ALGEBRA COMPETITION 2018

請同學以題組整體完成度為優先考量。

1. For $A \in M_n(\mathbb{Q})$, define $\widetilde{A} \in M_{2n}(\mathbb{Q})$ by
$$\widetilde{A} = \begin{pmatrix} 0 & I_n \\ A & 0 \end{pmatrix}.$$
   (a) Express the characteristic polynomial and the minimal polynomial of $\widetilde{A}$ in terms of those of A.
   (b) Suppose that $\widetilde{A}$ and $\widetilde{B}$ are similar in $M_{2n}(\mathbb{Q})$. Either prove that A and B are similar in $M_n(\mathbb{Q})$, or disprove this statement with a counter-example.

2. Let p be a prime, $\mathfrak{S}_p$ the symmetric group of degree p and G a transitive subgroup of $\mathfrak{S}_p$. Prove the following.
   (a) The order $|G|$ of G is divisible by p, and G contains a subgroup of order p.
   (b) If H is a non-trivial normal subgroup of G, then H is a transitive subgroup of $\mathfrak{S}_p$.
   (c) G is solvable if and only if G contains only one subgroup of order p.

3. Let $\rho : G \to \mathrm{GL}(V)$ be a faithful finite dimensional complex representation of a finite group G with character χ. (Being *faithful* means ρ is injective.) For every $n \in \mathbb{Z}_{\geq 0}$, denote by χ_n the character of $V^{\otimes n}$.

   Let ω be an irreducible complex character of G.
   (a) Let $a_n = \sum_{x \in G} \chi_n(x) \cdot \overline{\omega(x)}$. Show that the generating function
$$f(t) := \sum_{n \geq 0} a_n t^n \in \mathbb{C}[[t]]$$
   is a rational function and give a formula for the rational function $f(t)$ in terms of the characters χ and ω.
   (b) Determine the radius of convergence of the power series $f(t)$. Show that $f(t)$ is not identically equal to 0. Conclude that there are infinitely many $m \in \mathbb{Z}_{\geq 0}$ such that ω appears in $V^{\otimes m}$.

4. Let A be a commutative ring with unity and $\Lambda(A) = 1 + tA[[t]] \subset A[[t]]$. For $f = \sum_{k \geq 0} a_k(f) t^k \in \Lambda(A)$, let $f_n = \sum_{k=0}^{n} a_k(f) t^k$ be the truncation and formally write $f_n = \prod_{i=1}^{n} (1 - \xi_{n,i}(f) t)$. Thus $(-1)^k a_k(f)$ are elementary symmetric polynomials of $\xi_{n,i}(f)$. Let
$$\mu_n(f,g) = \prod_{i,j=1}^{n} (1 - \xi_{n,i}(f) \xi_{n,j}(g) t)$$
   Notice that the expansion of $\mu_n(f,g)$ is invariant under permutations of $\xi_{n,i}(f)$ and of $\xi_{n,j}(g)$ and we therefore regard $\mu_n(f,g)$ as in $A[\xi_{n,i}(f), \xi_{n,j}(g), t]^{\mathfrak{S}_n \times \mathfrak{S}_n} = A[t]$.

(a) Show that for a fixed integer $k \geq 0$, the coefficient of t^k in $\mu_n(f, g)$ is stationary as $n \to \infty$. Conclude that $\mu(f, g) := \lim_{n\to\infty} \mu_n(f, g)$ yields a well-defined binary operator on $\Lambda(A)$.

(b) Let $\alpha(f, g) = fg$, the multiplication in $A[[t]]$. Show that $\Lambda(A)$ with addition α and multiplication μ, defines a commutative ring with unity $(1 - t)$.

(c) Define $s_k : \Lambda(A) \to A, k \geq 1$, such that

$$\sum_{k\geq 1} s_k(f)t^k = -t\frac{d \log f}{dt} := -\frac{tf'}{f}.$$

Show that each s_k is a ring homomorphism.