt{(c-x)^2+b^2} \cdot V_2} = 0$

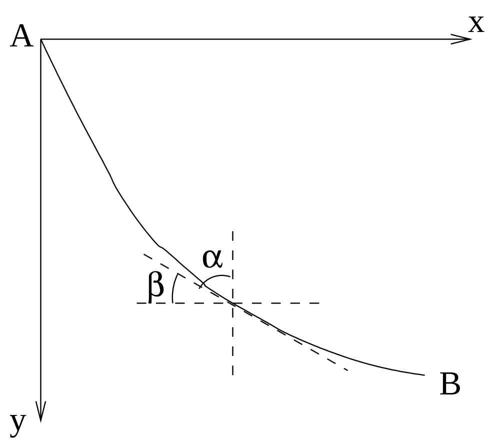
$\therefore \frac{x}{\sqrt{a^2+x^2}} \cdot \frac{1}{V_1} = \frac{c-x}{\sqrt{(c-x)^2+b^2}} \cdot \frac{1}{V_2}$

$\frac{\sin\alpha_1}{V_1} = \frac{\sin\alpha_2}{V_2}$

如果路徑變成如下:



$\because \frac{\sin\alpha_1}{V_1} = \frac{\sin\alpha_2}{V_2}$
$\frac{\sin\alpha_2}{V_2} = \frac{\sin\alpha_3}{V_3}$
$\frac{\sin\alpha_3}{V_3} = \frac{\sin\alpha_4}{V_4}$
$\therefore \frac{\sin\alpha_1}{V_1} = \frac{\sin\alpha_2}{V_2} = \frac{\sin\alpha_3}{V_3} = \frac{\sin\alpha_4}{V_4}$
如果將A,B之間愈切愈細, 則可以發現, 在每一點會有 $\frac{\sin\alpha}{v} = k$, k 是一個常數.

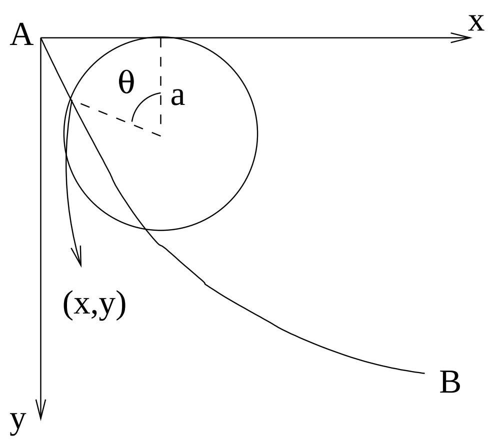


又在每一點的$V = \sqrt{2gy}$, y向下為正.
$\sin\alpha = \cos\beta = \frac{1}{\sec\beta} = \frac{1}{\sqrt{1+\tan^2\beta}} = \frac{1}{\sqrt{1+(\frac{dy}{dx})^2}}$
$k \cdot V = \frac{1}{\sqrt{1+(\frac{dy}{dx})^2}}$
$k^2 \cdot 2gy = \frac{1}{1+(\frac{dy}{dx})^2}$, let $\frac{1}{C} = 2gk^2$

$y \cdot (1 + (\frac{dy}{dx})^2) = C$
$(\frac{dy}{dx})^2 = \frac{c-y}{y}$
現在讓$\frac{dx}{dy} = \tan\phi \therefore (\tan\phi)^2 = \frac{y}{c-y}$
$\therefore y = c\sin^2\phi$
則$dy = 2c\sin\phi\cos\phi d\phi$

$$\begin{aligned} dx &= 2c\sin\phi\cos\phi\tan\phi d\phi \\ &= 2c\sin^2\phi d\phi \\ &= c(1-\cos 2\phi)d\phi \end{aligned}$$

$\therefore x = c(\phi - \frac{1}{2}\sin 2\phi) + c_1$
又有初始條件$x = y = 0$, as $\phi = 0$
$\therefore c_1 = 0$
$\therefore x = c(\phi - \frac{1}{2}\sin 2\phi) = \frac{c}{2}(2\phi - \sin 2\phi)$
$y = c\sin^2\phi = \frac{c}{2}(1-\cos 2\phi)$
Let $a = \frac{c}{2}$, $\theta = 2\phi$ 則$\left\{ \begin{array}{lcl} x &=& a(\theta - \sin\theta)\cdots(1) \\ y &=& a(1-\cos\theta)\cdots(2) \end{array} \right.$



又: a被B點的坐標(x,y)決定. $\theta - \frac{x}{a} = \sin\theta$
$1 - \frac{y}{a} = \cos\theta$
$(\theta - \frac{x}{a})^2 + (1-\frac{y}{a})^2 = 1$
$\theta = \frac{x}{a} + \sqrt{1-(1-\frac{y}{a})^2}$
$y = a\left(1 - \cos\left[\frac{x}{a} + \sqrt{1-(1-\frac{y}{a})^2}\right]\right)$
$\therefore$ 當$A = (0,0)$, 不同的B點位置會決定不同的a值.
註: 這種曲線稱為 **cycloid.**