Algebra Competition 2020

1. Let F be a field and n be a positive integer. Let A be an $n \times n$ matrix over F such that $A^n = 0$ but $A^{n-1} \neq 0$. Show that any $n \times n$ matrix B over F that commutes with A is contained in the F-linear span of $I, A, A^2, \ldots, A^{n-1}$.

2. For a prime power $q = p^n$, let $\mathrm{PSL}(2, \mathbb{F}_q)$ denote the projective special linear group of dimension 2 over $\mathbb{F}_q$.

(a) Prove that if $q \equiv \pm 1 \mod 8$, then $\mathrm{PSL}(2, \mathbb{F}_q)$ contains a subgroup isomorphic to S_4. (You may use without proof that a group presentation of S_4 is $\langle a, b, c : a^2 = b^3 = c^4 = abc = e \rangle$, where e denotes the identity element.)

(b) Prove that A_7 contains a subgroup isomorphic to $\mathrm{PSL}(2, \mathbb{F}_7)$.

3. Let D be an integral domain. A function $N : D \to \mathbb{Z}_{\geq 0}$ is said to be a Dedekind-Hasse norm on D if

(i) $N(0) = 0$,

(ii) $N(a) > 0$ if $a \neq 0$, and

(iii) for any nonzero elements a and b in D, either $b|a$ or there exist elements x and y in D such that $N(xa - yb) < N(b)$.

Also, a nonzero element d of D is said to be a universal side divisor if d is not a unit and has the property that for any a in D, either $d|a$ or there exists a unit u in D such that $d|(a - u)$.

(a) Prove that if an integral domain has a Dedekind-Hasse norm, then it is a principal ideal domain.

(b) Prove that if an integral domain is a Euclidean domain, but not a field, then it has a universal side divisor.

(c) Prove that $\mathbb{Z}[(1 + \sqrt{-19})/2]$ is a principal ideal domain, but not a Euclidean domain.

4. (a) Let K be a finite extension of $\mathbb{Q}$. Suppose that a, b, c are elements of K such that a is not a square in K and $a(b^2 - ac^2)$ is a square in K. Prove that $K(\sqrt{b + c\sqrt{a}})$ is a cyclic extension of degree 4 over K. (A finite extension L/K is said to be a cyclic extension if L/K is Galois and the Galois group is cyclic.)

(b) Let $\alpha = \sqrt{(2 + \sqrt{2})(2 + \sqrt{3})(3 + \sqrt{6})}$. Prove that the field $K = \mathbb{Q}(\alpha)$ is a Galois extension of $\mathbb{Q}$ and that the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ is isomorphic to the quaternion group of order 8.

5. Let A be a two-dimensional associative unital algebra over a field F. (That is, A is a vector space of dimension 2 over F that is also a ring with 1 such that the multiplication is F-bilinear.)

(a) Prove that A must in fact be commutative.

(b) Prove that if F is algebraically closed, then either $A \simeq F \times F$ or $A \simeq F[x]/(x^2)$.

(c) In the case $F = \mathbb{R}$, classify all possible A, up to isomorphisms.