第二章 極限

[2.6] 切線. 速度與變化率

3. The slope at D is the largest positive slope, followed by the positive slope at E. The slope at C is zero. The slope at B is steeper than at A(both are negative). In decreasing order, we have the slopes at: D, E, C, A, and B.

5.

(a) (i) Using Definition 1,

$$m = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to -3}\frac{f(x)-f(-3)}{x-(-3)} = \lim_{x\to a}\frac{(x^2+2x)-(3)}{x-(-3)}$$
$$= \lim_{x\to -3}\frac{(x+3)(x-1)}{x+3} = \lim_{x\to -3}(x-1) = -4$$

(ii) Using Definition 2,

$$m = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{f(-3+h)-f(-3)}{h}$$
$$= \lim_{h\to 0}\frac{[(-3+h)^2+2(-3+h)]-(3)}{h} = \lim_{h\to 0}\frac{9-6h+h^2-6+2h-3}{h}$$
$$= \lim_{h\to 0}\frac{h(h-4)}{h} = \lim_{h\to 0}(h-4) = -4$$

(b) Using the point-slope form of the equation of a line, an equation of the tangent line is $y-3=-4(x+3)$. Solving for y gives us $y=-4x-9$, which is the slope-intercept form of the equation of the tangent line.

9. Using (1) with $f(x)=\frac{x-1}{x-2}$ and $P(3,2)$,

$$m = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to 3}\frac{\frac{x-1}{x-2}-2}{x-3} = \lim_{x\to 3}\frac{\frac{x-1-2(x-2)}{x-2}}{x-3} = \lim_{x\to 3}\frac{3-x}{(x-2)(x-3)}$$
$$= \lim_{x\to 3}\frac{-1}{x-2} = \frac{-1}{1} = -1.$$

Tangent line: $y-2=-1(x-3) \Leftrightarrow y-2=-x+3 \Leftrightarrow y=-x+5$

11.

(a)

$$m = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{\frac{2}{x+3}-\frac{2}{a+3}}{x-a} = \lim_{x\to a}\frac{2(a+3)-2(x+3)}{(x-a)(x+3)(a+3)}$$
$$= \lim_{x\to a}\frac{2(a-x)}{(x-a)(x+3)(a+3)} = \lim_{x\to a}\frac{-2}{(x+3)(a+3)} = \frac{-2}{(a+3)^2}$$

(b)

(i) $a = -1 \Leftrightarrow m = \frac{-2}{(-1+3)^2} = -\frac{1}{2}$

(ii) $a = 0 \Leftrightarrow m = \frac{-2}{(0+3)^2} = -\frac{2}{9}$

(iii) $a = 1 \Leftrightarrow m = \frac{-2}{(1+3)^2} = -\frac{1}{8}$

13.

(a) Using (1),

$$\begin{aligned} m &= \lim_{x \to a} \frac{(x^3 - 4x + 1) - (a^3 - 4a + 1)}{x - a} = \lim_{x \to a} \frac{(x^3 - a^3) - 4(x - a)}{x - a} \\ &= \lim_{x \to a} \frac{(x - a)(x^2 + ax + a^2) - 4(x - a)}{x - a} = \lim_{x \to a} (x^2 + ax + a^2 - 4) = 3a^2 - 4 \end{aligned}$$

(b) At $(1, -2)$: $m = 3(1)^2 - 4 = -1$, so an equation of the tangent line is $y - (-2) = -1(x - 1) \Leftrightarrow y = -x - 1$. At $(2, 1)$: $m = 3(2)^2 - 4 = 8$, so an equation of the tangent line is $y - 1 = 8(x - 2) \Leftrightarrow y = 8x - 15$.

15.

(a) Since the slope of the tangent at $t = 0$ is 0, the car's initial velocity was 0.

(b) The slope of the tangent is greater at C than at B, so the car was going faster at C.

(c) Near A, the tangent lines are becoming steeper as x increases, so the velocity was increasing, so the car was speeding up. Near B, the tangent lines are becoming less steep, so the car was slowing down. The steepest tangent near C is the one at C, so at C the car had just finished speeding up, and was about to start slowing down.

(d) Between D and E, the slope of the tangent is 0, so the car did not move during that time.

16. Let a denote the distance traveled from 1:00 to 1:02, b from 1:28 to 1:30, and c from 3:30 to 3:33,where all the times are relative to $t = 0$ at the beginning of the trip.

21. The sketch shows the graph for a room temperature of 72^o and a refrigerator temperature of 38^o. The initial rate of change is greater in magnitude than the rate of change after an hour.

27.

(a)

(i) $\frac{\Delta C}{\Delta x} = \frac{C(105) - C(100)}{105 - 100} = \frac{6601.25 - 6500}{5} = \$20.25/\text{unit}.$
(ii) $\frac{\Delta C}{\Delta x} = \frac{C(101) - C(100)}{101 - 100} = \frac{6520.25 - 6500}{1} = \$20.25/\text{unit}.$

(b)

$$\frac{C(100+h)-C(100)}{h} = \frac{[5000+10(100+h)+0.05(100+h)^2]-6500}{h}$$
$$= \frac{20h+0.05h^2}{h} = 20+0.05h, h \neq 0$$

So the instantaneous rate of change is $\lim\limits_{h\to 0}\frac{C(100+h)-C(100)}{h} = \lim\limits_{h\to 0}(20+0.05h) = \$20/\text{unit}$.

第三章 導數

[3.1] 導數

9.(a) Using Definition 2 with $F(x)=x^3-5x+1$ and the point $(1,-3)$, we have

$$F'(1) = \lim_{h\to 0}\frac{F(1+h)-F(1)}{h} = \lim_{h\to 0}\frac{[(1+h)^3-5(1+h)+1]-(-3)}{h}$$
$$= \lim_{h\to 0}\frac{(1+3h+3h^2+h^3-5-5h+1)+3}{h} = \lim_{h\to 0}\frac{h^3+3h^2-2h}{h}$$
$$= \lim_{h\to 0}\frac{h(h^2+3h-2)}{h} = \lim_{h\to 0}(h^2+3h-2) = -2$$

So an equation of the tangent line at $(1,-3)$ is $y-(-3)=-2(x-1) \Leftrightarrow y=-2x-1$.

19. By definition 2, $\lim\limits_{h\to 0}\frac{(1+h)^{10}-1}{h} = f'(1)$, where $f(x)=x^{10}$ and $a=1$.
Or: $\lim\limits_{h\to 0}\frac{(1+h)^{10}-1}{h} = f'(0)$, where $f(x)=(1+x)^{10}$ and $a=0$.

31. $T'(10)$ is the rate at which the temperature is changing at 10:00 A.M. To estimate the value of $T'(10)$, we will average the difference quotients obtained using the times $t=8$ and $t=12$. Let

$$A = \frac{T(8)-T(10)}{8-10} = \frac{72-81}{-2} = 4.5 \text{ and } B = \frac{T(12)-T(10)}{12-10} = \frac{88-81}{2} = 3.5.$$

Then $T'(10) = \lim\limits_{h\to 10}\dfrac{T(t)-T(10)}{t-10} \approx \dfrac{A+B}{2} = \dfrac{4.5+3.5}{2} = 4^oF/h.$

35. Since $f(x)=x\sin(\frac{1}{x})$ when $x\neq 0$ and $f(0)=0$, we have

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h\sin(\frac{1}{h})-0}{h} = \lim_{h\to 0}\sin(\frac{1}{h}).$$

This limit does not exist since $\sin(\frac{1}{h})$ takes the values -1 and 1 on any inteval containing 0. (Compare with Example 4 in Section 2.2.)

36. Since $f(x) = x^2 \sin(\frac{1}{x})$ when $x \neq 0$ and $f(0) = 0$, we have

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin(\frac{1}{h}) - 0}{h} = \lim_{h\to 0} h \sin(\frac{1}{h}).$$

Since $-1 \leq \sin\frac{1}{h} \leq 1$, we have $-|h| \leq |h|\sin\frac{1}{h} \leq |h| \Rightarrow -|h| \leq h\sin\frac{1}{h} \leq |h|$. Because $\lim_{h\to 0}(-|h|) = 0$ and $\lim_{h\to 0}|h| = 0$, we know that $\lim_{h\to 0}(h\sin\frac{1}{h}) = 0$ by the Squeeze Theorem. Thus, $f'(0) = 0$.

[3.2] 導函數

4.

(a)'=II, since from left to right, the slopes of the tangents to graph (a) start out negative, become 0, then positive, then 0, then negative again. The actual function values in graph II follow the same pattern.

(b)'=IV, since from left to right, the slopes of the tangents to graph (b) start out at a fixed positive quantity, then suddenly become negative, then positive again. The discontinuities in graph IV indicate sudden changes in the slopes of the tangents.

(c)'=I, since the slopes of the tangents to graph (c) are negative for $x < 0$ and positive for $x > 0$, as are the function values of graph I.

(d)'=III, since from left to right, the slopes of the tangents to graph (d) are positive, then 0, then negative, then 0, then positive,then 0, then negative again, and the function values in graph III follow the same pattern.

25.

$$\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{1+2(x+h)} - \sqrt{1+2x}}{h} \left[\frac{\sqrt{1+2(x+h)} + \sqrt{1+2x}}{\sqrt{1+2(x+h)} + \sqrt{1+2x}}\right] \\
&= \lim_{h\to 0} \frac{(1+2x+2h) - (1+2x)}{h\left[\sqrt{1+2(x+h)} + \sqrt{1+2x}\right]} = \lim_{h\to 0} \frac{2}{\sqrt{1+2x+2h} + \sqrt{1+2x}} \\
&= \frac{2}{2\sqrt{1+2x}} = \frac{1}{\sqrt{1+2x}}
\end{aligned}$$

Domain of $g = [-\frac{1}{2}, \infty)$, domain of $g' = (-\frac{1}{2}, \infty)$.

35. f is not differentiable at $x = -1$ or at $x = 11$ because the praph has vertical tangents at those points; at $x = 4$, because there is a discontinuity there; and at $x = 8$, because the graph has a corner there.

40.(a)

$$g'(0) = \lim_{x\to 0} \frac{g(x)-g(0)}{x-0} = \lim_{x\to 0} \frac{x^{2/3}-0}{x} = \lim_{x\to 0} \frac{1}{x^{1/3}}$$

which does not exist.

(b)

$$\begin{aligned} g'(a) &= \lim_{x\to a} \frac{g(x)-g(a)}{x-a} = \lim_{x\to a} \frac{x^{2/3}-a^{2/3}}{x-a} \\ &= \lim_{x\to a} \frac{(x^{1/3}-a^{1/3})(x^{1/3}+a^{1/3})}{(x^{1/3}-a^{1/3})(x^{2/3}+x^{1/3}a^{1/3}+a^{2/3})} \\ &= \lim_{x\to a} \frac{x^{1/3}+a^{1/3}}{x^{2/3}+x^{1/3}a^{1/3}+x^{2/3}} \\ &= \frac{2a^{1/3}}{3a^{2/3}} = \frac{2}{3a^{1/3}} \text{or } \ \frac{2}{3}a^{-1/3} \end{aligned}$$

(c) $g(x) = x^{2/3}$ is continuous at $x = 0$ and $\lim_{x\to 0} |g'(x)| = \lim_{x\to 0} \frac{2}{3|x|^{1/3}} = \infty$. This shows that g has a vertical tangent line at $x = 0$.

47. In the right triangle in the diagram, let Δy be the side opposite angle ϕ and Δx the side adjacent angle ϕ. Then the slope of the tangent line l is $m = \Delta y/\Delta x = \tan\phi$. Note that $0 < \phi < \frac{\pi}{2}$. We know (see Exercise 17) that the derivative of $f(x) = x^2$ is $f'(x) = 2x$. So the slope of the tangent to the curve at the point $(1, 1)$ is 2. Thus, ϕ is the angle between 0 and $\frac{\pi}{2}$ whose tangent is 2; that is, $\phi = \arctan 2 \approx 63°$

[3.3] 微分公式

25.

$$\begin{aligned} F(y) &= \left(\frac{1}{y^2} - \frac{3}{y^4}\right)\left(y+5y^3\right) = (y^{-2}-3y^{-4})(y+5y^3) \\ F'(y) &= (y^{-2}-3y^{-4})(1+15y^3) + (y+5y^2)(-2y^{-3}+12y^{-5}) \\ &= (y^{-2}+15-3y^{-4}-45y^{-2}) + (-2y^{-2}+12y^{-4}-10+60y^{-2}) \\ &= 5+14y^{-2}+9y^{-4} \text{or } \ 5+14/y^2+9/y^4 \end{aligned}$$

33.

$$y = \frac{1}{x^4+x^2+1} \Rightarrow y' = \frac{(x^4+x^2+1)(0)-1(4x^3+2x)}{(x^4+x^2+1)^2} = -\frac{2x(2x^2+1)}{(x^4+x^2+1)^2}$$

41.

$$\begin{aligned} f(x) &= \frac{x}{x-c/x} \Rightarrow \\ f'(x) &= \frac{(x+c/x)(1)-x(1-c/x^2)}{\left(x+\frac{c}{x}\right)^2} = \frac{x+c/x-x+c/x}{\left(\frac{x^2+c}{x}\right)^2} = \frac{2c/x}{\frac{(x^2+c)^2}{x^2}} \cdot \frac{x^2}{x^2} = \frac{2cx}{(x^2+c)^2} \end{aligned}$$

53. $y = f(x) = x + \sqrt{x} \Rightarrow f'(x) = 1 + \frac{1}{2}x^{-1/2}$. So the slope of the tangent line at $(1, 2)$ is $f'(1) = 1 + \frac{1}{2}(1) = \frac{3}{2}$ and its equation is $y - 2 = \frac{3}{2}(x - 1)$ or $y = \frac{3}{2}x + \frac{1}{2}$.

55. (a)

$$y = f(x) = \frac{1}{1+x^2} \Rightarrow f'(x) = \frac{(1+x^2)(0) - 1(2x)}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2}. \quad (1)$$

So the slope of the tangent line at the point $(-1, \frac{1}{2})$ is $f'(-1) = \frac{2}{2^2} = \frac{1}{2}$ and its equation is $y - \frac{1}{2} = \frac{1}{2}(x+1)$ or $y = \frac{1}{2}x + 1$.

65. $y = f(x) = 1 - x^2 \Rightarrow f'(x) = -2x$, so the tangent in at $(2, -3)$ has slope $f'(2) = -4$. The normal line has slope $-\frac{1}{-4} = \frac{1}{4}$ and equation $y + 3 = \frac{1}{4}(x - 2)$ or $y = \frac{1}{4}x - \frac{7}{2}$.

84. f is clearly differentiable for $x < 2$ and for $x > 2$. For $x < 2$, $f'(x) = 2x$, so $f'_-(2) = 4$. For $x > 2$, $f'(x) = m$, so $f'_+(2) = m$. For f to be differentiable at $x = 2$, we need $4 = f'_-(2) = f'_+(2) = m$. So $f(x) = 4x + b$. We must also have continuity at $x = 2$, so $4 = f(2) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+}(4x + b) = 8 + b$. Hence, $b = -4$.

[3.5] 三角函數的導數

37.

$$\begin{aligned}\lim_{t\to 0}\frac{\sin 6t}{\sin 2t} &= \lim_{t\to 0}\left(\frac{\sin 6t}{t}\cdot\frac{1}{\cos 6t}\cdot\frac{t}{\sin 2t}\right) = \lim_{t\to 0}\frac{6\sin 6t}{6t}\cdot\lim_{t\to 0}\frac{1}{\cos 6t}\cdot\lim_{t\to 0}\frac{2t}{2\sin 2t}\\ &= 6\lim_{t\to 0}\frac{\sin 6t}{6t}\cdot\lim_{t\to 0}\frac{1}{\cos 6t}\cdot\frac{1}{2}\lim_{t\to 0}\frac{2t}{\sin 2t} = 6(1)\cdot\frac{1}{1}\cdot\frac{1}{2}(1) = 3\end{aligned}$$

40.

$$\begin{aligned}\lim_{t\to 0}\frac{\sin^2 3t}{t^2} &= \lim_{t\to 0}\left(\frac{\sin 3t}{t}\cdot\frac{\sin 3t}{t}\right) = \lim_{t\to 0}\frac{\sin 3t}{t}\cdot\lim_{t\to 0}\frac{3t}{t}\\ &= \left(\lim_{t\to 0}\frac{\sin 3t}{t}\right)^2 = \left(3\lim_{t\to 0}\frac{\sin 3t}{3t}\right)^2 = (3\cdot 1)^2 = 9\end{aligned}$$

46. Let $|PR| = x$. Then we get the following formulas for r and h in terms of θ and x: $\sin\frac{\theta}{2} = \frac{r}{x} \Rightarrow r = x\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2} = \frac{h}{x} \Rightarrow h = x\cos\frac{\theta}{2}$. Now $A(\theta) = \frac{1}{2}\pi r^2$ and $B(\theta) = \frac{1}{2}(2r)h = rh$. So

$$\begin{aligned}\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)} &= \lim_{\theta\to 0^+}\frac{\frac{1}{2}\pi r^2}{rh} = \frac{1}{2}\pi\lim_{\theta\to 0^+}\frac{r}{h} = \frac{1}{2}\pi\lim_{\theta\to 0^+}\frac{x\sin(\theta/2)}{x\cos(\theta/2)}\\ &= \frac{1}{2}\pi\lim_{\theta\to 0^+}\tan(\theta/2) = 0.\end{aligned}$$

47. By the definition of radian measure, $s = r\theta$, where r is the radius of the circle. By drawing the bisector of the angle θ, we can see that $\sin\frac{\theta}{2} = \frac{d/2}{r} \Rightarrow d = 2r\sin\frac{\theta}{2}$. So $\lim_{\theta\to 0^+}\frac{s}{d} = \lim_{\theta\to 0^+}\frac{r\theta}{2r\sin\theta/2} = \lim_{\theta\to 0^+}\frac{\theta/2}{\sin\theta/2} = 1$. [This is just the reciprocal of the limit $\lim_{\theta\to 0}\frac{\sin x}{x} = 1$ combined with the fact that as $\theta \to 0$, $\frac{\theta}{2} \to 0$ also.]