ALGEBRA COMPETITION 2017

請同學以題組整體完成度爲優先考量。

(1) A K_n representation $R = (V_1, V_2, \phi_a)$ consists of finite-dimensional complex vector spaces V_1 and V_2, and n linear maps $\phi_a : V_1 \to V_2$ for $a = 1, \cdots, n$. A morphism between K_n representations (V_1, V_2, ϕ_a) and (W_1, W_2, ψ_a) consists of linear map $f_i : V_i \to W_i$ for $i = 1, 2$ such that $\psi_a f_1 = f_2 \phi_a$ for $a = 1, \cdots, n$. An isomorphism(injection) of representations is a morphism (f_1, f_2) such that f_1 and f_2 are isomorphisms(injections). Given $\theta_1, \theta_2 \in \mathbb{R}$, we define $\Theta(R) = \big(\dim(V_1)\theta_1 + \dim(V_2)\theta_2\big)/\big(\dim(V_1) + \dim(V_2)\big)$. A representation R is called Θ-semistable if for every subrepresentation R' of R we have $\Theta(R') \leq \Theta(R)$.

(a) Show that a K_2 representation $R = (\mathbb{C}^k, \mathbb{C}^k, \phi_1, \phi_2)$ is indecomposable if and only if R is isomorphic to $(\mathbb{C}^k, \mathbb{C}^k, \mathrm{Id}_k, J_\lambda)$, where J_λ is a matrix in canonical Jordan form with only one block.

(b) Show that if $\theta_1 < \theta_2$ the space of Θ-semistable K_n representations with nonzero n_1 and n_2 is empty.

(c) Show that if $\theta_1 > \theta_2$ the space of isomorphic Θ-semistable K_n representations with dimension vector $(n_1, n_2) = (1, 1)$ is $\mathbb{CP}^n$.

(d) Suppose $\theta_1 > \theta_2$ and $(n_1, n_2) = (1, m)$ with $n > m > 1$. Please construct the space of isomorphic Θ-semistable K_n representations.

(2) Let $GL_3(\mathbb{F}_2)$ be the group of 3×3 invertible matrices with entries in $\mathbb{F}_2$.

(a) Consider the action of $GL_3(\mathbb{F}_2)$ on the nonzero vectors in $\mathbb{F}_2^3$. Show that there exists an injective group homomorphism from $GL_3(\mathbb{F}_2)$ to S_7.

(b) Let A be the set of 2-dimensional subspaces in $\mathbb{F}_2^3$ and B be the set

$$B = \{\{v_1, v_2, v_3\} \mid v_i \in \mathbb{F}_2^3, \quad v_1, v_2, v_3 \text{ are linearly independent over } \mathbb{F}_2\}.$$

Compute the cardinalities of A and B and show that $GL_3(\mathbb{F}_2)$ acts on A and B transitively.

(c) The resolvent $\Theta_g(y)$ of $g(x) \in \mathbb{Q}[x]$ of degree $n \geq 4$ with roots $\alpha_1, \cdots, \alpha_n$ is defined to be

$$\Theta_g(y) = \prod_{1 \leq i < j < k \leq n} \left(y - (\alpha_i + \alpha_j + \alpha_k)\right).$$

Let $f(x) \in \mathbb{Q}[x]$ be irreducible of degree 7 and $\Theta_f(y)$ be its resolvent, which we assume to be separable. Show that if the Galois group of $f(x)$ is isomorphic to $GL_3(\mathbb{F}_2)$, then $\Theta_f(y) = f_1(y) f_2(y)$, where $f_1(y), f_2(y) \in \mathbb{Q}[y]$ are irreducible of degree 7 and 28 respectively.

(3) Let D be a division ring and D^* be the multiplicative group of non-zero elements in D.

(a) Show that if $|D| < \infty$, then D is a field. (Hint: Consider the center F of D and the class equation for D^*.)

(b) Assume $\mathrm{char}(D) = p > 0$. Show that if G is a finite subgroup of D^*, then G is cyclic.

(c) Let D be the Hamilton's real quaternions $H = \{a_0 + a_1 i + a_2 j + a_3 k \ : \ a_i \in \mathbb{R}\}$. Show that $x^2 + 1 = 0$ has infinitely many roots in D and $\sum_{i=0}^{n} a_i x^i = 0$ (where $a_1, \ldots, a_n \in \mathbb{R}$ and $a_0 \in D$ but $a_0 \notin \mathbb{R}$) has at most n roots in D.

(4) Let $I = \langle f_1, f_2 \rangle$, where $f_1 = xz - y^2$ and $f_2 = x^3 - z^2$ in $\mathbb{C}[x, y, z]$.

(a) Let $f = -4x^2y^2z^2 + y^6 + 3z^5$. Determine if $f \in I$. (Justify your answer.)

(b) Let $g = xy - 5z^2 + x$. Determine if $g \in I$. (Justify your answer.)