臺大數學系新生暑假作業 2022/08

請書寫完整，於 08/31 22:00 前繳交至 Google 表單 https://forms.gle/r8XVZ5YXhU74gTik8，參加 09/02 新生日將使用並檢討本份作業。

1. (a) If a is a rational number and b an irrational number, show that $a + b$ is an irrational number. What about ab?

   (b) Show that $\sqrt[n]{2}$ is not a rational number for any integer $n \geq 2$.

   (c) Show that $\log_{10} 2$ is not a rational number.

2. Find all integer solutions $(x, y) \in \mathbb{Z}^2$ such that $2022x + 678y = 18$. (*Hint*: Use Euclidean algorithm.)

3. On $I = [a, b] \subset \mathbb{R}$, consider the statement P: "There exists $x \in I$ such that for every $y \in I$, $x \leqslant y$". Which of the following statements are $\neg P$?

   (a) For every $y \in I$, there exists $x \in I$ such that $x \leqslant y$.

   (b) There exists $y \in I$ such that for every $x \in I$, $x \leqslant y$.

   (c) For every $x \in I$, there exists $y \in I$ such that $x \leqslant y$.

   (d) There exists $x \in I$ such that for every $y \in I$, $x > y$.

   (e) There exist $x \in I$ and $y \in I$ such that $x > y$.

   (f) For every $x \in I$ and $y \in I$, $x > y$.

   (g) For every $y \in I$, there exists $x \in I$ such that $x > y$.

   (h) For every $x \in I$, there exists $y \in I$ such that $x > y$.

   (i) There exists $y \in I$ such that for every $x \in I$, $x > y$.

4. Determine if the following statements are true or false. Justify your answers.

   (a) $\forall\, x \in \mathbb{R}, \exists\, y \in \mathbb{R}$ such that $y^2 = 4x$.

   (b) $\exists\, y \in \mathbb{R}$ such that $\forall\, x \in \mathbb{R}$ we have $y^2 = 4x$.

   (c) $\forall\, y \in \mathbb{R}, \exists\, x \in \mathbb{R}$ such that $y^2 = 4x$.

   (d) $\exists\, x \in \mathbb{R}$ such that $\forall\, y \in \mathbb{R}$ we have $y^2 = 4x$.

   (e) For every subset $A \subseteq \mathbb{N}$, there exists $a \in A$ such that for every $x \in A$, $a \leqslant x$.

   (f) There is a subset $A \subseteq \mathbb{N}$ such that for every $a \in A$, there exists $x \in A$ such that $a < x$.

   (g) There is a subset $A \subseteq \mathbb{N}$ such that for every $a \in A$, there exists $x \in A$ such that $x < a$.

5. Let X be a set and $A, B, C \subseteq X$. Prove the following statements:

   (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

   (b) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ and $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

   (c) $A \cap B = \varnothing$ if and only if $A \subseteq B^c$.

6. The following problems deal with some relations between a mapping and operations on sets. Given a function $f \colon A \to B$, we have two definitions:

   - The **image** of a subset $X \subseteq A$ under the function f is the set:

   $$f(X) = \{f(x) \mid x \in X\}.$$

   - The **preimage** of a subset $Y \subseteq B$ under the function f is the set:

   $$f^{-1}(Y) = \{x \in A \mid f(x) \in Y\}.$$

   Prove the following statements:

   (a) For any subset $X \subseteq A$, we have $X \subseteq f^{-1}(f(X))$. Also, give an example to explain that the equality may not hold (namely, $X \subsetneq f^{-1}(f(X))$ is possible).

   (b) For any subset $Y \subseteq B$, we have $f(f^{-1}(Y)) \subseteq Y$. Also, give an example to explain that the equality may not hold (namely, $f(f^{-1}(Y)) \subsetneq Y$ is possible).

   (c) Assume that f is injective. Then, for any subset $X \subseteq A$, we have $f(A \setminus X) \subseteq B \setminus f(X)$. Does this statement hold if f is not assumed to be injective?

   (d) For any subsets $Y_1, Y_2 \subseteq B$, we have $f^{-1}(Y_1) \setminus f^{-1}(Y_2) = f^{-1}(Y_1 \setminus Y_2)$.

   (e) For a collection $\{X_i\}_{i=1}^n$ of subsets of A with $X_i \subseteq A$, we have:

      i. $f\left(\bigcup_{i=1}^n X_i\right) = \bigcup_{i=1}^n f(X_i)$

      ii. $f\left(\bigcap_{i=1}^n X_i\right) \subseteq \bigcap_{i=1}^n f(X_i)$. Also, give an example to explain that the equality may not hold (namely, $f\left(\bigcap_{i=1}^n X_i\right) \subsetneq \bigcap_{i=1}^n f(X_i)$ is possible).

   (f) For a collection $\{Y_j\}_{j=1}^m$ of subsets of B with $Y_j \subseteq B$, we have:

      i. $f^{-1}\left(\bigcup_{j=1}^m Y_j\right) = \bigcup_{j=1}^m f^{-1}(Y_j)$.

      ii. $f^{-1}\left(\bigcap_{j=1}^m Y_j\right) = \bigcap_{j=1}^m f^{-1}(Y_j)$.

7. Let $f : A \to B$ and $g : B \to C$ be maps.

   (a) Show that if both f and g are injections, then so is $g \circ f$.

   (b) Show that if both f and g are surjections, then so is $g \circ f$.

   (c) Show that if both f and g are bijections, then so is $g \circ f$.

   (d) Show that if $g \circ f$ is an injection, then f is also an injection. Give an example where g is not an injection but $g \circ f$ is.

   (e) Show that if $g \circ f$ is a surjection, then g is also a surjection. Give an example where f is not a surjection but $g \circ f$ is.

   (f) If $h : B \to A$ is a map such that $h \circ f = \mathbf{1}_A$ and $f \circ h = \mathbf{1}_B$, show that both f and h are bijections.

8. The unit sphere S in $\mathbb{R}^3$ is the set $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Let $N = (0, 0, 1) \in S$ and $\mathscr{M} = S \setminus \{N\}$. Consider the xy-plane $\mathscr{L} = \{(X, Y, 0) \mid X, Y \in \mathbb{R}\}$. The **stereographic projection** $f : \mathscr{M} \to \mathscr{L}$ is a mapping which sends a point $P \in \mathscr{M}$ to a point $P' \in \mathscr{L}$ where P' is the unique intersection point of the line through N and P with the plane $\mathscr{L}$.

   (a) Derive the stereographic projection formula:

   $$f(x, y, z) = \left(\frac{x}{1 - z}, \frac{y}{1 - z}, 0 \right).$$

   (b) Show that f is a bijection and find its inverse explicitly.

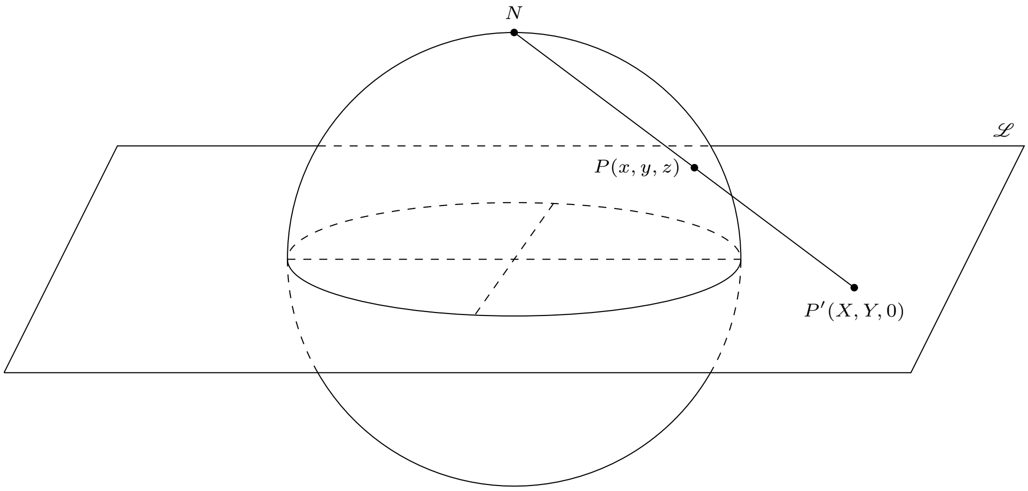

Figure 1: The stereographic projection

9. Fix $n \in \mathbb{N}$, and for any $a, b \in \mathbb{Z}$, let the relation $a \sim b$ mean $n \mid (a - b)$.

   (a) Show that $\sim$ is an equivalence relation on $\mathbb{Z}$.
   Remark: The equivalence class of a will be denoted by $\bar{a}$ or $\bar{a}_n$, and the set of these equivalence classes will be denoted by $\mathbb{Z}/n\mathbb{Z} := \{\bar{k} : k \in \mathbb{Z}\}$. This set has n elements.

   (b) Check that if $a \sim b$, $c \sim d$, then $a + c \sim b + d$.

   (c) Check that if $a \sim b$, $c \sim d$, then $ac \sim bd$.

Here, we will explain "well definedness" for the next exercise. This is an error hard to find when you are writing a proof.

You may want to define the following "taking the numerator" function:

$$f : \mathbb{Q} \to \mathbb{Z} : f(r) = p \quad \text{when } r = \frac{p}{q}.$$

This looks like a function but it is actually *not* a function, for a rational number $r = \frac{p}{q}$ may have another expression (say $\frac{2p}{2q}$), which makes r map to multiple values. In this situation, we say f is not well-defined.[1]

Let us see another example:

$$g : \mathbb{Z}/10\mathbb{Z} \to \mathbb{Z}/10\mathbb{Z} : f(\bar{a}) = \overline{a^2}.$$

Although $\bar{a}$ can be represented by many a (e.g. $a + 10$), such $\overline{a^2}$ are all the same equivalence class (by exercise 9(c)). In this situation, we say g is well-defined. That is, g maps every element in the domain to exactly one element.

The problem arises only when the element in the domain has several representations (like $r = \frac{p}{q} = \frac{kp}{kq}$) and we use these representations to define our function. For example, in defining a function $f : \mathbb{Q} \to \mathbb{Q}$, there is no danger in $f(r) = 1 + r^2$, but we have to be careful with $f(r) = \dfrac{q^2 + p^2}{q^2}$ for $r = \dfrac{p}{q}$.

When we say a "function"[2] f is **well-defined**, it means that f is *indeed* a function. That is, f maps every element in the domain to exactly one element.[3]

In mathematics, we check a function $f : X \to Y$ is well-defined by showing the following: If α, β represent the same element in X, then $f(\alpha) = f(\beta)$.

[1] You may ask p, q to be coprime and $q > 0$ in order to define this function well, but this is off-topic.

[2] Before the well definedness is checked, f is not yet a function!

[3] You can also treat addition as a function (with two variables). In the previous exercise, we checked that addition and multiplication on $\mathbb{Z}/n\mathbb{Z}$ is well-defined.

10. (a) Show that $f : \mathbb{Q} \to \mathbb{Q}$: $f(r) = \frac{p^2+q^2}{q^2}$ for $r = \frac{p}{q}$, is well-defined.

    (b) Show that $f : \mathbb{Z}/10\mathbb{Z} \to \mathbb{Z}/5\mathbb{Z}$: $f(\bar{a}) = \bar{a}$ is well-defined. (*Hint*: These two $\bar{a}$ are different!)

    (c) Show that $f : \mathbb{Z}/10\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$: $f(\bar{a}) = \bar{a}$ is not well-defined.

    (d) Is $f : \mathbb{Z}/10\mathbb{Z} \to \mathbb{Z}/20\mathbb{Z}$: $f(\bar{a}) = \overline{a^2}$ well-defined? Justify your answer.

11. Prove the following identities:

    (a) $\sin\alpha\cos\beta = \dfrac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}$, $\sin\alpha + \sin\beta = 2\sin\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}$.

    (b) $\cos\alpha\sin\beta = \dfrac{\sin(\alpha+\beta)-\sin(\alpha-\beta)}{2}$, $\sin\alpha - \sin\beta = 2\cos\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}$.

    (c) $\cos\alpha\cos\beta = \dfrac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}$, $\cos\alpha + \cos\beta = 2\cos\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}$.

    (d) $\sin\alpha\sin\beta = -\dfrac{\cos(\alpha+\beta)-\cos(\alpha-\beta)}{2}$, $\cos\alpha - \cos\beta = -2\sin\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}$.

    (e) $\dfrac{1}{2} + \cos x + \cos 2x + \cos 3x + \cdots + \cos nx = \dfrac{\sin\left[\left(n+\frac{1}{2}\right)x\right]}{2\sin\frac{x}{2}}$

12. (a) Find a number $\delta > 0$ such that $|x^2 - 4| < \dfrac{1}{2}$ if $|x-2| < \delta$.

    (b) Find a number $\delta > 0$ such that $|\sqrt{x^2+5} - 2| < 1$ if $|x-1| < \delta$.

    (You have to justify your answer.)

13. Use the mathematical induction to prove the following statements.

    (a) Prove that for all integers $n \geq 4$, we have $3^n > n^3$.

    (b) Define a sequence $\{c_n\}_{n=0}^{\infty}$ as follows:

$$\begin{cases} c_{n+1} = \frac{49}{8}c_n - \frac{225}{8}c_{n-2}, & n \geq 2, \\ c_0 = 0, c_1 = 2, c_2 = 16. \end{cases}$$

    Prove that $c_n = 5^n - 3^n$ for all $n \in \mathbb{N} \cup \{0\}$.

    (c) Suppose that $A = \begin{pmatrix} 7 & 12 \\ -2 & -3 \end{pmatrix}$. Prove that for any $n \in \mathbb{Z}$,

$$A^n = \begin{pmatrix} -2 & -6 \\ 1 & 3 \end{pmatrix} + 3^n \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}.$$

    Here $A^{-n} = (A^n)^{-1}$ is the inverse of A^n, and we follow the convention that $A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix.

14. (a) Show that for any matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$, we have the following relation:

$$\mathbf{A}^2 - \operatorname{tr}(\mathbf{A}) \cdot \mathbf{A} + \det(\mathbf{A}) \cdot \mathbf{I_2} = \mathbf{O_2}$$

where $\mathbf{I_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{O_2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\operatorname{tr}(\mathbf{A}) = a + d$ and $\det(\mathbf{A}) = ad - bc$.

(b) Let $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$. Evaluate the matrix $\mathbf{A}^5 - 4\mathbf{A}^4 - 3\mathbf{A}^3 - 7\mathbf{A}^2 - 16\mathbf{A} - 2\mathbf{I_2}$.

15. (Arithmetic and geometric means inequality)

(a) Show that for two non-negative real numbers $a, b \in \mathbb{R}_{\geqslant 0}$, we have

$$\frac{a+b}{2} \geqslant \sqrt{ab}$$

and that the equality holds if and only if $a = b$.

(b) Show that for any $k \in \mathbb{N}$, and 2^k non-negative real numbers $a_1, a_2, \ldots, a_{2^k} \in \mathbb{R}_{\geqslant 0}$, we have

$$\frac{1}{2^k} \sum_{i=1}^{2^k} a_i \geq \sqrt[2^k]{a_1 a_2 \cdots a_{2^k-1} a_{2^k}}$$

and that the equality holds if and only if $a_1 = a_2 = \cdots = a_{2^k}$.

(c) Suppose it is true that for any n ($n \geq 2$) non-negative real numbers $a_1, \ldots, a_n \in \mathbb{R}_{\geqslant 0}$, we have

$$\frac{1}{n} \sum_{i=1}^{n} a_i \geqslant \sqrt[n]{a_1 a_2 \cdots a_{n-1} a_n}$$

and that the equality holds if and only if $a_1 = a_2 = \cdots = a_n$. Show that the inequality remains true for $n - 1$ non-negative real numbers and that the equality holds if and only if they are equal. (*Hint*: Consider $a_n = \frac{1}{n-1}(a_1 + \cdots + a_{n-1})$.)

(d) Show that for any n non-negative real numbers $a_1, \ldots, a_n \in \mathbb{R}_{\geqslant 0}$, we have

$$\frac{1}{n} \sum_{i=1}^{n} a_i \geqslant \sqrt[n]{a_1 a_2 \cdots a_{n-1} a_n}$$

and that the equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

16. (Cauchy–Schwarz inequality) For real numbers $a_1, \ldots, a_n,\ b_1, \ldots, b_n \in \mathbb{R}$, show that

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leqslant (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)$$

and that the equality holds if and only if there exists some constants $c_1, c_2 \in \mathbb{R}$, which are not all 0, such that $c_1(a_1, \ldots, a_n) + c_2(b_1, \ldots, b_n) = (0, \ldots, 0)$. (*Hint*: The quadratic polynomial $\sum\limits_{i=1}^{n} (a_ix + b_i)^2$ in x is always non-negative.)

17. (Triangle inequality) Write $\|\mathbf{x}\| = \sqrt{\sum\limits_{k=1}^{n} x_k^2}$ for $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Show that

$$\|\mathbf{x} + \mathbf{y}\| \leqslant \|\mathbf{x}\| + \|\mathbf{y}\|$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and that the equality holds if and only if there exist some non-negative constants $c_1, c_2 \in \mathbb{R}$, which are not all 0, such that $c_1\mathbf{x} = c_2\mathbf{y}$.

(When $n = 1$, this inequality is just $|a + b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$.)

提醒：如果不理解題目該證明什麼，請至新生講義裡確認定義後再作答，並多多查詢英文單字字義。