ALGEBRA COMPETITION 2016

請同學以題組整體完成度爲優先考量。

1. (25%) Let G be the group consisting of invertible two by two matrices with entries from a finite field $\mathbb{F}_q$.
   (a) Compute the number of conjugacy classes of G.
   (b) What are the possible cardinalities for these conjugacy classes in G?

2. (25%)
   (a) Let $\omega = \frac{-1+\sqrt{-3}}{2}$, a root of the equation $x^2 + x + 1 = 0$. Show that $\mathbb{Z}[\omega]$ is a UFD.
   (b) Let p be a prime in $\mathbb{Z}$. Show that if $p \equiv 2 \pmod 3$, then p is a prime in $\mathbb{Z}[\omega]$
   (c) Let p be a prime in $\mathbb{Z}$. Show that if $p \equiv 1 \pmod 3$, then p is not a prime in $\mathbb{Z}[\omega]$.
   (d) Is 3 a prime in $\mathbb{Z}[\omega]$? (Justify your answer)

3. (25%) Let $K = \mathbb{C}(t)$, the field of rational functions in the variable t with complex coefficients. Let $\zeta \in \mathbb{C}$ be a primitive n-th root of unity. Consider the automorphism σ and τ of K over $\mathbb{C}$ defined by $\sigma(t) = t^{-1}$ and $\tau(t) = \zeta t$. Let G be the subgroup in $Aut(K/\mathbb{C})$ generated by σ and τ, and K^G be the fixed field of G.
   (a) Show that G is isomorphism to the dihedral group of order $2n$.
   (b) Compute the minimal polynomial of t over K^G.
   (c) Show that the fixed field K^G is $\mathbb{C}(u)$ for some u in $\mathbb{C}(t)$. Compute u explicitly.

4. (25%) Let A be a given finite abelian group. Let $\hat{A}$ be the set of all homomorphisms (characters) from A to the multiplicative groups of non-zero complex numbers.
   (a) Show that A and $\hat{A}$ have the same cardinality $\mid A \mid = N$.
   (b) Take N variables indexed by $a \in A$, say $\{X_a\}_{a \in A}$, and consider $\det(X_{ab^{-1}})$ as a homogeneous polynomial in these N variables. Prove the following factorization of $\det(X_{ab^{-1}})$ in $\mathbb{C}[X_a]$ as product of linear factors
   $$\det(X_{ab^{-1}}) = \prod_{\chi \in \hat{A}} \Big(\sum_{a \in A} \chi(a) X_a \Big).$$
   (c) Use (b) to write down the matrix $(X_{ab^{-1}})$ and the factorization of $\det(X_{ab^{-1}})$ explicitly in the case of $A = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.