胡達開先生紀念獎學金代數競賽

台大數學系

2019 年 5 月 17 日, 18:00 - 21:00

1. (20 pt.) Let K be a field with $\operatorname{char} K = p > 0$ and $A \in M_n(K)$. Suppose that

$$A^{p^k} = QAQ^{-1}$$

for some $k \in \mathbb{N}$ and $Q \in \mathrm{GL}_n(\overline{K})$. Show that $A^{p^m} = A$ for some $m \in \mathbb{N}$.

2. Prove that the following equations have only the assigned solutions in $\mathbb{Z}$:

(1) (10 pt.) $x^3 - y^2 = 1$, $(x, y) = (1, 0)$.

(2) (15 pt.) $x^3 - y^2 = 5$, no integer solutions.

3. Given a ring homomorphism $f : A \to B$, B is called an A-algebra if B is generated by $f(A)$ and $C_B(A) := C_B(f(A))$.

(1) (10 pt.) If P is a prime ideal[1] of B show that $f^{-1}P$ is a prime ideal of A if B is an A-algebra. Give a counterexample for general f.

(2) (15 pt.) For two A-algebras B and C, show that there is a unique natural A-algebra structure on $B \otimes_A C$. (Hint: consider $C_B(A) \otimes_{\mathbb{Z}} A \otimes_{\mathbb{Z}} C_C(A)$.)

4. Decompose the group rings $\mathbb{C}G$ and $\mathbb{Q}G$, either abstractly or explicitly, into product of simple factors for the following cases:

(1) (10 pt.) $G = C_n$, the cyclic group of order $n \in \mathbb{N}$.

(2) (10 pt.) $G = Q_8$, the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\} \subset \mathbb{H}$.

(3) (10 pt.) $G = D_n$, the dihedral group of order $2n$.

You may work on each part independently by assuming the previous parts in the same problem.

[1] An ideal $P \subsetneq R$ is prime if for any two ideals I, J of R, $IJ \subset P$ implies $I \subset P$ or $J \subset P$.